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Reallocating and Pricing Illiquid Capital: Two Productive Trees

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ABSTRACT

We develop a two-sector general-equilibrium model with capital accumulation and convex adjustment costs to analyze sectoral capital reallocation and asset pricing. Consumers weigh the diversification benefits of spreading investment across sectors against the productivity gains of concentrating capital in the more productive sector. We derive conditions under which sectoral capital reallocation shapes both sectoral and aggregate outcomes. Our framework highlights the importance of heterogeneity and capital liquidity—the ease of reallocating capital—in driving growth and asset prices, and uncovers a fundamental trade-off: while diversification enhances risk sharing, reallocation may dampen aggregate growth.

JEL Classification: D53, E22, E44, G12

1 | Introduction

Economic fluctuations often begin in a distinct sector, and then propagate throughout the economy. Yet equilibrium models in macro and finance typically assume a single representative firm, emphasizing the equilibrium response to aggregate shocks and de-emphasizing distribution and propagation. Sargent (1980) uses an irreversible investment version of the stochastic one-sector growth model as a vehicle for making some observations about the q theory of investment. Existing models that incorporate multiple sectors assume either that capital is perfectly liquid and can be reallocated frictionlessly, as in Cox et al. (1985, hereafter CIR), or that capital is completely illiquid and fixed, as in Lucas (1978) and multisector versions by Santos and Veronesi (2006), Cochrane et al. (2008, hereafter CLS), and Martin (2013). When capital is perfectly liquid as in CIR, Tobin's q is one at all times and heterogeneity plays no role in equilibrium. When capital is completely illiquid as in CLS, investment is zero at all times. We model capital reallocation and asset pricing jointly in a general equilibrium model with two sectors. In our model, investment drives both Tobin's q and the distribution of capital; these results fundamentally differ from both CIR and CLS due to the costly reallocation of capital.

We use convex capital adjustment costs to capture illiquidity. We show that the distribution of capital is the single state variable determining equilibrium capital reallocation and asset pricing in our framework. We first develop a baseline case with log utility to establish analytic findings in a simple benchmark case and then extend the model to a more general nonexpected utility framework.

When the two sectors are ex ante identical in the baseline log utility model, the economy tends toward a symmetric equilibrium, where the two sectors are of equal size and the consumer achieves maximum diversification and utility. With log utility, there is little variation in aggregate variables, regardless of the distribution of capital. However, when the two sectors are of equal size, the equilibrium interest rate attains the highest level as the consumer's precautionary saving demand is the highest. Within the sectors, there is substantial impact of the capital distribution. In particular, as one sector becomes small, its investment rate skyrockets—not because its marginal returns increase (we have constant-returns-to-scale), but rather because its cost of capital falls. This occurs because the small sector becomes virtually risk-free, and retaining the small sector and its potential for diversification becomes extremely valuable. This result demonstrates another advantage of the two-sector framework: even in the small-sector case, where the sector has a negligible feedback to equilibrium variables, the presence of the equilibrium affects sectoral decisions. We can therefore gauge the importance of general equilibrium on the results, by scaling from a negligibly small sector that approximates "partial equilibrium" to a single dominating sector that equates to standard one-sector general-equilibrium analyses.

We demonstrate the importance of adjustment costs for both growth and asset pricing. A high cost of reallocating capital acts as a tax on savings, so high adjustment costs deter savings, instead promoting consumption and dampening growth. Asset prices rise with adjustment costs because the rents to installed capital are higher, but the rates of return to capital investment are lower. These results give a hint of our findings when liquidity varies endogenously, expanding upon these comparative static results.

When we extend beyond log utility to recursive utility (Epstein and Zin 1989; Duffie and Epstein 1992a), the effect of the capital distribution becomes more pronounced—even for aggregate variables. With higher risk aversion, when the household is less diversified because of an unbalanced distribution of capital, the household responds by adjusting the consumption–savings decision: saving more and consuming less. This raises investment and growth, but also results in higher risk premia and expected returns, so asset prices fall. Hence, the distribution of capital affects both economic growth and asset prices.

Similarly, when the sectors are not ex ante identical or the two sectors have different adjustment costs (so that one is more liquid than the other), shocks to the distribution of capital affect the overall liquidity of the economy. When some of the economy's liquid capital is destroyed, even while liquid capital is rebuilt, aggregate investment and growth decline. The interest rate falls because the consumer has a greater incentive to save in the illiquid economy. The risk premium for liquid capital falls, but the risk premium on illiquid capital rises. Because the overall economy is now more illiquid, the aggregate risk premium rises and so does aggregate return volatility. Interestingly, changes in liquidity cause investment and Tobin's q to move in opposite directions: higher liquidity increases investment even while Tobin's q declines.

Our findings convey both the importance of heterogeneity and also caution about modeling heterogeneity. Initially, with symmetry and log utility, we establish a benchmark where the two sectors show interesting internal dynamics but have quantitatively small effects on the aggregate. However, when we depart from log utility or allow for asymmetry, the aggregate effects are substantial. First, even in the symmetric case, higher risk aversion (we consider risk aversion as high as four), the endogenous response of savings to an undiversified economy causes investment and growth to depend on the distribution of capital. Moreover, this generates a higher value of capital that is reflected in higher asset prices and lower rates of return. With ex ante asymmetry between the two sectors, changes in the distribution of capital determine the overall liquidity in the economy. These endogenous compositional changes in liquidity drive investment and growth, as well as asset prices and rates of return.

In our model, the benefits of diversification are counterbalanced by costs of reallocation. Eisfeldt and Rampini (2006) emphasize the costs of reallocation, showing in a quantitative setting that these costs must be time-varying and countercyclical in order to generate the observed pattern of capital reallocation. In our model, the cost of reallocation varies endogenously, generating large shifts in sectoral investment (e.g., as described above for a small sector) and reallocation of capital.

Since the initial release of our paper in 2008, several studies have built on key features of our two-sector equilibrium framework. Hansen et al. (2020) and Hansen and Sargent (2021) use an extended version of our two-sector model (with a richer stochastic structure) as a quantitative laboratory to study equilibrium effects of ambiguity aversion.[1] Kozak (2022) builds on our model and uses analytically tractable functional forms similar to ours to derive implications on bonds and stock return dynamics. Hansen et al. (2024) review a class of production equilibrium models with sluggish heterogenous capital stocks that extends on Eberly and Wang (2009, 2011). Bansal et al. (2017) analyze the roles that mean–variance and hedging demand play in accounting for sectoral shifts, extending a variant of our model to allow for goods to be imperfect substitutes and the preference for individual goods to vary over time. Li and Li (2025) develop a continuous-time two-sector equilibrium production model to demonstrate that mispricing of government credit support generates a downward bias in firm quality distribution that is self-perpetuating. Nguyen et al. (2025) generalize our two-sector q-theoretic model by incorporating disaster risks into a climate economic setting where the likelihood of climate change is an endogenous outcome of firm production.

The remainder of the paper is organized as follows. First, we present a baseline model with log utility in Section 2 and solve this baseline model in Section 3. We then use the analytical results to provide a quantitative analysis for a case with quadratic adjustment costs in Section 4. In Section 5, we extend the model to allow for a separation of risk aversion from elasticity of intertemporal substitution in order to have a better asset-pricing fit (Bansal and Yaron 2004), and then use this extended model to show how capital illiquidity, risk aversion, and ex ante sectoral asymmetry affect the aggregate and sectoral results. We conclude in Section 6. Appendices A and B provide supporting technical details for our baseline log utility and extended recursive utility models, respectively.

2 | Baseline Two-Sector Model

Consider an infinite-horizon continuous-time production economy. There are two productive sectors in the economy, sectors 0 and 1.

Let K_n, I_n, and Y_n denote the representative firm's capital stock, investment, and output, respectively, in sector n, where $n = 0, 1$. The representative firm in each sector has an "AK" production technology:

$$Y_n(t) = A_n K_n(t), \tag{1}$$

where $A_n > 0$ is the (constant) productivity in sector n. Capital accumulation is given by

$$dK_n(t) = \Phi_n(I_n(t), K_n(t))dt + \sigma_n K_n(t) dB_n(t)\,,\ n = 0, 1, \tag{2}$$

where $B_n(t)$ is a standard Brownian motion driving shocks to capital in sector n and the function $\Phi_n(I_n, K_n)$ measures the effectiveness of converting investment goods into installed capital in sector n.[2]

To ease exposition of our baseline model, we assume that (1) the volatility parameter for the two sectors is the same so that $\sigma_0 = \sigma_1 = \sigma > 0$ and (2) the correlation coefficient between the Brownian motions B_0 and B_1, denoted by ρ, is zero. Shocks appear in the capital accumulation dynamics (2) as in CIR and the endogenous growth models (see, e.g., the handbook chapter by Jones and Manuelli 2005). Similarly, Dumas (1992) considers shocks to allocations in a two-country model with linear adjustment (shipping) costs.

A representative consumer has a logarithmic utility given by

$$\mathbb{E}\left(\int_0^\infty e^{-\alpha t} \alpha \ln C(s)\, ds\right), \tag{3}$$

where $\alpha > 0$ is the subjective discount rate. We consider a more general recursive (Epstein–Zin) utility formulation in Section 5. The consumer is endowed with financial claims on the aggregate output from both sectors in the economy. Markets are complete.

As in Lucas and Prescott (1971), Hayashi (1982), and Jermann (1998), we assume that the adjustment technology in each sector is homogeneous of degree one in I and K, so we can write the installation function as follows:

$$\Phi_n(I_n, K_n) = \phi_n(i_n) K_n, \tag{4}$$

where $i_n \equiv I_n/K_n$ is the investment–capital ratio in sector n. We require $\phi_n'(\cdot) > 0$ and $\phi_n''(\cdot) < 0$. In an earlier paper, Eberly and Wang (2009), we use this specification in a deterministic model to examine the effects of capital reallocation on growth.

Now consider the market equilibrium. The representative consumer chooses his consumption and a complete set of financial claims to maximize (3). The representative firm in each sector takes the equilibrium stochastic discount factor (SDF) as given and maximizes firm value. All produced goods are either consumed or invested in one or the other of the two sectors, so the goods-market clearing condition holds:

$$C = Y_0 + Y_1 - I_0 - I_1. \tag{5}$$

In equilibrium, the representative consumer holds his financial claims on aggregate output in both sectors.

We conjecture and later verify that representative agent's value function $J(K_0, K_1)$ can be written in the following form:

$$J(K_0, K_1) = \ln\left[(K_0 + K_1)N(z)\right] = \ln(K_0 + K_1) + \ln N(z), \tag{6}$$

where z is the ratio between sector 1 capital K_1 and the aggregate capital $(K_0 + K_1)$:

$$z \equiv \frac{K_1}{K_0 + K_1} \tag{7}$$

and $N(z)$ is a function to be determined. Since physical capital is nonnegative, we have $0 \le z \le 1$. The state variables are the capital stocks in the two sectors. The homogeneity property of the model implies that the effective state variable is z after controlling for the size of the economy $K_0 + K_1$.

Interpretations of the Sectors. A natural question that arises is how to interpret the two sectors in our model. We highlight three illustrative applications. First, the framework can capture the transition from a traditional industrial economy to one increasingly driven by a fast-growing AI sector. In this context, the model allows for an explicit analysis of how AI-sector-specific shocks affect equilibrium capital reallocation. Second, the model lends itself to international settings in which the North and South differ in production technologies and capital accumulation dynamics. In such cases, capital illiquidity can reflect varying institutional constraints across regions. Third, the framework naturally extends to climate economics. For instance, the one-sector climate model of Barnett et al. (2020) can be generalized to a two-sector setting in which capital is endogenously allocated between green and brown sectors. In a related but distinct application, Nguyen et al. (2025) build on our model by incorporating disaster risks (Barro 2006; Hong et al. 2023a, 2023b) and other features to study how investment income taxation may incentivize capital reallocation from brown to green sectors.

3 | Solution

We first derive equilibrium allocations in Subsection 3.1 by solving a planner's problem. We derive asset-pricing implications in Subsection 3.2 and then obtain the decentralized market solution in Subsection 3.3 (see Appendix A.3 for details.) The one-sector solution (in Appendix A.1) serves as the boundary conditions for our two-sector model solution.[3]

3.1 | Planner's Solution

Let i denote the aggregate investment–capital ratio: the ratio between aggregate investment $(I_0 + I_1)$ and aggregate capital $(K_0 + K_1)$, so that $i \equiv (I_0 + I_1)/(K_0 + K_1)$. Using the definitions of z and sectoral investment–capital ratio $i_n = I_n/K_n$, we have

$$i(z) = (1 - z)i_0(z) + z i_1(z). \tag{8}$$

Scaling the goods-market equilibrium market condition in Equation (5) implies

$$c(z) + (1 - z)i_0(z) + z i_1(z) = A_0(1 - z) + A_1 z, \tag{9}$$

where $c(z) = C/(K_0 + K_1)$.

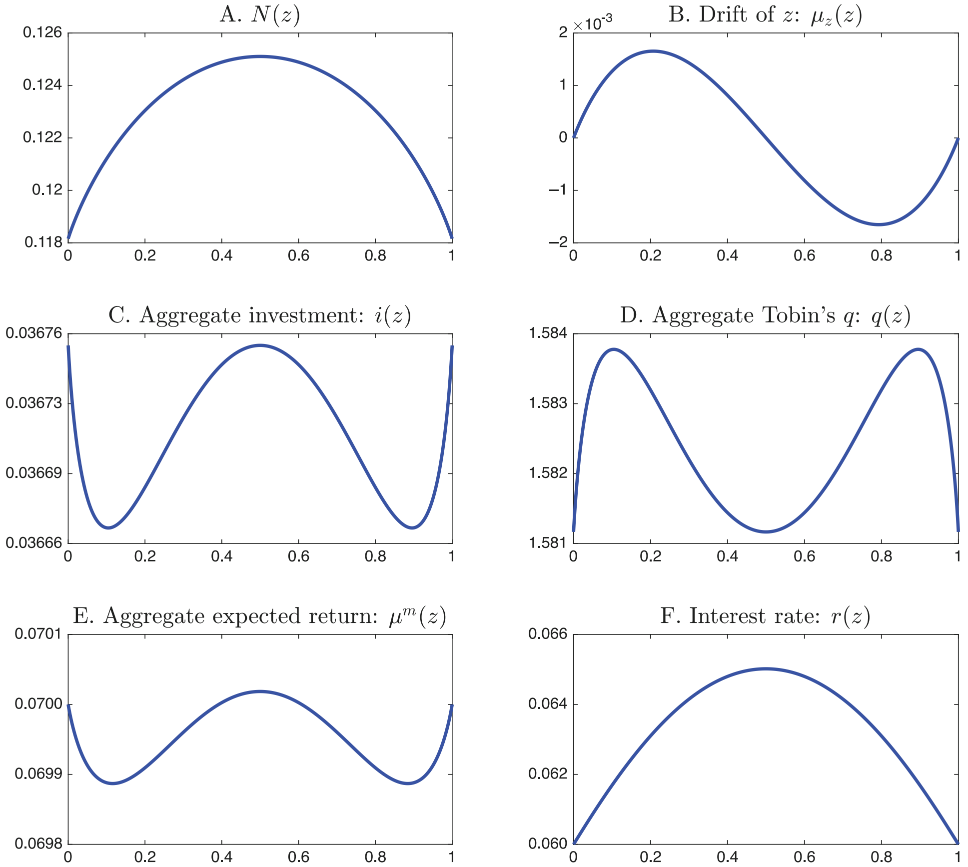

FIGURE 1 | Aggregate results (log utility).

3.1.1 | Investment and Endogenous Growth

Adjustment costs drive a wedge between gross investment I_n and the expected change in the capital stock in the economy $\Phi_n(I_n, K_n)$. The function $\Phi_n(I_n, K_n)$, which controls the effectiveness of converting investment goods into installed capital, allows for both depreciation, so that there is a difference between gross and net investment, and also investment adjustment costs so that investment goods are used up in the installation process. The expected growth rate $\phi_n(i_n)$ of capital nets out both depreciation and installation costs, so that the growth in the capital stock is less than both gross investment i_n and the traditional notion of net investment.

Let $g_n(z)$ denote the expected growth rate of capital in sector n. Using (2), we have $g_n(z) = \phi_n(i_n(z))$, which differs from sectoral gross investment $i_n(z)$. Let $g(z)$ denote the expected growth rate of aggregate capital $(K_0 + K_1)$. We thus have

$$g(z) = (1 - z)g_0(z) + zg_1(z). \tag{10}$$

3.1.2 | Endogenous Capital Reallocation

The evolution of z is given by

$$dz_t = \mu_z(z_t)dt + \sigma z_t(1 - z_t)(dB_1(t) - dB_0(t)), \tag{11}$$

where the drift $\mu_z(z)$ is given by

$$\mu_z(z) = z(1 - z)\left[g_1(z) - g_0(z) + (1 - 2z)\sigma^2\right]. \tag{12}$$

Adjustment costs make z potentially slow moving as seen from (12). Note that the drift $\mu_z(z)$ depends on $g_1(z) - g_0(z)$, the difference between the *endogenous* capital growth rates in the two sectors. The larger this difference, the more capital reallocation occurs in equilibrium. The sectoral growth rates will endogenously differ between the two sectors because of the "imbalance" between the two capital stocks (i.e., $z \neq 1/2$) even when the two sectors have the same technology: $g_1(z) = g_0(z)$ for $z \in [0, 1]$. Moreover, the persistence of z induced by adjustment costs expose the consumer to more sector-specific risk, *ceteris paribus*.[4] Note that the volatility of dz_t is a quadratic function in z which attains its highest value at $z = 1/2$ and becomes zero at $z = 0, 1$ (i.e., the one-sector economy is absorbing), as in the two-tree pure-exchange model of CLS.

3.1.3 | Value Function and Optimal Policies

The following Hamilton–Jacobi–Bellman (HJB) equation describes the planner's problem:[5]

$$0 = \max_{I_0, I_1} \ \alpha \ln C - \alpha J + \Phi_0(I_0, K_0)J_0 + \Phi_1(I_1, K_1)J_1 + \frac{1}{2}\sigma^2 K_0^2 J_{00} + \frac{1}{2}\sigma^2 K_1^2 J_{11}. \tag{13}$$

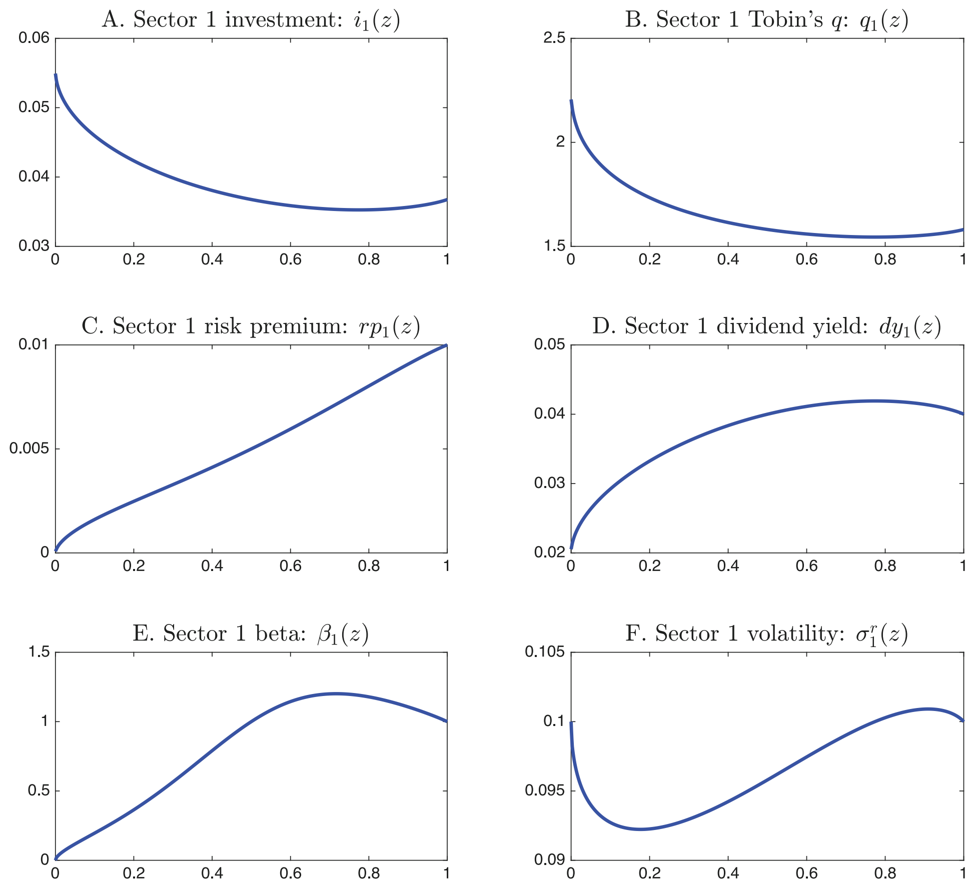

FIGURE 2 | Sectoral implications (log utility).

Using the conjectured value function (6), we obtain the following two first-order conditions (FOCs) with respect to the sectoral investment–capital ratios i_0 and i_1:

$$\frac{c^*(z)}{N(z)} = \frac{\alpha}{\phi_0'(i_0^*(z))} \frac{1}{N(z) - zN'(z)}, \tag{14}$$

$$\frac{c^*(z)}{N(z)} = \frac{\alpha}{\phi_1'(i_1^*(z))} \frac{1}{N(z) + (1-z)N'(z)}, \tag{15}$$

where $c^*(z)$ is the optimal aggregate consumption–capital ratio satisfying (9).

Substituting (6) and the FOCs (14) and (15) for i_0 and i_1 into the HJB equation (13), we obtain the nonlinear ordinary differential equation (ODE) for z:

$$0 = \alpha \ln\left(\frac{c^*(z)}{N(z)}\right) + \phi_0(i_0^*(z))L_0(z) + \phi_1(i_1^*(z))L_1(z) - \frac{\sigma^2}{2}\left[L_0(z)^2 + L_1(z)^2\right] + \sigma^2 M(z), \tag{16}$$

where functions $L_0(z)$, $L_1(z)$, and $M(z)$ are defined as

$$L_0(z) = (1-z)\left[1 - z\frac{N'(z)}{N(z)}\right], \tag{17}$$

$$L_1(z) = z\left[1 + (1-z)\frac{N'(z)}{N(z)}\right], \tag{18}$$

$$M(z) = \frac{z^2(1-z)^2 N''(z)}{N(z)}. \tag{19}$$

Note that $L_0(z) + L_1(z) = 1$. When $z = 0$ and $z = 1$, we obtain the one-sector solution (see Appendix A.1) that serves as the boundary conditions:

$$N(0) = p_0 \quad \text{and} \quad N(1) = p_1, \tag{20}$$

where $p_n = (A_n - i_n)\exp\left[\frac{1}{\alpha}\left(\phi_n(i_n) - \frac{\sigma^2}{2}\right)\right]$ and i_n solves $(A_n - i_n)\phi_n'(i_n) = \alpha$.

3.2 | Asset-Pricing Implications

Our production model features endogenous growth and has direct asset-pricing implications. For logarithmic utility given in (3), the equilibrium SDF is $\xi_t = e^{-\alpha t}\alpha / C_t^*$. Applying Itô's Lemma to ξ_t and using $C_t^* = c^*(z_t)(K_0(t) + K_1(t))$ and the equilibrium asset-pricing restriction that the drift of $d\xi_t/\xi_t$ is $-r_t = -r(z_t)$, we obtain

$$\frac{d\xi_t}{\xi_t} = -r(z_t)dt - \eta_0(z_t)dB_0(t) - \eta_1(z_t)dB_1(t), \tag{21}$$

where $r(z_t)$ is the equilibrium interest rate, and $\eta_0(z_t)$ and $\eta_1(z_t)$ represent the market prices of risk for the two diffusion risks $B_0(t)$ and $B_1(t)$, respectively. In Appendix A.2, we show

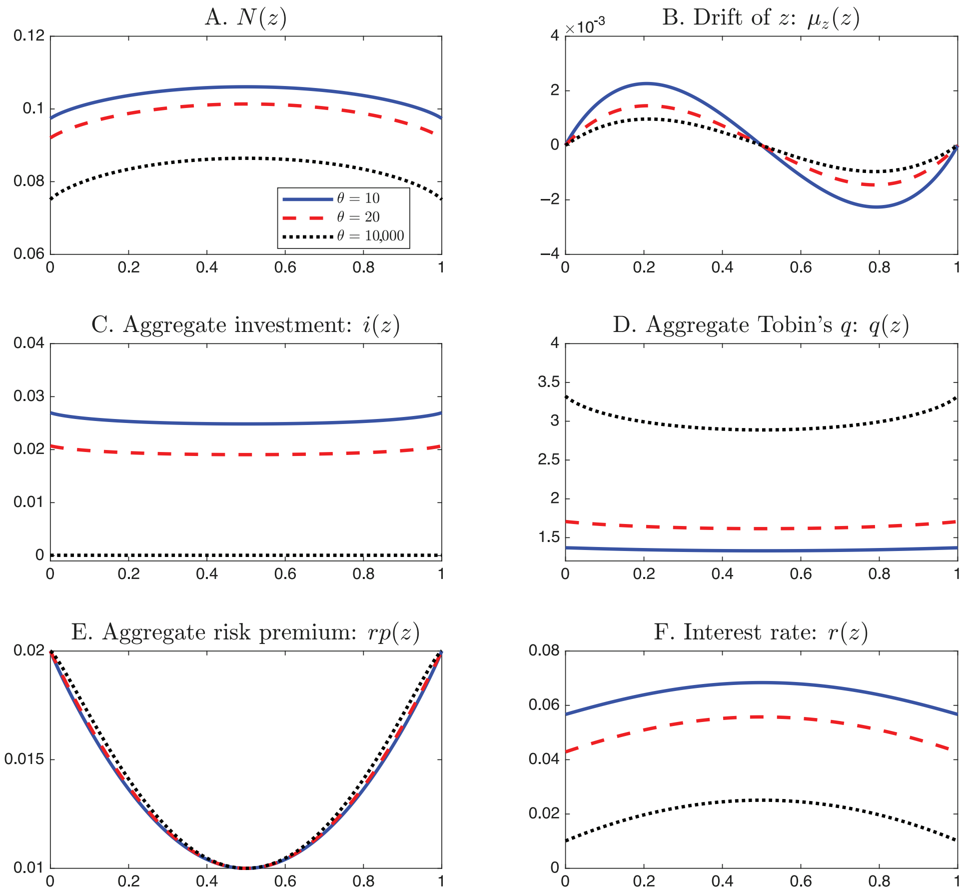

FIGURE 3 | Aggregate implications with risk aversion $\gamma = 2$ and elasticity $\psi = 0.5$.

$$r(z) = \alpha + \mu_C(z) - \sigma^2(1 - z_t + \varepsilon(z_t))^2 - \sigma^2(z_t - \varepsilon(z_t))^2, \tag{22}$$

$$\eta_0(z) = \sigma(1 - z + \varepsilon(z)), \quad \text{and} \quad \eta_1(z) = \sigma(z - \varepsilon(z)), \tag{23}$$

where

$$\varepsilon(z) = -\frac{c^{*\prime}(z)}{c^*(z)} z(1 - z), \tag{24}$$

$$\mu_C(z) = g(z) - \varepsilon(z)\left[\phi_1(i_1^*(z) - \phi_0(i_0^*(z)\right] + \sigma^2\varepsilon^2(z) + \sigma^2 z^2(1 - z)^2 \frac{d^2}{dz^2} \ln c(z). \tag{25}$$

3.3 | Decentralized Market Solution

3.3.1 | Investment and Tobin's *q*

Let $V_n(K_n; z)$ denote firm value in sector n. Using the homogeneity property, we have

$$V_n(K_n; z) = q_n(z) K_n, \quad n = 0, 1, \tag{26}$$

where Tobin's q in sector n is given by

$$q_n(z) = \frac{1}{\phi_n'(i_n(z))}. \tag{27}$$

Intuitively, a unit of investment increases capital stock by $\phi_n'(i_n)$ and each unit of capital is valued at $q_n(z)$. Therefore, $\phi_n'(i_n) \cdot q_n(z)$ must equal the marginal cost of investing, which is one. Equation (27) is the standard FOC for q theory of investment.

The market value of aggregate capital is $V(z) = V_0(z) + V_1(z) = q(z)(K_0 + K_1)$, where Tobin's q for the aggregate capital stock is given by

$$q(z) = (1 - z) q_0(z) + z q_1(z). \tag{28}$$

3.3.2 | Consumption and Dividend Yield

With complete markets and logarithmic utility, the aggregate consumption–wealth ratio $C(z)/V(z)$ equals the discount rate α, so that $c(z) = \alpha q(z)$, as in the one-sector setting. While the aggregate dividend yield (i.e., consumption/wealth ratio) is constant and equals the discount rate α, the sectoral dividend yield dy_n is stochastic and is given by

$$dy_n(z) = \frac{A_n - i_n(z)}{q_n(z)}, \quad n = 0, 1. \tag{29}$$

The above formula implies that sectoral dividend yield can be negative if $i_n(z) > A_n$ for sector n. Unlike a one-sector model, the sectoral dividend yield can be negative when the consumer's investment incentive is high for that sector. We show that this

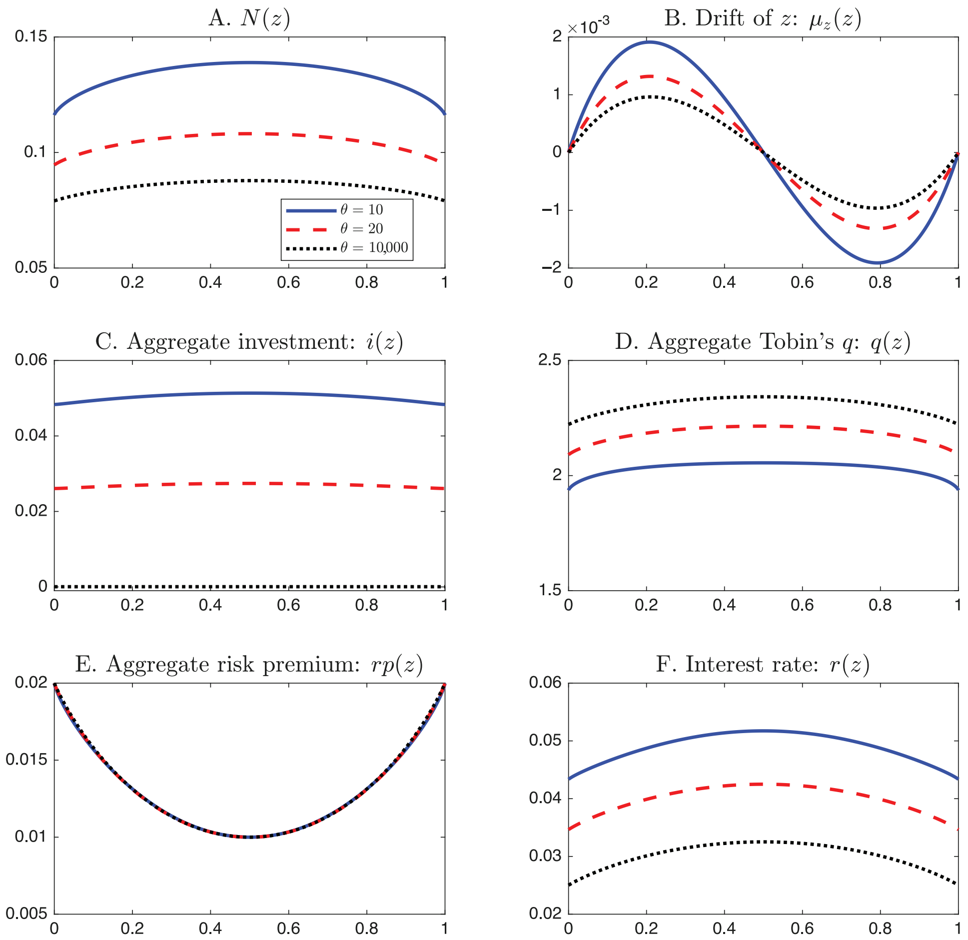

FIGURE 4 | Aggregate implications with risk aversion $\gamma = 2$ and elasticity $\psi = 2$.

negative dividend yield, which in effect implies equity issuance to finance investment, may indeed occur when that sector is small and provides large diversification benefits.

4 | Quantitative Analysis

Up to now, none of our results depend on the particular functional form of adjustment costs, $\phi_n(i)$. To further illustrate the properties of the model, we now specify a quadratic adjustment cost function as follows:

$$\phi_n(i) = i - \frac{\theta_n}{2} i^2 - \delta_n, \qquad (30)$$

where $\theta_n \geq 0$ is the adjustment cost parameter. When $\theta_n = 0$, the expected growth rate of capital is $\phi_n(i) = i - \delta_n$. We may naturally interpret δ_n as the expected rate of depreciation.

To ease exposition, in this section we set the production and adjustment technologies in the two sectors to be the same. Despite the identical technologies, the two sectors price investment differently and carry different risk premia because of differences in their capital stocks, even with constant-returns-to-scale production.

Parameter Choices. We choose model parameters to generate sensible aggregate predictions and to highlight the impact of endogenous investment and growth on equilibrium pricing and capital reallocation. The annual subjective discount rate is $\alpha = 0.04$. Annual volatility is $\sigma = 0.10$ and the annual productivity parameter is $A = 0.10$. Finally, we choose the adjustment cost parameter $\theta = 10$ and $\delta = 0$. The correlation coefficient is $\rho = 0$. In the one-sector economy (i.e., $z = 0, 1$), we have $i = 0.0368$, $q = 1.58$, and $g = \phi(i) = 0.03$.

4.1 | Sectoral Distribution of Capital z and the Value Function

Figure 1 has six panels, arranged in three rows and two columns. Panel A of Figure 1 shows $N(z)$, the logarithm of the representative consumer's value function per unit of aggregate capital ($K_0 + K_1$), as a function of $z = K_1/(K_0 + K_1)$. Intuitively, we expect that $N(z)$ is maximized at $z = 1/2$, where the consumer achieves the maximally attainable level of diversification between the two sectors.

Panel B of Figure 1 plots the drift of z, $\mu_z(z)$ given in Equation (12). There is a natural tendency for z to move toward the center (i.e., when $z < 1/2$, $\mu_z(z) > 0$ and hence on average z increases toward 1/2). This mean reversion effect of $\mu_z(z)$ in z is also

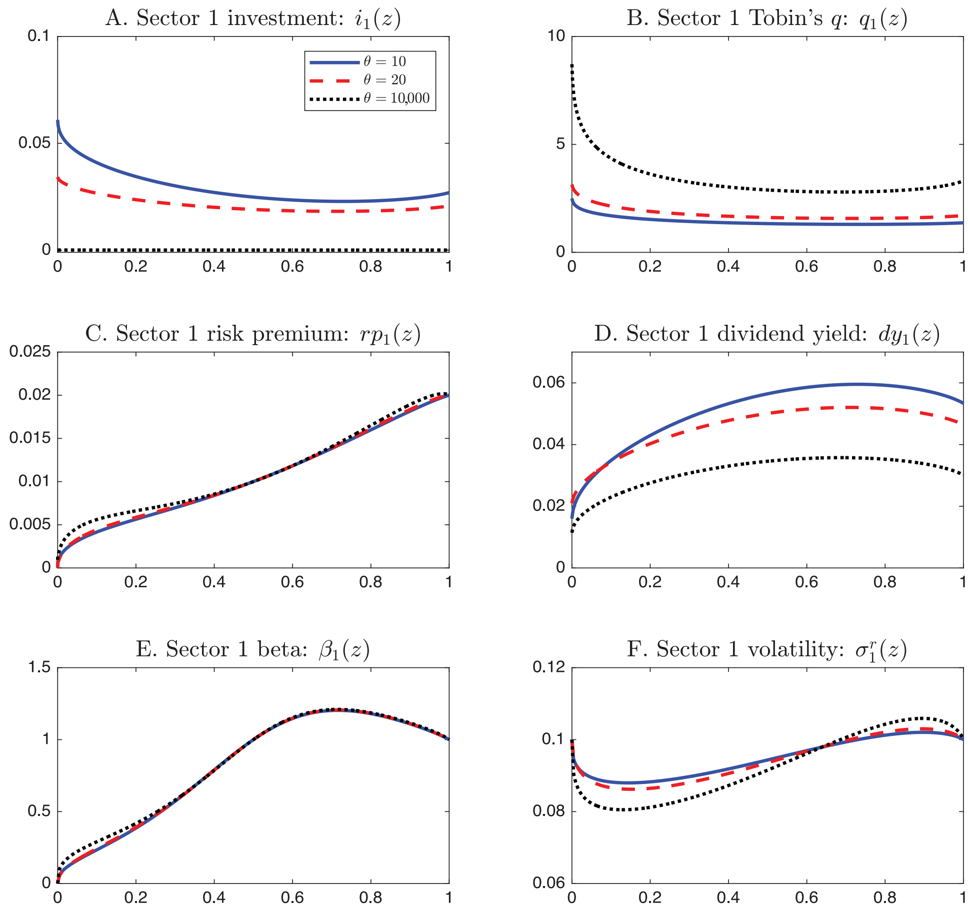

FIGURE 5 | Sectoral implications with risk aversion $\gamma = 2$ and elasticity $\psi = 0.5$.

present in CLS due to the definition of z. Unlike CLS, however, the "central tendency" is stronger in our production economy due to endogenous investment and growth. Controlling for size (i.e., per unit of capital), the consumer has greater demand for the smaller sector, and hence invests more per unit of capital, *ceteris paribus*. For example, when $0 < z < 1/2$, sector 1 is the smaller one, so the firm invests and grows at a faster rate, and $i_1(z) > i_0(z)$ and $g_1(z) > g_0(z)$. The flexibility to adjust capital growth enhances the "central tendency" of $\mu_z(z)$ due to endogenous growth. In contrast, in the CIR model with no adjustment costs and hence unit marginal q, the economy frictionlessly responds to shocks and shifts capital between two sectors, always maintaining half of its capital stock in each sector.

4.2 | Aggregate Implications

In Panels C and D of Figure 1, we plot the aggregate investment–capital ratio $i(z)$ and the aggregate Tobin's q as functions of z. Note that neither $i(z)$ nor $q(z)$ are monotonic in z.

The symmetry between the two sectors allows us to focus on the region $0 \le z \le 1/2$. First, aggregate investment $i(z)$ decreases with z due to the adjustment cost. After reaching the lowest value at around $z = 0.10$, investment $i(z)$ starts to increase with z until it peaks at $z = 1/2$. Intuitively, when z is close to zero (i.e., sector 0 is effectively the only one), diversification has little value added, but the adjustment costs of having two sectors may be high. Aggregate investment therefore falls. However, for sufficiently high z, the diversification benefits outweigh the costs of adjusting capital stock. As a result, aggregate investment increases and peaks again at $z = 1/2$.

Perhaps surprisingly, aggregate Tobin's q moves in the *opposite* direction of aggregate investment $i(z)$ due to general equilibrium. With logarithmic utility, consumption is proportional to firm value, that is, $c(z) = \alpha q(z)$, where α is the agent's subjective discount rate. In equilibrium with symmetry, we have $c(z) + i(z) = A$. Therefore, a unit increase in $i(z)$ implies a unit decrease in $c(z)$ and hence Tobin's q decreases α units. Aggregate consumption $c(z)$ and hence $q(z)$ first increase with z and then falls with z for $z \le 1/2$. Therefore, our model predicts a negative (or weak) relation between i and q in aggregate data, even though the neoclassic q theory of investment hold perfectly in the model. Heterogeneity and equilibrium aggregation have first-order effects and potentially overturn the conventional wisdom. However, for the illustration, the quantitative effects of sectoral distribution z on aggregate investment are small. This is due to both consumption smoothing[6] and the investment adjustment cost which encourages the firm to smooth its investment. In fact, we show in Eberly and Wang (2009) in the deterministic case, and extend to uncertainty in the Appendix, that aggregate values are immune to the sectoral distribution of capital in the case with log utility and log capital installation costs. Deviations from

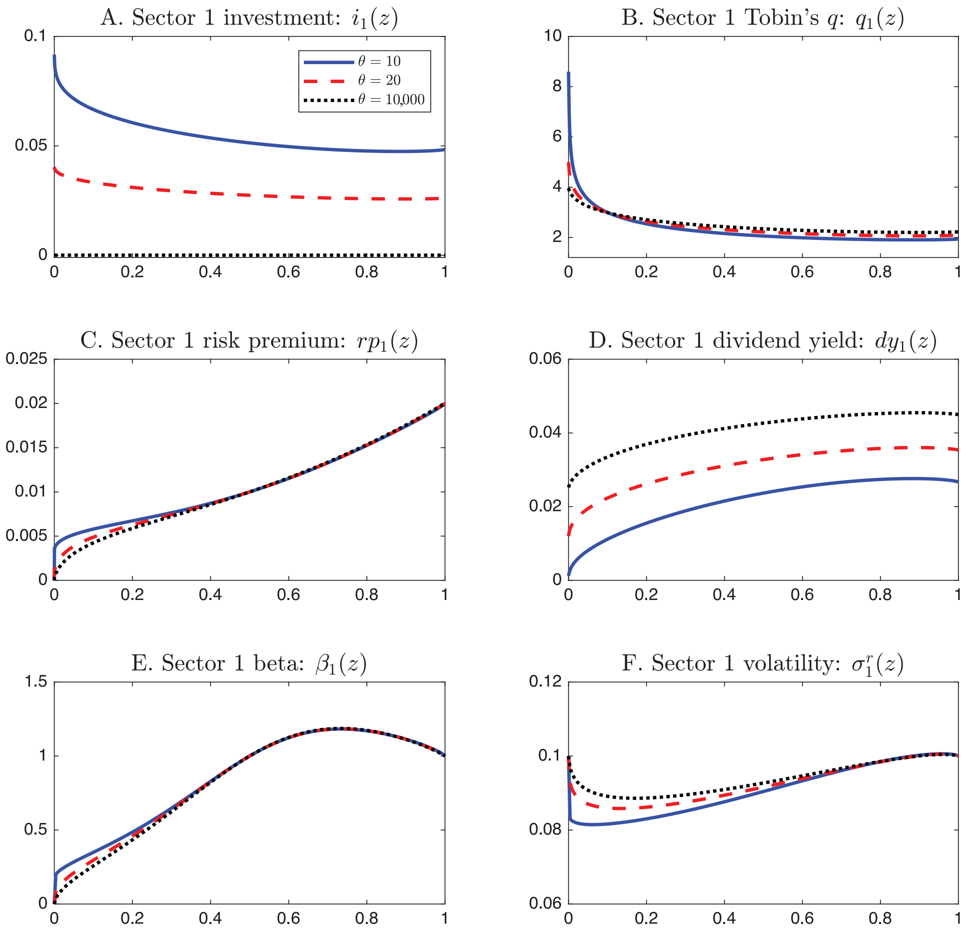

FIGURE 6 | Sectoral implications with risk aversion $\gamma = 5$ and elasticity $\psi = 2$.

these assumptions exhibit larger distributional effects, which we explore when we depart from the log utility case in Section 5 and subsequent analysis.

Next, we turn to the asset-pricing implications. In Panels E and F of Figure 1, we plot the expected return of the market portfolio (aggregate wealth), $\mu^m(z)$, and the equilibrium interest rate $r(z)$.

The expected return on the market portfolio, $\mu^m(z)$, closely tracks the expected aggregate investment $i(z)$ and aggregate growth rate $g(z)$. This is consistent with the standard asset-pricing result that growth increases the expected rate of return on the risky asset. Its shape again resembles a "W" as a function of z.

The risk-free rate $r(z)$ depends on both the expected growth rate of aggregate consumption and the volatility of aggregate consumption growth. Quantitatively, the precautionary saving motive, measured by the variance of the market portfolio $\sigma_m^2(z)$, varies much more with z than does aggregate growth. Hence, the precautionary motive determines the dependence of the equilibrium interest rate on z. Note that $r(z)$ reaches its maximum at $z = 1/2$, where diversification achieves the highest possible level and precautionary saving demand is lowest. A high interest rate is necessary to encourage saving in equilibrium when the economy is well-diversified and the precautionary saving motive is weak.

4.3 | Sectoral Implications

The next set of figures shows sectoral values; in each panel we graph results for sector 1 only for brevity since results are symmetric for sector 0. Panels A and B of Figure 2, respectively, plot the investment–capital ratio $i_1(z)$ and Tobin's q in sector 1.

4.3.1 | Sectoral Investment and *q*

Recall that Tobin's q in sector 1 is given by $q_1(z) = 1/\phi'(i_1(z)) = [1 - \theta i_1(z)]^{-1}$. Therefore, Tobin's q is monotonically increasing in its sectoral investment–capital ratio and hence Tobin's q and investment convey essentially the same information for a given adjustment cost function. Note that both investment–capital ratio $i_1(z)$ and Tobin's q, $q_1(z)$, decrease before z reaches 0.80. Sectoral Tobin's q and investment capital ratio i become significantly larger as the sector becomes smaller, because the consumer values the smaller sector more for diversification benefits, *ceteris paribus*. Recall that there are constant-returns-to-scale in production, so this relationship between q and sector size is not due to decreasing marginal returns in production. Rather, the diversification benefits of keeping the small sector "alive" with the potential to grow are very valuable. Upon vanishing, the sector will never be reborn, and the economy (with only the one surviving sector) will be significantly riskier thereafter.

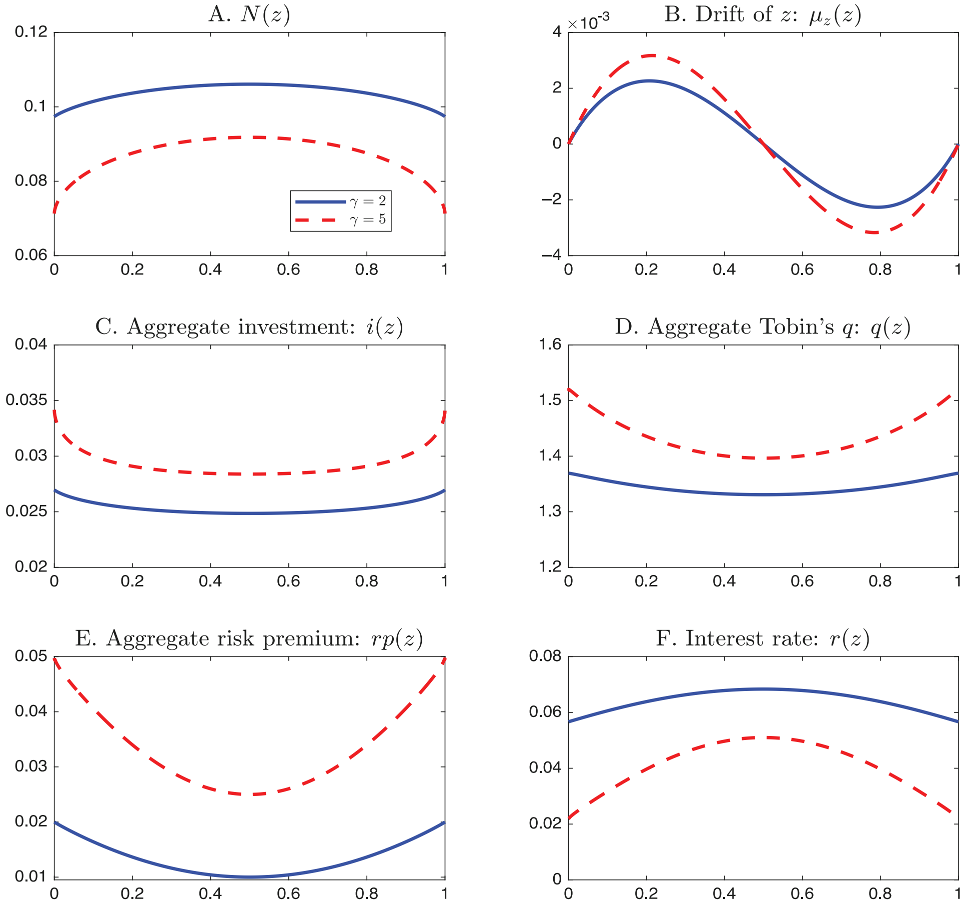

FIGURE 7 | Aggregate implications with different levels of risk aversion $\gamma = 2, 5$ and elasticity $\psi = 0.5$.

Note that $i_1(z)$ and $q_1(z)$ increase with z for sufficiently high z. The representative agent's consumption-smoothing motive in equilibrium requires consumption and hence aggregate investment to be relatively smooth and not too volatile. When sectors are sufficiently imbalanced, the contribution by the dwindling sector to total investment is negligible. Hence, to maintain a sufficient level of aggregate investment for the purpose of consumption smoothing, the investment–capital ratio in the larger sector (i.e., sector 1 when z is high) must rise as its share z increases. This explains the increasing behavior of the investment–capital ratio $i_1(z)$ and Tobin's q in z at the right side of the graph.

4.3.2 | Sectoral Risk Premium and Dividend Yield

Panels C and D of Figure 2 graph the sectoral risk premium $rp_1(z)$ and dividend yield $dy_1(z)$. The risk premium of a miniscule sector is effectively zero, because this sector carries almost no weight in aggregate consumption, and the correlation ρ between the two shocks is zero. The same intuition applies in the pure-exchange economy (e.g., CLS). Recall that the interest rate is lowest at $z = 0$ and $z = 1$, therefore, the discount rate, the sum of the interest rate and the risk premium, for a sector is lowest when it is vanishing. Intuitively, in equilibrium, the preferences for consumption smoothing and risk diversification lower the risk premium and the discount rate for the shrinking sector. Since the physical production technology remains unchanged, the vanishing sector invests at the highest rate $i_1(0)$ to take advantage of its lowest cost of capital.

To finance this high level of investment around $z = 0$, the firm lowers its dividend yield. The dividend yield for the dwindling sector is positive in our example. However, for other parameter values, the firm may choose to issue equity and hence the dividend yield may be negative. Tobin's q reaches the maximal level $q_1(0)$ at $z = 0$ despite the low dividend yield. Note that the high valuation of capital for the vanishing sector is primarily driven by the discount rate effect induced by diversification benefits. Unlike the aggregate dividend yield, which is equal to the subject discount rate α for log utility, the sectoral dividend yield varies significantly with z.

4.3.3 | Sectoral β and Volatility

We now turn to sectoral risk measures. In Panels E and F of Figure 2, we plot β and return volatility $\sigma_1^r(z)$ in sector 1. The β for sector 1 is given by

$$\beta_1(z) = \frac{z}{z^2 + (1-z)^2}\left[1 + \frac{q_1'(z)}{q_1(z)}(1-z)(2z-1)\right]. \tag{31}$$

The nonmonotonic behavior of β can be understood by considering several benchmarks. First, zero risk premium for the disappearing sector implies $\beta_1(0) = 0$. Second, with increases in

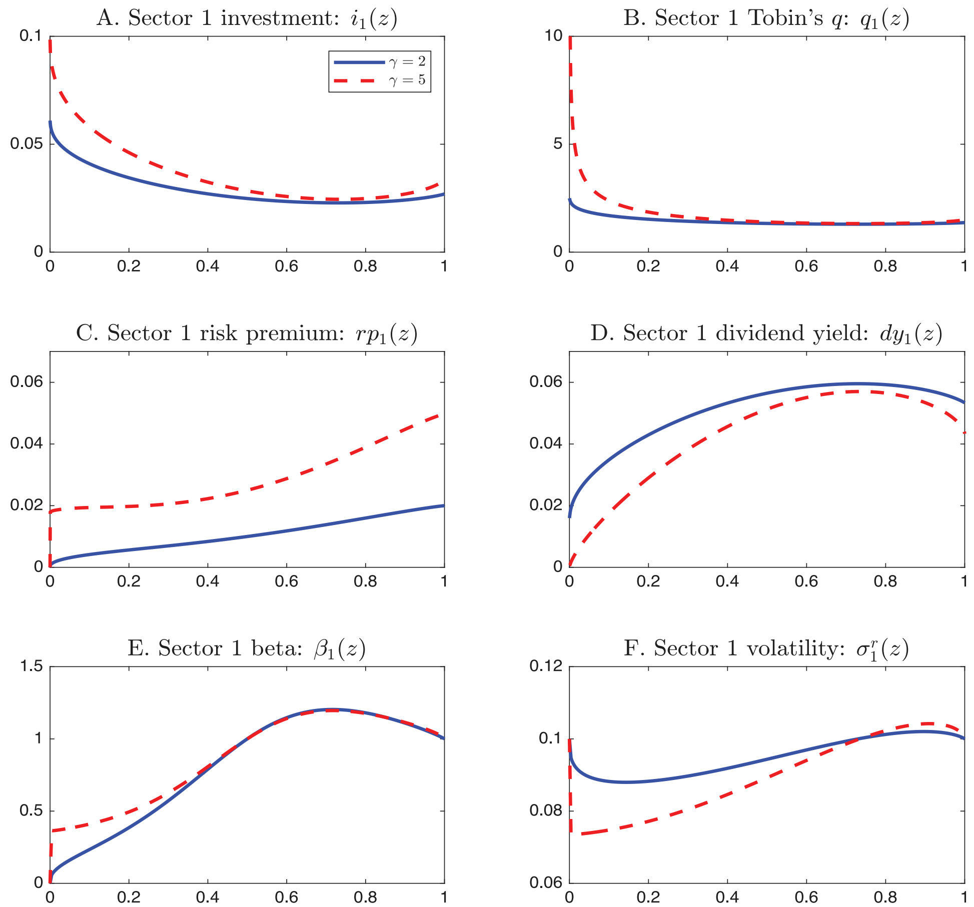

FIGURE 8 | Sectoral implications with different levels of risk aversion $\gamma = 2, 5$ and elasticity $\psi = 0.5$.

the share of capital z, more consumption is financed out of sector 1's output and hence $\beta_1(z)$ rises. Third, $\beta(1/2) = 1$, which follows from symmetry between the two sectors and $\beta = 1$ for the market portfolio by definition. When z increases above $1/2$, $\beta_1(z)$ exceeds one, because the other sector becomes smaller and carries smaller β (again by symmetry). Therefore, the bigger sector is riskier, *ceteris paribus*. Finally, when the sector becomes sufficiently large (i.e., low enough z), β has to fall as the sector becomes effectively the market portfolio, which has $\beta = 1$ by definition. Indeed, in the limit, when sector 1 comprises the whole economy ($z = 1$), $\beta_1 = 1$.

Now consider return volatility for sector 1, $\sigma_1^r(z)$. We have

$$\sigma_1^r(z) = \sigma\sqrt{\left(\frac{q_1'(z)}{q_1(z)}z(1-z)\right)^2 + \left(1 + \frac{q_1'(z)}{q_1(z)}z(1-z)\right)^2}. \quad (32)$$

While $\sigma_1^r(z)$ also varies nonmonotonically with z, its behavior is rather different from $\beta_1(z)$. At $z = 0$, $\beta_1(z)$ is zero and hence all return volatility comes from the idiosyncratic component because the sector carries no weight in the aggregate. Since total return volatility is the same as capital stock growth volatility σ for the miniscule sector, we have $\sigma_1^r(0) = \sigma = 0.10$. When $z = 1$, sector 1 is the whole economy and hence the aggregate volatility is also $\sigma = 10\%$. It is thus natural to expect a nonmonotonic relation between sectoral return volatility $\sigma_1^r(z)$ and sectoral distribution of capital stock z.

5 | Model and Solution With Recursive Utility

We now extend our baseline model to allow for more flexible preferences and sectoral asymmetry. Specifically, we consider the setting where the representative consumer has a homothetic preference featuring both constant relative risk aversion (CRRA) and constant elasticity of intertemporal substitution (Epstein and Zin 1989 and Weil 1990). We use the continuous-time formulation of this recursive utility introduced by Duffie and Epstein (1992a). That is, the agent has a recursive preference defined as follows:

$$J_t = \mathbb{E}_t\left[\int_t^\infty f(C_s, J_s)ds\right], \quad (33)$$

where $f(C, J)$ is known as the normalized aggregator for consumption C and the agent's continuation value J. Duffie and Epstein (1992a) show that $f(C, J)$ for Epstein–Zin nonexpected (homothetic) utility is given by

$$f(C, J) = \frac{\alpha}{1-\psi^{-1}} \frac{C^{1-\psi^{-1}} - ((1-\gamma)J)^{\omega}}{((1-\gamma)J)^{\omega-1}}, \quad (34)$$

where

$$\omega = \frac{1-\psi^{-1}}{1-\gamma}. \quad (35)$$

The parameter $\psi \geq 0$ measures the elasticity of intertemporal substitution, and the parameter $\gamma \geq 0$ is the coefficient of relative risk

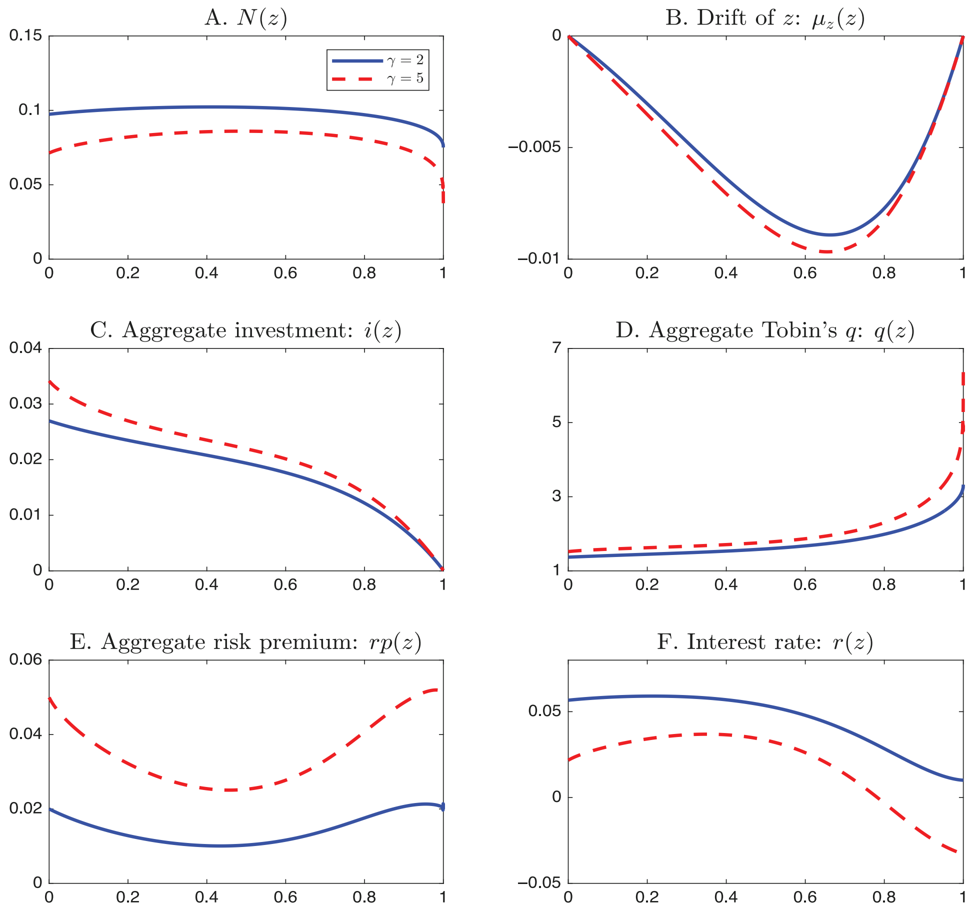

FIGURE 9 | Aggregate implications with asymmetric sectors: $\theta_0 = 10$ and $\theta_1 = 10,000$.

aversion. The parameter $\alpha > 0$ is the entrepreneur's subjective discount rate. The widely used time-additive separable CRRA utility is a special case of the above Duffie–Epstein–Zin–Weil recursive utility specification where the coefficient of relative risk aversion γ is equal to the inverse of the elasticity of intertemporal substitution ψ, that is, $\gamma = \psi^{-1}$ and hence $\omega = 1$.

For the general recursive utility given in Equations (33) and (34), the scale-invariance property proves useful in keeping our model analysis tractable (see Duffie and Epstein 1992b for example). Using this preference, we can quantify both the effect of intertemporal substitution and that of risk aversion on equilibrium resource allocation and asset pricing.

5.1 | Solution

We conjecture and later verify that representative agent's value function $J(K_0, K_1)$ is homothetic in sectoral capital stocks K_0 and K_1 and can be written in the following form:

$$J(K_0, K_1) = \frac{1}{1-\gamma}((K_0 + K_1)N(z))^{1-\gamma}, \tag{36}$$

where $z = K_1/(K_0 + K_1)$ and $N(z)$ is a function to be determined.

For a two-sector economy, sectoral investment–capital ratios i_0 and i_1 jointly solve the following implicit equations as functions of $z = K_1/(K_0 + K_1)$:

$$\left(\frac{c^*(z)}{N(z)}\right)^{1/\psi} = \frac{\alpha}{\phi_0'(i_0^*(z))}\frac{1}{N(z) - zN'(z)}, \tag{37}$$

$$\left(\frac{c^*(z)}{N(z)}\right)^{1/\psi} = \frac{\alpha}{\phi_1'(i_1^*(z))}\frac{1}{N(z) + (1-z)N'(z)}, \tag{38}$$

where $c^*(z)$ is the optimal aggregate consumption–capital ratio: $c^*(z) = C^*/(K_0 + K_1)$. Naturally, (9) continues to hold.

Equations (37), (38), and (9) and the following ODE jointly give the solution for sectoral investment–capital ratios:

$$0 = \frac{\alpha}{1-\psi^{-1}}\left[\left(\frac{c^*(z)}{N(z)}\right)^{1-\psi^{-1}} - 1\right] + \phi_0(i_0^*(z))L_0(z) + \phi_1(i_1^*(z))L_1(z)$$
$$- \frac{\gamma}{2}\left[\sigma_0^2 L_0^2(z) + \sigma_1^2 L_1^2(z) + 2\rho\sigma_0\sigma_1 L_0(z)L_1(z)\right] + \frac{\sigma_0^2 - 2\rho\sigma_0\sigma_1 + \sigma_1^2}{2}M(z), \tag{39}$$

where $L_0(z)$, $L_1(z)$, and $M(z)$ are given in (17), (18), and (19), respectively.

Both $z = 0$ and $z = 1$ are absorbing barriers. They correspond to the one-sector model solution. The boundary conditions are

$$N(0) = p_0, \quad \text{and} \quad N(1) = p_1, \tag{40}$$

where p_n is given by (B.3) in Appendix B, evaluated with parameters and the optimal investment–capital ratios in sector n.

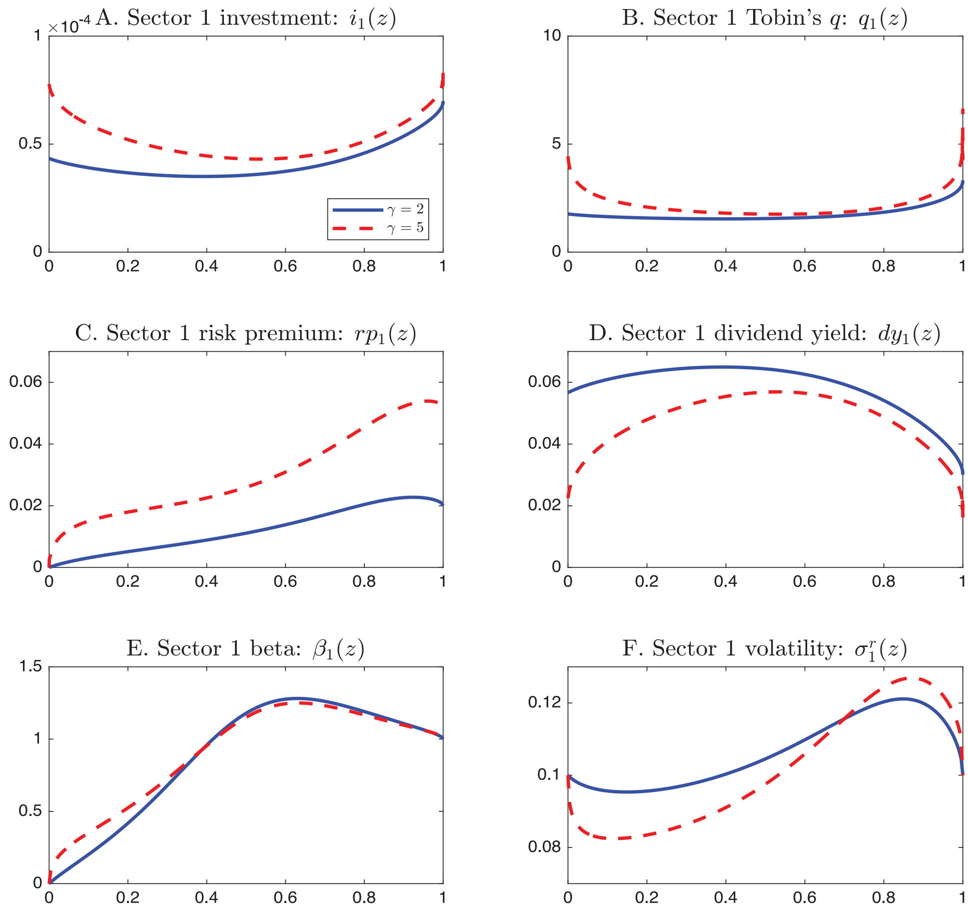

FIGURE 10 | Sector 1 implications with asymmetric sectors: $\theta_0 = 10$ and $\theta_1 = 10,000$.

Using Ito's formula, the dynamics of $z = K_1/(K_0 + K_1)$, which govern endogenous capital reallocation, are given by

$$dz_t = \mu_z(z_t)dt + z_t(1 - z_t)\sigma_1 dB_1(t) - z_t(1 - z_t)\sigma_0 dB_0(t), \quad (41)$$

where the drift of z, $\mu_z(z)$, is given by

$$\mu_z(z) = z(1 - z)\left[\phi_1(i_1(z)) - \phi_0(i_0(z)) + (1 - z)\sigma_0^2 - z\sigma_1^2 - (1 - 2z)\rho\sigma_0\sigma_1\right]. \quad (42)$$

5.2 | Quantitative Analysis

As in the baseline case, the analytic results with recursive utility are independent of the functional form for adjustment costs, $\phi_n(i)$. In order to calculate quantitative results we now use the baseline quadratic adjustment cost function given in (30).

To understand the role of capital liquidity, we now consider a comparative static change in the efficiency of reallocating capital. In standard equilibrium models, this experiment is not possible, since capital reallocation is either frictionless (CIR) or ruled out in pure-exchange settings (CLS). In this section, we analyze the aggregate and sectoral effects of changing the adjustment cost parameter θ. We choose three levels of the adjustment cost parameters: $\theta = 10, 20$, and 10,000 for Figures 3–6. The higher the value of θ, the more illiquid is physical capital. The extreme value of $\theta = 10,000$ corresponds to essentially completely illiquid capital. Without investment, the economy essentially behaves as a pure-exchange economy (CLS). We set the agent's coefficient of relative risk aversion $\gamma = 2$. All other parameter values are the same as in Section 4.

5.2.1 | Effects of Capital Illiquidity

The adjustment costs impose direct resource costs (hence lowering the welfare) and also discourage savings and investment. The higher the adjustment cost parameter θ is, the lower welfare $N(z)$ is, and the higher consumption is. In fact, for the highest adjustment cost, the consumer consumes virtually the entire dividend and does not save; in this case, no direct resource costs are incurred at all, but the high adjustment cost gives rise to the misallocation of resources.

Aggregate Implications. Figures 3 and 4 plot aggregate implications for two settings where the elasticity of intertemporal substitution ψ is set at $\psi = 0.5$ and $\psi = 2$, respectively. We consider these two values of elasticities because there is much debate about the magnitude of elasticity of intertemporal substitution. In the macrofinance literature (where long-run risk is a key input), a high value of elasticity ψ is often chosen.[7]

First, we show that regardless of elasticity of intertemporal substitution, welfare $N(z)$ decreases with the adjustment cost. For a given θ, $N(z)$ is higher when the sectors are more balanced.

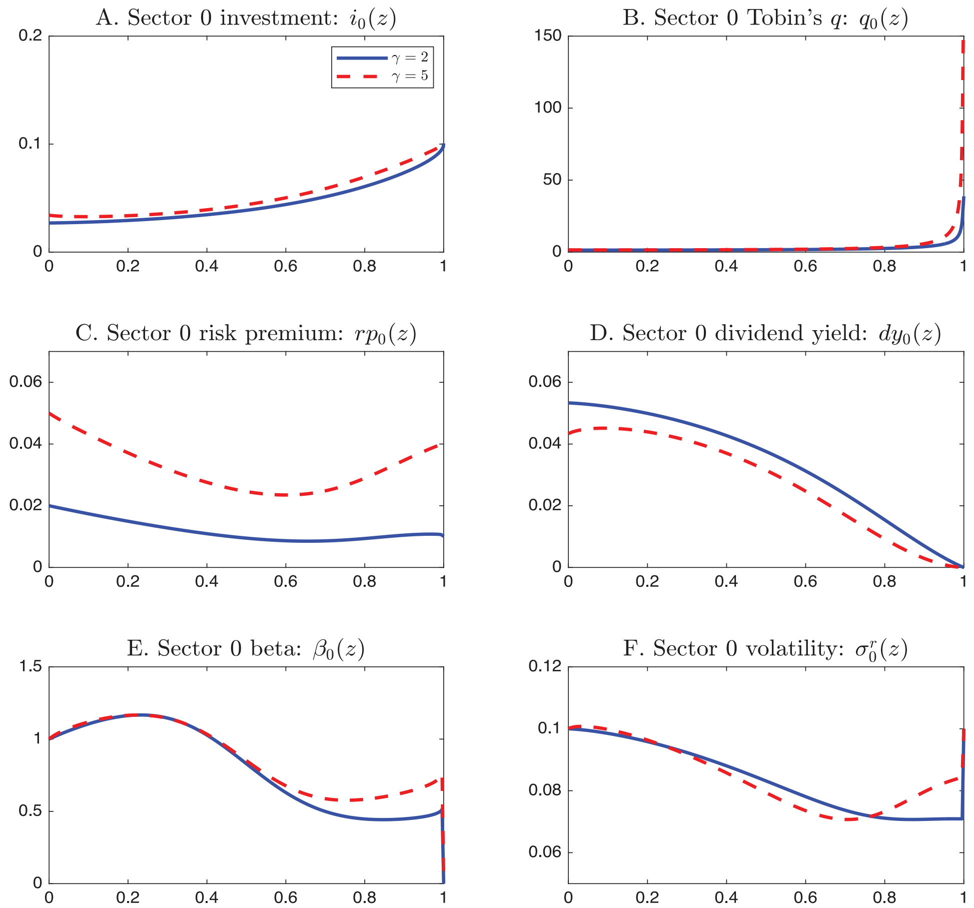

FIGURE 11 | Sector 0 implications with asymmetric sectors: $\theta_0 = 10$ and $\theta_1 = 10,000$.

Second, investment decreases with the adjustment cost and hence consumption must increase with the adjustment cost in the short run: aggregate output is either invested or consumed via dividends. Third, the higher the adjustment cost, the higher the rents to installed capital and hence the higher is Tobin's q.

Fourth, the higher the adjustment cost, the smaller is adjustment and thus on average the smaller is magnitude of the change in z, $|\mu_z(z)|$. When the adjustment cost is lower, the consumer actively reallocates capital to drive the allocation of capital back to the optimal value of $z = 0.5$, so the central tendency in Figures 3 and 4 is dramatically strengthened as capital becomes more liquid, that is, the adjustment cost declines. Fifth, the aggregate risk premium $rp(z)$ is virtually independent of the adjustment cost. This is perhaps counterintuitive. We provide intuition in two steps: first, with one sector only (i.e., $z = 0, 1$), this is expected because the volatility of the shock to capital is exogenously given. Adjustment costs enter via the expected change, that is, the drift, in capital accumulation. This in turn translates into the implication that only the drift $\mu_z(z)$, not the volatility, of the capital stock share z (the key state variable), depends on the adjustment cost specification. Since the risk premium depends on the volatility of z and the pricing kernel, we naturally do not expect much variation of the aggregate risk premium with respect to the adjustment cost. Moreover, for a given value of θ, the agent's incentive to consume is highest when the two sectors are more balanced because the systematic risk is smaller (due to diversification).

Finally, the adjustment cost has a significant effect on the level of the interest rate and hence also the expected aggregate market return.[8] Intuitively, the more liquid capital is, the more attractive and hence the higher is investment. In order to clear the goods market, we need to encourage the consumer to save so that investment can be financed. As a result, the equilibrium interest rate is higher in a more liquid economy, as we see for both levels of elasticities. This is often viewed as one undesirable effect of introducing production into equilibrium asset-pricing models because it pushes up the equilibrium interest rate.

Sectoral Implications. Figures 5 and 6 plot the corresponding sectoral results for the same two settings, that is, elasticity of intertemporal substitution ψ is set at $\psi = 0.5$ and $\psi = 2$. In the low adjustment cost economy, the incentive to save and reallocate capital is strong, and hence investment is high at the sectoral level. This effect is also reflected in the lower value of Tobin's q when adjustment costs are low. When adjustment costs are so high as to prohibit investment almost entirely, the value of Tobin's q for a vanishing sector increases sharply, as the marginal value of reviving the shrinking sector skyrockets.

The sectoral risk premium and sectoral β are almost independent of the adjustment costs provided that the sector is not too small. However, when the sector is small enough (low z for sector 1), the properties of the sectoral risk premium and β differ depending on the elasticity of intertemporal substitution and the adjustment costs. In those situations, the diversification incentive is very strong.

With a relatively small elasticity of intertemporal substitution (e.g., $\psi = 0.5$), for low values of z, the higher the adjustment cost is, the larger the sectoral risk premium and β are. Intuitively, a more costly adjustment process makes the smaller sector riskier.

With a large elasticity of intertemporal substitution (e.g., $\psi = 2$), the representative agent's incentive to smooth consumption over time is very high. As a result, the value of the dwindling sector skyrockets and investment increases substantially when the adjustment cost is relatively low. Intuitively, if ψ is high (e.g., $\psi = 2$), for the dwindling sector, the sensitivity of Tobin's q with respect to changes in sectoral distribution z is quite high when the adjustment cost is low. As a result, the sectoral risk premium and sectoral β are higher in dwindling sectors when the adjustment cost is low.

5.2.2 | Effects of Risk Aversion

Since diversification across the two sectors plays an important role in capital allocation in the model, we now consider changing risk aversion in order to explore the quantitative impact of risk. We already discussed the log utility case ($\gamma = 1$) in Figures 1 and 2, so now we graph the results for different values of risk aversion, $\gamma = 2, 5$, holding the other parameters of the model fixed. Since we have now developed the model with nonexpected utility, we hold the intertemporal elasticity of substitution, ψ, fixed and equal to one.

Aggregate Implications. In Figure 7, we plot the aggregate implications of the model with two values of risk aversion. In all cases, higher risk aversion lowers utility and raises investment. Note, however, the greater curvature in utility as a function of z, the distribution of capital, for higher values of γ. The more risk-averse consumer responds more to changes in the distribution of capital, which determine how well-diversified the household is. When his risk aversion is high, the consumer cuts consumption more as he becomes less diversified (z closer to zero or one), and instead engages in more precautionary savings. Panel C of Figure 7 shows that this savings response translates into higher investment and growth near the boundary values of z, compared to $z = 0.5$. This greater response of investment to the distribution of capital increases the mean reversion in the model, evidenced in the drift in z, $\mu_z(z)$. The higher is risk aversion, the greater is the central tendency in z. These effects are also evident in the interest rate, which also shows more curvature in the high risk-aversion case. With higher risk aversion, the model also generates higher expected returns and risk premia, consistent with the lower asset prices and investment, compared to the low risk-aversion case.

Sectoral Implications. In Figure 8, we plot the sectoral implications for the model for two different values of risk aversion. As in the aggregate case, the higher value of risk aversion is associated with high investment and Tobin's q in each sector, as well as a higher risk premium. The dividend yield is lower when risk aversion is high because of the strong investment response to higher risk aversion, depleting the dividend.

These results indicate that risk aversion, even for modest values of γ, substantially enhances the effect of the distribution of capital on aggregate variables. In particular, the central tendency in the model is much stronger as risk aversion increases, since the consumer is more sensitive to departures from the well-diversified economy. The enhanced central tendency drives greater investment and growth for extreme values of z. As in the single sector model, greater risk aversion increases risk premia, but lowers expected returns owing to the lower interest rate resulting from greater precautionary saving. Hence, asset prices rise with risk aversion in equilibrium.

5.2.3 | Asymmetric Sectors, Varying Risk Aversion

So far, we have only considered cases with two symmetric sectors, so the natural equilibrium is maximum diversification at $z = 0.5$. Now we allow for the two sectors to have different values of the adjustment cost parameter θ, so that one sector is relatively liquid and the other illiquid; this allows us to examine shocks to liquidity, as the economy endogenously moves between high and low liquidity with variation in z. In Figures 9 through 11, we plot results for the model with $\theta_0 = 10$, and $\theta_1 = 10,000$.[9]

Aggregate Implications. When z is low, sector 1 (the illiquid sector) is small and the economy overall is relatively liquid. As z rises, the economy has more illiquid capital. Now, the value function in Figure 9 achieves its maximum at a value of z less than 0.5, since the consumer prefers to hold more of the relatively liquid capital. Because sector 1 is completely illiquid, investment in this sector is always zero, so the drift in z must always be negative (reflecting investment in sector 0, which reduces z), and there is more reallocation for higher values of risk aversion. Panel C of Figure 9 shows that investment falls monotonically as z increases: as capital becomes less liquid on average, savings and hence aggregate investment falls monotonically. Similarly, consumption rises monotonically as z increases and capital is less liquid. Paradoxically, Tobin's q rises with z as investment falls. This effect is consistent with the fact that the value of installed capital rises as adjustment costs generate rents to installed capital. Again, changes in liquidity cause investment and Tobin's q to move in opposite directions: higher liquidity (higher values of z) increases investment while Tobin's q declines. Panel E of Figure 9 shows that the aggregate risk premium is highest when the economy is less well-diversified, especially for higher risk aversion. The interest rate reaches its peak to the left of $z = 0.5$, where it is also relatively flat. In this region, the economy is very liquid so there is little change in precautionary savings as z varies. On the right-hand side, however, where the economy is illiquid, precautionary savings and hence the interest rate are more sensitive to changes in z, especially for higher risk aversion.

Sectoral Implications. The next two figures, Figures 10 and 11, show the sectoral values as functions of z in the economy; since the two sectors are no longer symmetric, we now graph the two sectors separately. Figure 10 shows values for sector 1, the illiquid sector with high adjustment costs. Panel A of Figure 10 shows that investment is always near zero (because of the prohibitively high adjustment costs), so Tobin's q varies with z, especially as sector 1 becomes very large. The middle panel shows that the risk premium in sector 1 tends to rise as that sector becomes a larger share of the economy, consistent with our earlier discussion of the symmetric sectors. Since investment is zero for all values of z, the dividend yield simply reflects the ratio of A to value, or the inverse of Tobin's q. Similar to what we saw in the symmetric case, the bottom panel shows that sectoral β rises and then falls as sector 1 becomes larger.

Figure 11 for the liquid sector shows the properties of liquid capital in the model. Panel A of Figure 11 shows that both investment and Tobin's q in sector 0 rise with z, as the economy becomes less liquid on average. In this example, as z approaches unity and the liquid sector tends to disappear, the value of liquid capital (measured by Tobin's q) exceeds 100, since the agent places such a high value on resuscitating the liquid sector. The second panel of the charts shows that the risk premium in the liquid sector does not generally increase with the size of the sector, especially for higher risk aversion. As z rises and sector 0 shrinks, its risk premium initially falls, as expected, but then it rises again (before falling abruptly when the sector becomes insignificant), especially when risk aversion is high. In the range where the liquid sector is a relatively small part of the economy, sector 0 is nonetheless the only sector with an "adjustable" capital stock. Thus, it buffers all shocks (even shocks to the other sector) to provide consumption smoothing. This can be seen in Panels E and F of the figure, where sector 0 volatility increases to the right of the graph. Moreover, sector 0's β also falls, and then increases again (before going to zero) as sector 0 shrinks. Thus, as sector 0 becomes very small, the value of its capital (Tobin's q) shoots up because the overall cost of capital falls, but sector 0 itself becomes riskier and more volatile.

This version of the model provides a useful lens to consider a shock to liquidity. A negative shock to K_0 destroys liquid capital and increases z. This shock causes the overall economy to become less liquid: this is an endogenous change in liquidity in contrast to the comparative static change in θ we considered in Section 5.2. Starting from z at its utility-maximizing value, a negative shock to liquidity causes aggregate investment and growth to decline, even though investment in liquid capital increases to rebuild the liquid sector. The interest rate falls because the consumer has a greater incentive to save in the illiquid economy. Interestingly, the risk premium for liquid capital falls initially (and may rise if z becomes large enough), but the risk premium on illiquid capital rises. Because the overall economy is now less liquid, the aggregate risk premium rises and so does aggregate return volatility. This set of circumstances is remarkably like descriptions of a liquidity shock during the global financial crisis. In the model, the results are endogenously generated by the equilibrium response to a relative scarcity of liquid capital.

6 | Conclusion

We develop a parsimonious two-sector equilibrium q-theory of investment with costly sectoral capital reallocation. The model features a trade-off between the benefits of sectoral diversification (after accounting for sectoral growth differences) and the costs of reallocating capital. While a balanced capital stock adjusted for sectoral growth heterogeneity maximizes diversification, maintaining this balance requires costly capital reallocation. As a result, the agent optimally sacrifices some efficiency and growth to manage risk through diversification.

The desire for diversification induces mean reversion in the relative size of the two sectors, but this force is moderated by adjustment costs, leading to time-varying and stochastic sectoral reallocation. When the economy becomes unbalanced with one sector large and the other small, the incentive to invest in the smaller sector to restore diversification intensifies. As the smaller sector contracts, its cost of capital approaches the risk-free rate, its Tobin's q and investment surge, and its dividend yield may even turn negative. These dynamics arise even when both sectors share identical constant-returns-to-scale production technologies and capital adjustment costs.

We see several directions for future research. Although our model is fully dynamic and yields quantitative implications, it can be extended to enable a more systematic investigation of the trade-off between sectoral diversification and costly capital reallocation. Such extensions may also illuminate the business cycle and asset-pricing implications of this trade-off (see, e.g., Hansen et al. 2024 for a review of recent developments). As noted in the penultimate paragraph of the Introduction, a range of applications including climate economics, international contexts, and the AI economy, naturally build on the two-sector equilibrium production framework developed here. These research questions inherently involve costly capital reallocation and thus naturally call for a multisector equilibrium approach.

Acknowledgments

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Endnotes

[1] Hansen and Sargent "view the resulting model as an excellent environment for endogenizing and analyzing sources of shocks with long-run consequences that complicate the choices facing investors" (see Hansen and Sargent 2021, 244).

[2] Brunnermeier and Sannikov (2014) analyze equilibrium dynamics with financial frictions, employing the same capital accumulation equation as equation (2) in our model. Whereas their heterogeneous-agent framework features risk-neutral specialists and households, our model assumes a representative risk-averse household. An alternative specification of capital accumulation replaces the diffusion shocks in equation (2)—which are proportional to sectoral capital—with investment-specific diffusion shocks that scale with sectoral investment, as in Albuquerque and Wang (2008).

[3] Pindyck and Wang (2013) extend our one-sector setting by incorporating catastrophes (Barro, 2006) with the goal of analyzing their economic consequences within an equilibrium q-theoretic framework. Bolton, Wang, and Yang (2019) use the one-sector q-theoretic setting to analyze the role of the inalienability of human capital in corporate finance, investment, and valuation. Rebelo, Wang, and Yang (2022) build on

the one-sector q-theoretic setting to study the impact of financial development on sovereign debt and default.

[4] Dai, Giroud, Jiang, and Wang (2024) develop a q-theoretic model of a financially constrained firm with two divisions, each characterized by its own capital accumulation process and adjustment costs. Similar to our z, the relative size of the two divisions in their framework endogenously emerges as a persistent state variable, governing dynamic capital reallocation across divisions within the firm's internal capital market.

[5] We use $J_n(K_0, K_1)$ to denote a first-order derivative of the agent's value function $J(K_0, K_1)$ with respect to K_n, capital stock in sector $n = 0, 1$, and $J_{mn}(K_0, K_1)$ to denote a second-order derivative of $J(K_0, K_1)$ with respect to capital stocks in sectors n and m.

[6] For log utility, the wealth effect offsets the substitution effect.

[7] Bansal and Yaron (2004) argue that the elasticity of intertemporal substitution is larger than one and use 1.5 in their long-run risk model. Attanasio and Vissing-Jorgensen (2003) estimate that the elasticity of intertemporal substitution is higher than unity for stockholders. Hall (1988) uses aggregate consumption data, obtains an estimate near zero. Using micro and macro evidence, Guvenen (2006) aims to reconcile the different estimates and finds that the elasticity depends on wealth.

[8] This is because aggregate risk premium is effectively independent of the adjustment cost as we have argued in the preceding paragraph and documented in Figures 3 and 4.

[9] In our earlier paper, Eberly and Wang (2009), we studied asymmetric productivity, A, in a deterministic setting.

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Appendices

We first provide technical details for the baseline model with logarithmic utility in Appendix A and then for the general case with recursive utility in Appendix B.

Appendix A: Baseline Model With Logarithmic Utility

A.1 | Solution: One-Sector Economy

The one-sector economy serves as a benchmark and also is the solution to the model in the extreme case where all capital is invested in one sector. In this case, the sectoral capital stock is the aggregate capital, which is the single state variable in this economy.

Let $J(K_n)$ denote the value function associated with the utility maximization problem (3) in a one-sector economy. The following Hamilton–Jacobi– Bellman (HJB) equation holds:

$$0 = \max_{I_n} \quad \alpha \ln C - \alpha J(K_n) + \Phi_n(I_n, K_n) J'(K_n) + \frac{1}{2}\sigma_n^2 K_n^2 J''(K_n). \quad \text{(A.1)}$$

Using $C = Y_n - I_n$, the first-order condition (FOC) for investment implies $\alpha / C = \phi'(i_n) J'(K_n)$.

We conjecture and later verify that the value function is of the from

$$J(K_n) = \ln(p_n K_n), \quad \text{(A.2)}$$

where p_n is a constant to be determined. The FOC for investment implies $c^* = \alpha q_n$, where $q_n = 1/\phi_n'(i_n^*)$ is Tobin's q. Using $A_n - c = i_n$, we can determine the optimal investment–capital ratio as the solution to the equation: $(A_n - i_n^*)\phi'(i_n^*) = \alpha$. Substituting (A.2) into (A.1) and simplifying the expression, we obtain

$$p_n = (A_n - i_n^*) \exp\left[\frac{1}{\alpha}\left(\phi_n(i_n^*) - \frac{\sigma_n^2}{2}\right)\right]. \quad \text{(A.3)}$$

The equilibrium of the one-sector economy features stochastic growth, where the stochastic growth rates of consumption, investment, and capital, and output are all equal. Moreover, these growth rates are independently and identically distributed.

The equilibrium interest rate r is given by $r = \alpha + \phi(i_n) - \sigma^2$, the sum of the subjective discount rate α and the expected growth rate $\phi(i_n)$, minus the standard precautionary saving term for logarithmic utility. The expected return of a financial claim on aggregate output is $\alpha + \phi(i_n)$ implying that the aggregate risk premium is equal to σ^2. When capital is perfectly liquid ($\phi_n'(i_n) = 1$), $q_n = 1$.

A.2 | Asset-Pricing Implications

The stochastic discount factor (SDF) is given by $\xi_t = \alpha e^{-\alpha t}/C_t^*$. Noting that $C_t^* = c^*(z_t)(K_0(t) + K_1(t))$ and using Itô's lemma, we derive the equilibrium dynamics of aggregate consumption

$$\frac{dC_t^*}{C_t^*} = \mu_C(z_t)dt + \sigma(1 - z_t + \varepsilon(z_t))dB_0(t) + \sigma(z_t - \varepsilon(z_t))dB_1(t), \quad \text{(A.4)}$$

where $\varepsilon(z)$ and $\mu_C(z)$ are given in (24) and (25), respectively. The equilibrium dynamics of the SDF is given by

$$\begin{aligned}\frac{d\xi_t}{\xi_t} &= -\alpha dt - \frac{dC_t^*}{C_t^*} + \frac{dC_t^*}{C_t^*}\frac{dC_t^*}{C_t^*} \\ &= \left[-\alpha - \mu_C(z_t) + \sigma^2(1 - z_t + \varepsilon(z_t))^2 + \sigma^2(z_t - \varepsilon(z_t))^2\right]dt \\ &\quad - \sigma[(1 - z_t + \varepsilon(z_t))dB_0(t) + (z_t - \varepsilon(z_t))dB_1(t)].\end{aligned} \quad \text{(A.5)}$$

Since the drift of $d\xi_t/\xi_t$ is given by $-r_t = -r(z_t)$, comparing it with (A.5) allows us to identify the interest rate $r(t)$ as specified in (22).

A.3 | Decentralized Market Solution

A.3.1 | Firm Value Maximization

Taking the unique SDF process $\{\xi_s;\ s \geq 0\}$ as given, the representative firm in sector n maximizes its value by choosing I_n to solve

$$\max_{I_n} \mathbb{E}\left[\int_0^\infty \frac{\xi_s}{\xi_0}(A_n K_n(s) - I_n(s))ds\right] \quad \text{(A.6)}$$

subject to the production technology (1) and capital accumulation equation (2).

Let $V_n(K_n; z)$ denote firm value in sector n. Using Itô's Lemma, we have the following dynamics for V_n:

$$\begin{aligned}dV_n = &\left[\Phi_n \frac{\partial V_n}{\partial K_n} + \frac{\sigma^2}{2}K_n^2\frac{\partial^2 V_n}{\partial K_n^2} + \mu_z(z)\frac{\partial V_n}{\partial z} + \sigma^2 z^2(1-z)^2\frac{\partial^2 V_n}{\partial z^2} + (-1)^{n+1}\sigma^2 z(1-z)K_n\frac{\partial^2 V_n}{\partial K_n \partial z}\right]dt \\ &+ \sigma K_n \frac{\partial V_n}{\partial K_n}dB_n + \sigma z(1-z)\frac{\partial V_n}{\partial z}(dB_1 - dB_0).\end{aligned}$$

Under the optimal investment I_n^*, we have $\xi_t V_n(K_n(t); z(t)) = \mathbb{E}_t\left[\int_t^\infty \xi_s(A_n K_n(s) - I_n^*(s))ds\right]$. Note that

$$\xi_t V_n(K_n(t); z(t)) + \int_0^t \xi_s(A_n K_n(s) - I_n^*(s))ds = \mathbb{E}_t\left[\int_0^\infty \xi_s(A_n K_n(s) - I_n^*(s))ds\right]. \quad \text{(A.7)}$$

Therefore, the left side of (A.7) is a martingale. Applying Itô's lemma to the left side of (A.7) and using the fact that its drift is zero, we obtain the HJB equation:

$$\begin{aligned}rV_n = &\max_{I_n}(A_n K_n - I_n) + \Phi_n\frac{\partial V_n}{\partial K_n} + \frac{\sigma^2}{2}K_n^2\frac{\partial^2 V_n}{\partial K_n^2} + \mu_z(z)\frac{\partial V_n}{\partial z} + \sigma^2 z^2(1-z)^2\frac{\partial^2 V_n}{\partial z^2} \\ &+ (-1)^{n+1}\sigma^2 z(1-z)K_n\frac{\partial^2 V_n}{\partial K_n\partial z} - \sigma\eta_n K_n\frac{\partial V_n}{\partial K_n} - \sigma z(1-z)(\eta_1 - \eta_0)\frac{\partial V_n}{\partial z}.\end{aligned} \quad \text{(A.8)}$$

Using the conjecture $V_n(K_n; z) = q_n(z)K_n$, we can obtain the simplified HJB equation:

$$\begin{aligned}rq_n(z) = &\max_{i_n}(A_n - i_n) + [\phi_n(i_n) - \sigma\eta_n(z)]q_n(z) \\ &+ [\mu_z(z) - \sigma z(1-z)(\eta_1(z) - \eta_0(z) + (-1)^n\sigma)]q_n'(z) \\ &+ \sigma^2 z^2(1-z)^2 q_n''(z).\end{aligned} \quad \text{(A.9)}$$

The FOC for i_n gives the Tobin's q in sector n:

$$q_n(z) = \frac{1}{\phi_n'(i_n(z))}. \quad \text{(A.10)}$$

Let $D_n = A_n K_n - I_n$ denote the dividends of sector n. Using the dynamics of V_n given in (A.7) and $V_n(K_n; z) = q_n(z)K_n$, we can derive the cum-dividend return of sector n:

$$\frac{dV_n + D_n dt}{V_n} = \mu_n(z)dt + \sigma dB_n + \sigma z(1-z)\frac{q_n'(z)}{q_n(z)}(dB_1 - dB_0), \quad \text{(A.11)}$$

where

$$\mu_n(z) = \phi_n(i_n(z)) + dy_n(z) + \left[\mu_z(z) + (-1)^{n+1}\sigma^2 z(1-z)\right]\frac{q_n'(z)}{q_n(z)} + \sigma^2 z^2(1-z)^2\frac{q_n''(z)}{q_n(z)}, \tag{A.12}$$

where $dy_n(z)$ is the sectoral dividend yield given by

$$dy_n(z) = \frac{D_n}{V_n} = \frac{A_n - i_n(z)}{q_n(z)}. \tag{A.13}$$

A.3.2 | Consumer Optimality

Let W denote the representative household's total marketable wealth and π_n the fraction allocated to the stock of the firm in sector n. The representative household dynamically chooses consumption, and asset allocation among the risky assets and the risk-free asset to maximize life-time utility:

$$\max_{C,\pi_0,\pi_1} \mathbb{E}\left(\int_0^\infty e^{-\alpha t}\alpha \ln C(s)ds\right) \tag{A.14}$$

subject to the wealth accumulation

$$\begin{aligned} dW_t =& [r(z_t) + (\mu_0(z_t) - r(z_t))\pi_0(z_t) + (\mu_1(z_t) - r(z_t))\pi_1(z_t)]W_t dt - C_t dt \\ &+ \sigma W_t[\Gamma_0(z_t)dB_0 + \Gamma_1(z_t)dB_1], \end{aligned} \tag{A.15}$$

where

$$\Gamma_n(z) = \pi_n(z) - (-1)^n z(1-z)\left(\pi_0(z)\frac{q_0'(z)}{q_0(z)} + \pi_1(z)\frac{q_1'(z)}{q_1(z)}\right). \tag{A.16}$$

Let $H(W;z)$ denote the household's value function. Using the standard dynamic programming approach, we can derive the HJB equation:

$$\begin{aligned} \alpha H = \max_{C,\pi_0,\pi_1} & \alpha \ln C + [rW + (\mu_0 - r)\pi_0 W + (\mu_1 - r)\pi_1 W - C]H_W + \frac{\sigma^2 W^2}{2}\left(\Gamma_0^2 + \Gamma_1^2\right)H_{WW} \\ &+ \mu_z H_z + \sigma^2 z^2(1-z)^2 H_{zz} + \sigma^2 z(1-z)W(\Gamma_1 - \Gamma_0)H_{Wz}. \end{aligned} \tag{A.17}$$

The FOCs for consumption C and for π_n $(n = 0, 1)$ are, respectively:

$$\frac{\alpha}{C} = H_W, \tag{A.18}$$

$$\begin{aligned} (\mu_n - r)WH_W =& -\left[\Gamma_n + (\Gamma_1 - \Gamma_0)z(1-z)\frac{q_n'(z)}{q_n(z)}\right]\sigma^2 W^2 H_{WW} \\ &-\left[2z(1-z)\frac{q_n'(z)}{q_n(z)} - (-1)^n\right]z(1-z)\sigma^2 W H_{Wz}. \end{aligned} \tag{A.19}$$

Conjecture that the household's value function is of the form

$$H(W;z) = \ln(W\widetilde{N}(z)), \tag{A.20}$$

where $\widetilde{N}(z)$ is a function of z to be determined. Substituting (A.20) into (A.18) and (A.19), we obtain:

$$C^* = \alpha W, \tag{A.21}$$

$$\pi_0^* = \frac{1}{F_0 F_1 - E^2}\frac{(\mu_0 - r)F_1 - (\mu_1 - r)E}{\sigma^2}, \tag{A.22}$$

$$\pi_1^* = \frac{1}{F_0 F_1 - E^2}\frac{(\mu_1 - r)F_0 - (\mu_0 - r)E}{\sigma^2}, \tag{A.23}$$

where

$$E = z(1-z)\left[\frac{q_0'(z)}{q_0(z)} - \frac{q_1'(z)}{q_1(z)} + 2z(1-z)\frac{q_0'(z)}{q_0(z)}\frac{q_1'(z)}{q_1(z)}\right], \tag{A.24}$$

$$F_0 = 1 - 2z(1-z)\frac{q_0'(z)}{q_0(z)} + 2z^2(1-z)^2\left[\frac{q_0'(z)}{q_0(z)}\right]^2, \tag{A.25}$$

$$F_1 = 1 + 2z(1-z)\frac{q_1'(z)}{q_1(z)} + 2z^2(1-z)^2\left[\frac{q_1'(z)}{q_1(z)}\right]^2. \tag{A.26}$$

A.3.3 | Market Equilibrium

In equilibrium, (i) the goods-market clearing condition holds: $C = Y_0 + Y_1 - I_0 - I_1$; and (ii) the financial markets clearing condition holds: $\pi_0 + \pi_1 = 1$, which implies $W = V_0(K_0; z) + V_1(K_1; z) \equiv V(K_0, K_1; z)$. We can verify that the equilibrium is consistent with the social planner's solution.

Appendix B: Extended Model With Recursive Utility

B.1 | One-Sector Economy

The one-sector economy defines the boundaries of the two-sector economy as one sector's capital stock shrinks to zero. The one-sector equilibrium investment–capital ratio i_n^* for sector n solves the following nonlinear implicit equation:

$$A_n - i_n^* = \frac{1}{\phi_n'(i_n^*)}\left[\alpha + (\psi^{-1} - 1)\left(\phi_n(i_n^*) - \frac{\gamma\sigma_n^2}{2}\right)\right]. \tag{B.1}$$

Note that the left side of (B.1) is also the equilibrium consumption–capital ratio. Note that in equilibrium, the firm's optimality implies that Tobin's q is given by $q_n^* = 1/\phi_n'(i_n^*)$. Therefore, the marginal propensity to consume out of wealth C/V, that is, the dividend yield on aggregate wealth, is given by

$$dy_n = \frac{A_n - i_n^*}{q_n^*} = \alpha + (\psi^{-1} - 1)\left(\phi_n(i_n^*) - \frac{\gamma\sigma_n^2}{2}\right). \tag{B.2}$$

The equilibrium value function coefficient p_n for a one-sector economy is given by

$$p_n = \frac{\alpha}{\phi_n'(i_n^*)}\left[1 + \frac{\psi^{-1} - 1}{\alpha}\left(\phi_n(i_n^*) - \frac{\gamma\sigma_n^2}{2}\right)\right]^{1/(1-\psi)}. \tag{B.3}$$

B.2 | Planner's Resource Allocation for a Two-Sector Economy

The following HJB equation describes the planner's problem:

$$0 = \max_{I_0, I_1} f(C,J) + \phi_0(i_0)K_0 J_0 + \phi_1(i_1)K_1 J_1 + \frac{1}{2}\sigma_0^2 K_0^2 J_{00} + \rho\sigma_0\sigma_1 K_0 K_1 J_{01} + \frac{1}{2}\sigma_1^2 K_1^2 J_{11}. \tag{B.4}$$

Using the conjectured value function (36), we obtain the FOCs with respect to the sectoral investment–capital ratios i_0 and i_1, as given in (37) and (38).

With the optimal $c^*(z)$, the normalized aggregator of Duffie–Epstein utility for the agent is given by

$$f(C^*, J) = \frac{\alpha}{1 - \psi^{-1}}\left[\left(\frac{c^*(z)}{N(z)}\right)^{1-\psi^{-1}} - 1\right]((K_0 + K_1)N(z))^{1-\gamma}. \tag{B.5}$$

As a special case, if $\psi = 1$, we have $f(C^*, J) = \alpha\left[\ln c^*(z) - \ln N(z)\right]((K_0 + K_1)N(z))^{1-\gamma}$.

Using the conjectured value function (36), and substituting the normalized aggregator (B.5) as well as the optimal sectoral investment–capital ratios i_0^* and i_1^*, determined by the FOCs (37) and (38), into the HJB equation (B.4), we obtain the nonlinear differential equation (39) in the main text.

Similarly, we apply our one-sector results to $z = 0$ and $z = 1$, we obtain the boundary conditions given by (40) and (B.3) at the absorbing boundaries: $z = 0, 1$.

The aggregate capital accumulation dynamics is

$$\frac{d(K_0(t)+K_1(t))}{K_0(t)+K_1(t)} = g(z_t)dt + (1-z_t)\sigma_0 dB_0(t) + z_t\sigma_1 dB_1(t), \tag{B.6}$$

where the aggregate growth (capital accumulation) rate $g(z)$ is given by

$$g(z) = (1-z)\phi_0(i_0^*(z)) + z\phi_1(i_1^*(z)). \tag{B.7}$$

The volatility of the aggregate growth (capital accumulation) rate is given by

$$\sigma(z) = \sqrt{\sigma_0^2(1-z)^2 + 2\rho\sigma_0\sigma_1(1-z)z + \sigma_1^2 z^2}. \tag{B.8}$$

The aggregate Tobin's q is

$$q(z) = \frac{V(K_0,K_1)}{K_0+K_1} = (1-z)q_0(z) + zq_1(z). \tag{B.9}$$

B.3 | Asset-Pricing Implications

Let ξ denote the equilibrium SDF. Using the results in Duffie and Epstein (1992b), we have

$$\xi_t = \exp\left[\int_0^t f_J(C_s^*,J_s)\,ds\right] f_C(C_t^*,J_t). \tag{B.10}$$

We have

$$f_J(C^*,J) = \left(\frac{\alpha}{1-\psi^{-1}}\right)\left[(\psi^{-1}-\gamma)\left(\frac{c^*(z)}{N(z)}\right)^{1-\psi^{-1}} - (1-\gamma)\right], \tag{B.11}$$

$$f_C(C^*,J) = \frac{\alpha(C^*)^{-\psi^{-1}}}{((1-\gamma)J(K_0,K_1))^{\omega-1}} = \frac{\delta(N(z)(K_0+K_1))^{\psi^{-1}-\gamma}}{(C^*)^{\psi^{-1}}}. \tag{B.12}$$

The equilibrium dynamics of the SDF is given by

$$d\xi_t = -r(z_t)\xi_t dt - \eta_0(z_t)\xi_t dB_0(t) - \eta_1(z_t)\xi_t dB_1(t), \tag{B.13}$$

where the equilibrium interest rate as a function of z is given by

$$\begin{aligned} r(z) = {} & \alpha + \alpha\left(\frac{\psi^{-1}-\gamma}{1-\psi^{-1}}\right)\left[1-\left(\frac{c^*(z)}{N(z)}\right)^{1-\psi^{-1}}\right] + \gamma g(z) \\ & - (\gamma+1)\left[(1-z)\sigma_0^2 - z\sigma_1^2 - (1-2z)\rho\sigma_0\sigma_1\right]\varepsilon(z) - \varepsilon(z)\left(\phi_1(i_1^*(z)) - \phi_0(i_0^*(z))\right) \\ & + \frac{\psi^{-1}}{2}\left[\frac{d^2}{dz^2}\ln c(z) - (1-\gamma\psi)\frac{d^2}{dz^2}\ln N(z)\right] z^2(1-z)^2(\sigma_0^2 - 2\rho\sigma_0\sigma_1 + \sigma_1^2) \\ & - \frac{(\gamma+1)\gamma}{2}\left(\sigma_0^2(1-z)^2 + \sigma_1^2 z^2 + 2\rho\sigma_0\sigma_1 z(1-z)\right) - \frac{(\sigma_0^2 - 2\rho\sigma_0\sigma_1 + \sigma_1^2)\varepsilon^2(z)}{2}, \end{aligned} \tag{B.14}$$

and the equilibrium market prices of risk for two diffusion risks $B_0(t)$ and $B_1(t)$, $\eta_0(z)$ and $\eta_1(z)$, are, respectively, given by

$$\eta_0(z) = \sigma_0\varepsilon(z) + \gamma\sigma_0(1-z), \tag{B.15}$$

$$\eta_1(z) = -\sigma_1\varepsilon(z) + \gamma\sigma_1 z, \tag{B.16}$$

where

$$\varepsilon(z) = \psi^{-1}\left(-\frac{c^{*\prime}(z)}{c^*(z)} + (1-\gamma\psi)\frac{N'(z)}{N(z)}\right)z(1-z). \tag{B.17}$$

The market-to-book ratio, average q, is also equal to marginal q, for sector n is given by

$$q_n(z) = \frac{V_n}{K_n} = \frac{1}{\phi_n'(i_n^*(z))}, \quad n = 0, 1. \tag{B.18}$$

The dividend yields in sector n, dy_n, is given by

$$dy_n(z) = \frac{D_n}{V_n} = \frac{A_n - i_n^*(z)}{q_n(z)}. \tag{B.19}$$

Next, we derive the dynamics for the rate of return from investing in sector 0, $dR_0(t)$, which is given by the sum of sector 0 dividend yield $D_0(t)dt/V_0(t) = dy_0(z_t)dt$ and the expected rate of capital gains $dV_0(t)/V_0(t)$. Using Ito's formula, we obtain:

$$\begin{aligned} dR_0(t) &= \frac{D_0(t)dt + dV_0(t)}{V_0(t)} = dy_0(z_t)dt + \frac{dq_0(z_t)}{q_0(z_t)} + \frac{dK_0(t)}{K_0(t)} + \frac{dq_0(z_t)}{q_0(z_t)}\frac{dK_0(t)}{K_0(t)} \\ &= \mu_0^r(z_t)dt + \frac{q_0'(z_t)}{q_0(z_t)} z_t(1-z_t)(\sigma_1 dB_1(t) - \sigma_0 dB_0(t)) + \sigma_0 dB_0(t), \end{aligned} \tag{B.20}$$

where the expected rate of return in sector 0, $\mu_0^r(z)$, is given by

$$\begin{aligned} \mu_0^r(z) = {} & dy_0(z) + \phi_0(i_0(z)) + z(1-z)[\phi_1(i_1(z)) - \phi_0(i_0(z))]\frac{q_0'(z)}{q_0(z)} \\ & - z^2(1-z)\left(\sigma_0^2 - 2\rho\sigma_0\sigma_1 + \sigma_1^2\right)\left[\frac{q_0'(z)}{q_0(z)} - \frac{1}{2}\frac{q_0''(z)}{q_0(z)}(1-z)\right]. \end{aligned} \tag{B.21}$$

Let $\sigma_0^r(z)$ denote the return volatility in sector 0. We may calculate $\sigma_0^r(z)$ as follows:

$$\sigma_0^r(z) = \left[\left(\frac{q_0'(z)}{q_0(z)}z(1-z)\right)^2\left(\sigma_0^2 - 2\rho\sigma_0\sigma_1 + \sigma_1^2\right) - 2z(1-z)\frac{q_0'(z)}{q_0(z)}\left(\sigma_0^2 - \rho\sigma_0\sigma_1\right) + \sigma_0^2\right]^{1/2}. \tag{B.22}$$

Similarly, the instantaneous rate of return $dR_1(t)$ including both the dividend yield and capital gains in sector 1 is given by

$$\begin{aligned} dR_1(t) &= \frac{D_1(t)dt + dV_1(t)}{V_1(t)} = dy_1(z_t) + \frac{dq_1(z_t)}{q_1(z_t)} + \frac{dK_1(t)}{K_1(t)} + \frac{dq_1(z_t)}{q_1(z_t)}\frac{dK_1(t)}{K_1(t)} \\ &= \mu_1^r(z_t)dt + \frac{q_1'(z_t)}{q_1(z_t)} z_t(1-z_t)(\sigma_1 dB_1(t) - \sigma_0 dB_0(t)) + \sigma_1 dB_1(t), \end{aligned} \tag{B.23}$$

where the expected rate of return in sector 1, $\mu_1^r(z)$, is given by

$$\begin{aligned} \mu_1^r(z) = {} & dy_1(z) + \phi_1(i_1(z)) + z(1-z)[\phi_1(i_1(z)) - \phi_0(i_0(z))]\frac{q_1'(z)}{q_1(z)} \\ & + z(1-z)^2\left(\sigma_0^2 - 2\rho\sigma_0\sigma_1 + \sigma_1^2\right)\left[\frac{q_1'(z)}{q_1(z)} + \frac{1}{2}\frac{q_1''(z)}{q_1(z)}z\right]. \end{aligned} \tag{B.24}$$

Let $\sigma_1^r(z)$ denote the return volatility in sector 1. We may calculate $\sigma_1^r(z)$ as follows:

$$\sigma_1^r(z) = \left[\left(\frac{q_1'(z)}{q_1(z)}z(1-z)\right)^2\left(\sigma_0^2 - 2\rho\sigma_0\sigma_1 + \sigma_1^2\right) + 2z(1-z)\frac{q_1'(z)}{q_1(z)}\left(\sigma_1^2 - \rho\sigma_0\sigma_1\right) + \sigma_1^2\right]^{1/2}. \tag{B.25}$$

The sectoral risk premium is then given by $rp_n(z) = \mu_n^r(z) - r(z)$, for $n = 0, 1$.

Using the portfolio argument, we obtain the following dynamics for the rate of return on the market portfolio $dR^m(t)$:

$$dR^m(t) = \mu^m(z_t)dt + \sigma_0^m(z_t)dB_0(t) + \sigma_1^m(z_t)dB_1(t), \tag{B.26}$$

where the expected return of the market portfolio is then given by

$$\mu^m(z) = \frac{1}{q(z)}\left[(1-z)q_0(z)\mu_0^r(z) + zq_1(z)\mu_1^r(z)\right], \tag{B.27}$$

and the volatility functions are given by

$$\sigma_1^m(z) = \frac{\sigma_1 z}{q(z)}\left[q_1(z) + (1-z)^2 q_0'(z) + z(1-z)q_1'(z)\right], \tag{B.28}$$

$$\sigma_0^m(z) = \frac{\sigma_0(1-z)}{q(z)}\left[q_0(z) - z(1-z)q_0'(z) - z^2 q_1'(z)\right]. \tag{B.29}$$

The market return volatility is therefore given by

$$\sigma^m(z) = \sqrt{\left(\sigma_0^m(z)\right)^2 + 2\rho\sigma_1^m(z)\sigma_0^m(z) + \left(\sigma_1^m(z)\right)^2}. \tag{B.30}$$

The aggregate market risk premium is given by $rp^m(z) = \mu^m(z) - r(z)$.

Sectoral betas are defined in the standard way, that is, $\beta_0(z_t) = \mathrm{Cov}_t(dR_0, dR^m)/\mathrm{Var}_t(dR^m)$. The betas for sectors 0 and 1, $\beta_0(z)$ and $\beta_1(z)$, are given by

$$\beta_0(z) = \frac{\sigma_0\sigma_0^m(z) + \rho\sigma_0\sigma_1^m(z)}{(\sigma^m(z))^2} + \frac{(\sigma_1 - \rho\sigma_0)\sigma_1^m(z) - (\sigma_0 - \rho\sigma_1)\sigma_0^m(z)}{(\sigma^m(z))^2} z(1-z)\frac{q_0'(z)}{q_0(z)}, \tag{B.31}$$

$$\beta_1(z) = \frac{\sigma_1\sigma_1^m(z) + \rho\sigma_1\sigma_0^m(z)}{(\sigma^m(z))^2} + \frac{(\sigma_1 - \rho\sigma_0)\sigma_1^m(z) - (\sigma_0 - \rho\sigma_1)\sigma_0^m(z)}{(\sigma^m(z))^2} z(1-z)\frac{q_1'(z)}{q_1(z)}. \tag{B.32}$$

The instantaneous correlation between $dR_0(t)$ and $dR_1(t)$ is calculated as follows:

$$\begin{aligned}\chi(z) = {} & \frac{z(1-z)}{\sigma_0^r(z)\sigma_1^r(z)}\left[\sigma_0^2\frac{q_1'(z)}{q_1(z)}\left(z(1-z)\frac{q_0'(z)}{q_0(z)} - 1\right) + \sigma_1^2\frac{q_0'(z)}{q_0(z)}\left(z(1-z)\frac{q_1'(z)}{q_1(z)} + 1\right)\right] \\ & - \frac{\rho\sigma_0\sigma_1}{\sigma_0^r(z)\sigma_1^r(z)}\left[2z^2(1-z)^2\frac{q_0'(z)}{q_0(z)}\frac{q_1'(z)}{q_1(z)} + z(1-z)\left(\frac{q_0'(z)}{q_0(z)} - \frac{q_1'(z)}{q_1(z)}\right) - 1\right].\end{aligned} \tag{B.33}$$