Leverage and Credit Risk Dynamics under Costly Equity Issuance*

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Abstract

We develop a dynamic game-theoretic model of leverage and credit risk pricing, featuring downward jump cash-flow shocks and costly external equity. The firm begins with a low target leverage and rolls over its debt at equilibrium interest rates, allowing leverage to drift in response to realized earnings. As leverage rises, the firm proactively deleverages through recapitalization, preserving financial slack and mitigating default risk. Default occurs unpredictably following a downward jump shock and becomes more likely as leverage increases, causing equilibrium credit spreads to rise with leverage. The firm is endogenously risk-averse at low leverage but becomes risk-seeking at high leverage.

Keywords: jump shocks, credit spreads, costly external equity, debt capacity, tradeoff theory, pecking order, financial flexibility, risk seeking

JEL Codes: G32, G33, G35, E22

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1 Introduction

"How did you go bankrupt? Two ways. Gradually, then suddenly."

Ernest Hemingway, The Sun Also Rises.

In his AFA presidential address DeMarzo (2019) states that "Capital structure is not static, but rather evolves over time as an aggregation of sequential decisions." New decisions are called for in response to realized profits and losses, or changes in the firm's investment opportunity set. We take such a dynamic perspective and propose a tractable model that accounts for CFOs' stated top four concerns, as reported in Graham and Harvey (2001) and Graham (2022): 1.) maintain financial flexibility, 2.) target credit ratings, 3.) preserve internal funds, and 4.) reduce earnings volatility.

We show how these considerations come into play in a dynamic model with classic tradeoffs—namely, the informational and tax benefits of debt and financial distress costs—as in Miller and Modigliani (1961), Miller (1977), Myers (1984), and Leland (1994). The informational benefits of debt and the external financing costs of equity arise from both adverse-selection–induced equity dilution costs, as in Myers and Majluf (1984), and differential transaction and flotation costs. Although these costs give debt a funding advantage over external equity in the near term, they can lead to excessively high leverage over time. This dynamic prompts firms to proactively deleverage by issuing equity at a cost to postpone an even more costly default.

To capture the rich state-contingent financial tradeoffs firms face, we develop a model of the intertwined dynamics of leverage and credit risk pricing under a jump-diffusion process for corporate earnings before interest and taxes (EBIT). The introduction of jump risk is critical, because only with jumps will a firm face (sudden) bankruptcy, consistent with the evidence. Anticipating the firm's default strategy, credit markets price the firm's debt with an equilibrium credit spread. This sets up a dynamic game between the firm and its creditors until the firm defaults.

Just as jump risk is critical, the assumption of costly external equity is necessary to generate state-contingent equilibrium default. If equity issuance were costless, the model would counterfactually predict a constant target leverage ratio, excessively frequent equity issuance and equity payouts, and a time-invariant credit spread. In contrast, under costly external equity, when a downward cash-flow jump shock hits a firm, how it responds depends on both the size of this shock and its current leverage: The firm may choose to simply absorb the shock by increasing its borrowing (for a small jump), or it may undertake a costly recapitalization by issuing equity (for a medium-size jump), or it may default (for a large jump). What qualifies as a small, medium, or large jump in this context depends on the firm's current leverage. Taking these optimal state-contingent responses into account, we turn to the equilibrium pricing of debt and equity.

Following Black and Scholes (1973) and Merton (1973), debt in our model is short term, maturing instantaneously (so that all outstanding debt matures before the firm issues any new debt). In equilibrium, default can only be triggered by sufficiently large downward jumps. Unlike in structural credit-risk models with term debt à la Leland (1994), diffusion shocks cannot trigger default when debt is instantaneous, because the firm continuously adjusts its liabilities. Consequently, the equilibrium credit spread is characterized by a closed-form integral rather than by a differential equation as in Leland-style diffusion-based structural models. Using this integral expression for the credit spread, we obtain a tractable integro-differential equation for the firm's equity value, substantially simplifying our joint analysis of leverage and pricing.

At any given time, the firm may find itself in one of four regions for the leverage state variable, depending on its history of past earnings shocks and financing decisions (see Figure 1). From the left to the right of this figure, we have: 1.) A payout region, where the firm pays a lumpy dividend and reaches an optimal target leverage; 2.) A target zone, the interval between the optimal target leverage and the proactive deleveraging threshold, where debt and leverage evolve in response to profits and losses; 3.) An equity-issuance zone, the interval between the proactive deleveraging threshold and the firm's equilibrium debt capacity, where the firm recapitalizes by issuing equity to proactively delever its balance sheet and to bring leverage to the recapitalization target level; and 4.) A default region (to the right of the maximum debt limit), where the firm optimally defaults.

In the leverage target zone, the firm behaves as a credit-card revolver by simply letting its debt balance (on a line of credit) move up or down in response to negative or positive earnings

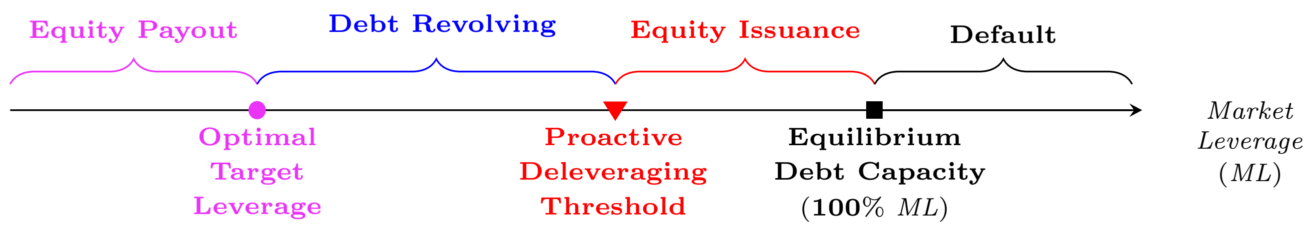

Figure 1: FOUR-REGION SOLUTION FOR MARKET LEVERAGE (ML). In the equity payout region, the firm makes a lumpy payout to reach optimal target leverage. In the debt revolving region, the firm retains earnings and rolls over its debt as a creditor-card revolver. In the equity issuance region, the firm proactively deleverages. In the default region, the firm's ML would have exceeded one without default. The three thresholds, target leverage, proactive deleveraging threshold, and debt capacity, are endogenously determined.

shocks, at an equilibrium interest rate that reflects the firm's changing credit risk. This, in effect, means that leverage is a (persistent) state variable.

To fend off the risk of bankruptcy, when following a downward jump, leverage increases to a moderately high level, the firm proactively issues equity rather than waiting until the last minute, when it has exhausted its debt capacity as in existing dynamic models, e.g., Hennessy and Whited (2007) and Bolton, Chen, and Wang (2011), respectively. Given that both the (risk-adjusted) default probability and the loss given default (LGD) are lumpy as a result of jumps, the expected cost of default, the product of these two, is also lumpy. To counterbalance this lumpy cost, the firm generates a lumpy benefit by proactively issuing equity, giving rise to an *equity-issuance zone* as an additional region of defense against default.

These four regions are connected via market leverage dynamics ($\Delta\, ML$) defined as follows:

$$\Delta\, ML = \textbf{equity payout} \; - \; \textbf{retained earnings} \; - \; \textbf{equity issuance} \; + \; (\mathbf{100\%} - ML)\,,$$

where each of the four terms on the right side indicates the key margin the firm uses to manage its leverage and corresponds to the color-matched region in Figure 1. This figure also reveals the preferred order in which the firm finances itself: First, retained earnings, second, debt, and third, equity. This prediction has a pecking-order flavor (Myers, 1984), but with two key differences. First, the firm proactively issues costly equity as a preemptive move to reduce the risk of future default. Second, the firm reverts to a recapitalization target, which is generally different from the optimal target leverage. This is because the firm also faces a marginal cost when issuing external equity.

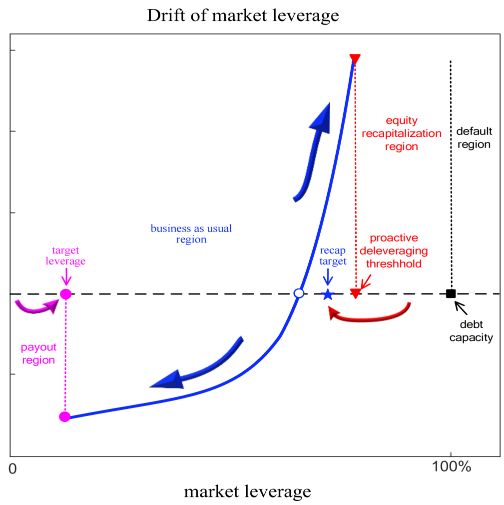

Figure 2: Graphical illustration of market leverage dynamics.

Figure 2 depicts the resulting highly non-linear, non-monotonic leverage dynamics with four coloured arrows. The red arrow describes how a firm transitions from the equity-issuance region to the recapitalization target (the blue star $\star$) in the business-as-usual region, where debt is the marginal source of financing. The two blue arrows going in opposite directions describe the non-monotonic leverage dynamics when business is as usual: When market leverage ML is to the left of the open circle, it tends to drift down towards target leverage, but when ML is to the right of the open circle, it is expected to drift upward. The latter captures the scenarios where the firm is likely to be stuck in a debt spiral, as debt begets more debt (not because of dilution incentives, but because of rising interest servicing costs). Finally, if leverage is too low, the firm issues a lumpy amount of debt to reach the target leverage (see the magenta arrow) and if leverage is too high (to the right of the black square), the firm defaults.

What is more, due to the interaction between jumps and fixed equity issuance costs, firm value is concave for low values of leverage, so that the firm is then locally endogenously risk-averse, and convex for high values of leverage, so that the firm is then endogenously risk-seeking. This is a prediction that is absent in other dynamic models, e.g., Hennessy and Whited (2007) and Bolton, Chen, and Wang (2011). In sum, our model highlights that the firm's dynamic debt policy cannot be reduced to the concept of a single target leverage ratio.

Our quantitative exercise (in Sections 4 and 5) predicts an average market leverage of 23%, within the range of estimates reported in Strebulaev and Whited (2012), even when we allow for a substantial tax advantage of debt. This result is driven by the interaction between downward jumps and costly external equity. What is more, we show that paradoxically the higher is the equity issuance cost the lower is the firm's optimal target zone for leverage. Even a small fixed equity issuance cost generates a wide target zone, where leverage changes passively and significantly lowers both target and average leverage. When equity issuance is so costly that the firm never wants to issue equity, we show that the firm chooses a leverage target that is even more conservative, i.e., very close to zero (Strebulaev and Yang, 2013). We also show that target leverage is sensitive to cash flow risk, in particular jump shocks, but is only moderately sensitive to changes in the corporate income tax rates, and largely insensitive to changes in the liquidation recovery value of assets. Our model thus explains the prudent leverage management observed for listed companies (Graham and Harvey, 2001).

We extend our baseline model along four directions in Section 6. First, we show that the firm can be a net saver if its EBIT process is also subject to contemporaneous shocks. Second, we allow the firm to manage risk using hedging positions in futures markets. Third, we show that whether the firm is taxed symmetrically or not on its profits and losses has a negligible effect on firm value and leverage policy. Fourth, we introduce personal equity income taxation and show that it has both important qualitative and quantitative implications.

We further generalize our earnings-based model to allow for growth options and investment in Section 7. We find that the firm's growth option exercising and investment decisions are nonmonotonic in leverage. When leverage is high or low, the firm exercises the growth option when it arrives, but bypasses the growth option when leverage is at an intermediate level. This is because the exercise of growth options and investment serve two roles, value creation and prudent debt management. These investment policies are affected by debt overhang (Myers, 1977) and risk shifting (Jensen and Meckling, 1976) effects, the latter being driven by the interaction between downward jumps (with stochastic losses) and fixed equity issuance costs.

Related literature. Our framework is related to three strands of the literature on dynamic corporate finance. First, the contingent-claim leverage dynamics and credit risk literature, initiated by Leland (1994), assumes that equity issuance is costless.[1] Building on Leland (1994) and allowing the firm to continuously adjust its debt, DeMarzo and He (2021) show that a firm that cannot commit to a term debt issuance policy *ex ante* tends to frequently issue debt and never retires its debt.[2] They refer to this equilibrium prediction as debt ratcheting.

Extending the setting of DeMarzo and He (2021) to allow for downward EBIT jumps and for Markov perfect equilibrium punishment strategies should the firm deviate from the announced debt policy, Malenko and Tsoy (2025) show that the optimal time-consistent debt policy features two regimes: a stable regime and a distress regime. The optimal time-consistent debt strategy requires the firm to (1.) continuously and actively manage its liabilities in the stable regime so that its debt-EBIT ratio stays at the optimal (constant) target and (2.) passively allow its leverage to evolve in response to EBIT shocks in the distress regime, into which the firm may transition after being hit by downward jump shocks.

Benzoni et al. (2022) show that when fixed debt issuance costs are introduced into the DeMarzo and He (2021) setting, the firm's incentive to rachet its debt is mitigated. Extending DeMarzo and He (2021), Hu, Varas, and Ying (2023) analyze a setting in which the cash-flow growth rate can transition from a high to a low state and where firm can also issue both short-term and long-term debt. They highlight the tradeoff between the commitment value of short-term debt and the hedging benefits of long-term debt.[3]

Unlike the models discussed above, our model features short-term debt, where the firm fully repays all its maturing outstanding debt before issuing any new debt, thus protecting current creditors from future dilution.[4] Another key difference is our assumption of costly external

[1] In Leland (1994, 1994b), the dynamics of firm leverage are driven by an exogenous stochastic firm value process as the firm pre-commits to a debt structure. Goldstein, Ju, and Leland (2001) work with a fixed rollover rule in a cash-flow-based setting (also see Strebulaev, 2007).

[2] In their model, all debts are *pari passu* with the same expected maturity. They show that the firm's incentive to dilute existing debt dissipates all value gains from adjusting leverage.

[3] Hu (2023) incorporates a differential tax treatment of debt versus equity in a setting similar to DeMarzo and He (2021).

[4] Abel (2018) considers a dynamic tradeoff model with costless equity issuance, short-term debt, and a cashflow process that is constant for an exogenous random duration of time and changes to a new value (drawn from a time-invariant distribution) at a stochastic moment governed by a Poisson process. Shareholders never voluntarily reduce debt in Abel (2018).

equity issuance, which makes the firm endogenously risk averse in the business-as-usual region and risk seeking when leverage is so high that the firm is close to default. This nonlinear and non-monotonic endogenous risk attitude is a key new prediction of our model.

Turning to the credit-risk pricing in this literature, there are broadly two ways of analyzing credit risk: (1) the structural approach in the tradition of Merton (1974) and Leland (1994) and (2) the jump-intensity reduced-form approach following Duffie and Singleton (2003). The structural models are well grounded in dynamic optimization and equilibrium analysis, but these models face significant empirical challenges. For example, they predict too low credit spreads, especially for bonds close to maturity. In our model, credit risk can be large even though debt is short term, because the firm constantly faces downward jump shocks. Our model is tractable enough that we are able to derive an essentially closed-form formula for the equilibrium credit spread and for equity value in terms of an integro-differential equation. The pricing equation for equity value in our model involves an ordinary integro-differential equation, in which the integral part arises from jump-indued credit risk. This is in sharp contrast with diffusion-shock based structural models, in which equity value is characterized by an ordinary differential equation without an integral part.

As in Duffie and Singleton (2003), the default process in our model is doubly stochastic.[5] Unlike Duffie and Singleton (2003) and related credit risk pricing models, however, the equilibrium credit risk in our model is an outcome of a dynamic game-theoretic interaction between an optimizing firm and creditors who break even in bond markets. In summary, our model combines the strengths of the structural and reduced-form credit risk approaches, and highlights the critical importance of a full analysis of the firm's future capital structure policy for the accurate pricing of credit risk at any point in time.

The second related literature is the continuous-time corporate liquidity management literature.[6] In our model, short-term debt plays the role of both corporate liquidity and leverage. One key difference between our model and this literature is that we frame the firm's problem in terms of a

[5] This is also known as a Cox process, as both the timing and likelihood of default are stochastic, see, e.g., Duffie (2001).

[6] See Bolton, Chen, and Wang (2011), Décamps, Mariotti, Rochet, and Villeneuve (2011), Hugonnier, Malamud, and Morellec (2015), Décamps et al. (2016), Della Seta, Morellec, and Zucchi (2020), Gryglewicz et al. (2022), Abel and Panageas (2023), and Hartman-Glaser, Mayer, and Milbradt (2025), for example.

dynamic game between the firm's shareholders and creditors, whereas the models in the liquidity management literature do not consider any game-theoretic interactions between shareholders and creditors. For example, in Bolton, Chen, and Wang (2011), both debt capacity and the borrowing cost (credit spread) are exogenous. Unlike in this literature, in our model corporate leverage and equilibrium credit risk, together with an endogenous debt capacity, are jointly determined in a dynamic game between the firm and creditors.

Our paper is also related to a third literature: the discrete-time dynamic corporate finance literature, following Hennessy and Whited (2005, 2007) and Riddick and Whited (2009).[7] The model in this literature that is most closely related to ours is Hennessy and Whited (2007) as it also features equilibrium default and credit risk.[8] Other than this shared model feature, however, the analysis in Hennessy and Whited (2007) and ours are very different.

The use of continuous-time modeling makes a clear distinction between diffusion shocks, which are continuous, and discrete jump shocks. This distinction, which is not feasible in discrete-time models, allows us to clearly identify jumps as the only source of default in our model, and thus to derive a particularly tractable expression for the credit spread. Furthermore, this jump-based default mechanism naturally gives rise to a two-step solution procedure: first solving the equilibrium credit-risk spread characterized by a closed-form integral formula, and then solving a tractable integro-differential equation for the firm's equity value, which fully incorporates the credit spread information via the integral part of the equation. This two-step procedure has the following advantages.

First, this procedure decouples the credit spread calculation from the firm's equity value calculation. This decoupling offers a substantial computational advantage as the credit spread calculation, which only involves a closed-form integral, is pinned down by a single number: the equilibrium default threshold for the debt-EBIT ratio.

[7] Important early work on dynamic investment under costly external financing and financial constraints includes Cooley and Quadrini (2001) and Gomes (2001).

[8] Moyen (2004) also features equilibrium default and credit risk pricing. As debt is risk free in Hennessy and Whited (2005), this paper has no implications on credit risk, a key focus of our analysis. Riddick and Whited (2009) focus on corporate savings and contains no credit-risk analysis.

Second, the firm's equity value is fully characterized by a tractable ordinary integro-differential equation with economically intuitive boundary conditions.[9] This pricing equation sharply characterizes the model's key economic predictions by mapping the firm's equity value and optimal policies into four mutually exclusive regions for the firm's debt-EBIT ratio: an equity payout region, a debt revolving region, an equity issuance region, and a default region, divided by three endogenously determined thresholds: payout boundary, proactive equity issuance boundary, and equilibrium debt capacity.

Third, the tight mapping between the economic mechanism and the mathematical structure of our model enables a globally efficient numerical solution method for a tractable nonlinear ordinary integro-differential equation with the debt-EBIT ratio as a single state variable (see Appendix B for details.)

The discrete-time literature, e.g., Hennessy and Whited (2007), does not allow for such a two-step solution procedure. Rather, the equilibrium credit spread and the firm's equity value have to be jointly solved via a "loop-within-a-loop" iterative procedure, which is numerically much more challenging than ours. This is because both the equilibrium credit spread and equity value are interdependent implicit numerical integral functions with three state variables: productivity shock, debt balance, and capital stock. See Strebulaev and Whited (2012) for a discussion of numerical complexity in the discrete-time literature.[10]

In summary, by exploiting the tractability of our model and using a two-step procedure, we parsimoniously characterize the highly nonlinear joint dynamics of leverage, credit risk, and equity value, together with the firm's non-monotonic risk taking, equity issuance, and equilibrium default policies. In contrast, the numerical analysis in the discrete-time literature is both more challenging and more opaque, which can make it difficult to understand the model's economic

[9] Methodologically, our model is the first to apply the variational-inequality method to analyze equilibrium credit spreads in a jump-diffusion setting where the firm and creditors play a dynamic game. Technically, we can summarize the model's global solution using a variational inequality. Friedman (1982) provides a mathematical treatment of the variational-inequality approach for partial differential equations that arise from various dynamic optimization problems.

[10] On page 120, the authors write: *"To solve the model, one needs to know firm value in order to calculate the default states and thus the interest rate. However one needs to know the interest rate to calculate firm value. One feasible solution algorithm is a "loop-within-a-loop," ... The complexity of these models is one disadvantage they have with respect to contingent claims models, in which default thresholds can often be found analytically and can almost always be characterized as a differential equation."*

mechanism. To be sure, Stebulaev and Whited (2012) note that, *"Although detail adds to the realism and rigor of the research, the unintended consequence is that the models appear to be black boxes with many indistinguishable moving internal parts."*

Finally, in terms of specific research questions, Hennessy and Whited (2007) and our work are complementary to each other. Their main focus is to infer the magnitude of financing costs using simulated method of moments to estimate their model. We develop a tractable, intuitive model to analyze the joint leverage and credit risk dynamics,[11] and then use the model to generate new testable quantitative and empirical predictions.

The remainder of this paper is organized as follows. Section 2 introduces the baseline model. Section 3 characterizes the baseline model solution. Using the solution, Section 4 provides a quantitative analysis and Section 5 discusses the key determinants of leverage via a set of comparative static analyses. Section 6 extends the baseline model by incorporating savings and hedging margins. Section 7 incorporates growth option and capital accumulation into our baseline model. Finally, Section 8 concludes.

2 Baseline Model

Investors are risk averse and demand a risk premium for bearing systematic risk. We thus specify the firm's EBIT on a risk-adjusted basis, i.e., under the risk-neutral measure $\mathbb{Q}$. Working directly under $\mathbb{Q}$ is for ease of exposition, allowing us to discount stochastic cashflows at the risk-free rate. In the paper, unless otherwise stated, all calculations involving stochastic processes are performed under $\mathbb{Q}$.

The EBIT process $\{Y_t\}$ includes downward discrete jump shocks and associated jump risk premia. We capture both jumps and jump risk premia by assuming that under $\mathbb{Q}$:

$$\frac{dY_t}{Y_{t-}} = \mu dt + \sigma d\mathcal{B}_t - (1-Z)d\mathcal{J}_t \,, \quad Y_0 > 0 \,, \tag{1}$$

where μ is the drift parameter, σ is the diffusion-volatility parameter, $\mathcal{B}$ is a standard Brownian motion process, and $\mathcal{J}$ is a pure jump process with a constant arrival rate λ under the risk-neutral

[11] The discrete-time literature generally does not explore equilibrium credit-risk pricing.

measure.[12] The inclusion of downward jumps generates negatively skewed and fat-tailed EBIT growth.

If a jump arrives at time t ($d\mathcal{J}_t = 1$), the firm's EBIT decreases from Y_{t-} to $Y_t = ZY_{t-}$, where $Z \in [0,1]$ is a random variable drawn from a distribution function $F(\,\cdot\,)$. As long as the firm remains solvent, there is no limit to the number of these shocks; the occurrence of a jump shock does not change the likelihood of another in the future. The expected percentage EBIT loss upon a jump arrival is $(1 - \mathbb{E}(Z))$, where $\mathbb{E}[\,\cdot\,]$ is the expectation operator and the expected EBIT growth rate, g, is given by:

$$g = \mu - \lambda(1 - \mathbb{E}(Z)) \; . \tag{2}$$

The firm's financing choices involve both internal and external sources of funds. Internal funds come from internally generated cash flows that are retained over time. External funds include short-term debt, which we assume to be costless for simplicity,[13] and equity. Critically, external equity is costly in our model (and in reality). We denote by X_t the firm's debt balance at time t.

We assume that debt is short-term. Let $C_t dt$ denote the (promised) debt interest payment over a small time interval $(t, t + dt)$. Conditional on not defaulting, the firm has to pay its outstanding debt balance (principal) X_t and interests $C_t dt$ at time $t + dt$.

Note that our model features a dynamic game in which the firm and creditors continuously interact until the firm defaults. Because debt matures instantaneously, both principal and interest must be repaid at every instant absent default. This is why short-term debt is issued at par. Following a large EBIT loss, shareholders may declare default. With competitive markets, creditors price this default risk *ex ante*.

[12] In Appendix B, we derive the EBIT process (1) under the risk-neutral measure by starting with a jump-diffusion EBIT process under the physical measure and specifying risk premia for jump and diffusion shocks. The jump-diffusion process (1) generalizes the geometric Brownian motion process commonly used in the contingent-claim capital structure literature, e.g., Fischer, Heinkel, and Zechner (1989), Leland (1994, 1994b), and Goldstein, Ju, and Leland (2001). Malenko and Tsoy (2025) use this jump-diffusion EBIT process to study optimal time-consistent debt policy.

[13] While in practice firms incur transaction costs in issuing debt or securing a line of credit, these costs are small relative to the costs firms incur when they issue equity. What is important for our analysis is that equity issuance costs are higher than debt issuance costs, not whether debt issuance is costly or not. This is why we set debt issuance costs to zero for simplicity.

Equilibrium debt pricing. Let η_{t-} denote the equilibrium credit spread. Without loss of generality, we can write the interest payment C_{t-} as:

$$C_{t-} = (r + \eta_{t-})X_{t-}. \tag{3}$$

Creditors receive the firm's entire liquidation proceeds following default, but under limited liability creditors do not fully recover what is owed to them. To compensate debtholders for the credit risk they bear, the following expression for η_{t-} has to hold:

$$\eta_{t-} = \lambda \mathbb{E}_{t-}\left[\underbrace{\left(1 - \frac{L_t}{X_{t-}}\right)}_{\text{fractional LGD}} \mathbf{1}_t^{\mathcal{D}} \right], \tag{4}$$

where $\mathbf{1}_t^{\mathcal{D}} = 1$ if the firm defaults at t and $\mathbf{1}_t^{\mathcal{D}} = 0$ otherwise. Note that the credit spread η_{t-} depends on the firm's default policy. This is where the strategic interaction between the firm and creditors appears. We discuss the economics of credit spread in the next section. In Appendix A, we provide a derivation of the credit risk pricing equation (4).

Corporate taxes and earnings. In practice, corporate profits and losses have asymmetric tax treatments. We capture this feature in our model by assuming that the firm pays income taxes at a constant rate ($\tau > 0$), when making a profit ($Y_{t-} > C_{t-}$), and pays no taxes when incurring a loss ($Y_{t-} \leq C_{t-}$). That is, the firm's tax payment Θ_{t-} is given by $\Theta_{t-} = \Theta(C_{t-}, Y_{t-}) = \theta(c_{t-})Y_{t-}$, where $c_{t-} = C_{t-}/Y_{t-}$ and

$$\theta(c) = \tau(1 - c)\mathbf{1}_{c \leq 1}, \tag{5}$$

is the scaled tax payment. The indicator function $\mathbf{1}_{c \leq 1}$ equals one if the firm makes a profit ($c \leq 1$) and zero otherwise (i.e., $c > 1$). Next, we obtain earnings, E_{t-}, by deducting interest payments $C_{t-} = (r + \eta_{t-})X_{t-}$ and taxes Θ_{t-} from EBIT Y_{t-}:

$$E_{t-} = \underbrace{Y_{t-}}_{\text{EBIT}} - \underbrace{(r + \eta_{t-})X_{t-}}_{\text{interest payments}} - \underbrace{\Theta_{t-}}_{\text{taxes}}. \tag{6}$$

As X_{t-} and Y_{t-} determine taxes and interest payments, they pin down earnings E_{t-}.

Financial distress costs. Following Leland (1994) and the tradeoff theory literature, we assume that bankruptcy causes deadweight losses and specify the firm's recovery value at $T^{\mathcal{D}}$ as follows:

$$L_{T^{\mathcal{D}}} = \ell Y_{T^{\mathcal{D}}} \,, \tag{7}$$

where the multiple ℓ gives the firm's liquidation recovery per unit of EBIT.[14] Unless ℓ is very high, the firm prefers operating over defaulting. Since the APR applies in bankruptcy and liquidation is inefficient, equityholders default only when it is no longer in their interest to continue operating the firm—that is, when the firm's equity value reaches zero.

Costly external equity issuance. Firms rarely issue external equity, and when they do, they incur significant costs, due to a combination of asymmetric information (Myers and Majluf, 1984), managerial incentive issues, and underwriting costs. For convenience we capture adverse selection costs, underwriting fees, and other floatation costs with a simple reduced form function. A large empirical literature has sought to quantify these costs, including the costs arising from the negative stock price reaction in response to the announcement of a new equity issue.[15]

Building on the findings of this literature, we assume that the firm incurs both fixed and proportional costs when it issues equity. Fixed costs are what generates the observed infrequent and lumpy equity issuance. As we will show, the proportional cost gives rise to two distinct targets for leverage: "optimal target leverage" when debt and earnings retention are the marginal sources of financing and the "recapitalization target" when external equity is the marginal source of financing. Only when fixed equity issuance costs are present do firms have an incentive to seek risk, causing firm value to become convex in the high-leverage region, as we demonstrate later.[16]

We assume that the fixed cost is proportional to Y_t, so that $h_0 Y_t$ denotes the total fixed equity-issuance cost. This formulation ensures that the firm never grows out of the fixed cost, preserving the first-order consideration of equity issuance costs and the model's homogeneity property. The firm also incurs variable issuance costs: For each dollar the firm raises it incurs a

[14] We can gauge the magnitude of ℓ as in Leland (1994). Suppose that the firm's liquidation value $L_{T^{\mathcal{D}}}$ equals a fraction $\varphi \in (0,1)$ of its first-best value: $\Pi(Y) = \frac{Y}{r-g}$, where g is the growth rate given in (2). To ensure that $\Pi(Y)$ is finite, we require that $r > g$. $L_{T^{\mathcal{D}}} = \varphi \Pi(Y_{T^{\mathcal{D}}})$. Then, $\ell = \varphi \pi = \varphi/(r-g)$.

[15] Altinkilic and Hansen (2000) estimate underwrite fee schedules. Asquith and Mullins (1986) measure the indirect costs of external equity using seasoned equity offerings announcements. Hennessy and Whited (2007) use simulated method of moments to infer the magnitude of financing costs and their estimates support the view that external equity issuances involve large indirect costs and firms are sensitive to these costs.

[16] In Appendix F, we analyze different effects of fixed versus proportional equity issuance costs, using a few helpful special cases of our model.

marginal cost h_1. Therefore, when raising net proceeds $M_t \geq 0$, the firm incurs a total equity issuance cost of $h_0 Y_t + h_1 M_t$. Let $dH_t = h_0 Y_t + h_1 M_t \geq 0$ denote this cost so that $\{H_t; t \geq 0\}$ is the cumulative (undiscounted) nondecreasing equity issuance cost process.

Equity payouts. In addition to managing debt and equity issuance, the firm also decides if and when to pay dividends. We denote by U_t the (undiscounted) cumulative payout to equityholders up to time t. In our model the optimal target leverage is closely tied to the firm's payout decision to shareholders. Next, we introduce an accounting identity that is key to understanding the firm's dynamic financial management.

Sources of funds equal uses of funds. Conditional on no default, the firm's sources of funds must at all times equal its uses:

$$\underbrace{Y_{t-}dt}_{\text{EBIT}} + \underbrace{(X_{t-} + dX_t)}_{\text{(gross) debt issuance}} + \underbrace{dN_t}_{\text{external equity}} = \underbrace{\Theta_{t-}dt}_{\text{taxes}} + \underbrace{(X_{t-} + C_{t-}dt)}_{\text{debt repayment}} + \underbrace{dU_t}_{\text{payouts}}, \tag{8}$$

where $dN_t \geq 0$ and $dU_t \geq 0$ denote the firm's net external equity issuance and its incremental payout over $(t, t+dt)$, respectively. Note that dN_t and dU_t are two separate non-negative control margins, with N_t and U_t denoting respectively the cumulative (undiscounted) non-decreasing external equity issuance and payouts up to time t. The second term on the left side of (8) represents the firm's gross debt issuance, which can be written as the sum of its balance due X_{t-} and incremental borrowing dX_t with no loss of generality, as dX_t can be arbitrarily chosen (positive or negative, smooth or lumpy). The three terms on the left side of (8) describe the sources of funds.

The three terms on the right side of (8) describe the uses of funds. In addition to paying taxes (the first term) and repaying its debt (the second term), the firm also decides whether to make a payout to shareholders or not (the third term).

Debt balances X_{t-} appear on both sides of (8). By eliminating X_{t-} on the two sides and rearranging terms, we obtain the following accounting identity for a solvent firm:

$$\underbrace{dU_t}_{\text{payouts}} = \underbrace{E_{t-}dt}_{\text{earnings}} + \underbrace{dX_t}_{\text{net debt issuance}} + \underbrace{dN_t}_{\text{external equity}}, \tag{9}$$

where E_{t-} represents earnings, as defined in (6). Equation (9) states that at all t, the cash outflows of a solvent firm (payouts) must equal the cash inflows, which arise from a combination of earnings, net debt issuance, and equity issuance. Importantly, out of the three policies: net debt issuance (dX_t), net equity issuance (dN_t), and payouts (dU_t), a firm can freely choose only two. The third must adjust as the residual to satisfy (9).

Because external equity is costly and a dollar dividend payment to shareholders is only worth a dollar, issuing equity to pay dividends is suboptimal. Therefore, a solvent firm can only be in one of the three mutually exclusive scenarios: (a) issuing equity ($dN_t > 0$) but paying no dividends ($dU_t = 0$); (b) issuing no equity ($dN_t = 0$) but paying dividends ($dU_t > 0$); (c) neither issuing equity nor paying dividends ($dN_t = dU_t = 0$). In all three cases, the net debt issuance dX_t is uniquely pinned down by (9).[17]

Next, we state the firm's optimization problem.

Firm optimality. Let $S = (T^{\mathcal{D}}, \{N_t, U_t; t \in (0, T^{\mathcal{D}})\})$ denote the firm's (default timing, equity issuance, and equity payout) policies. The firm chooses its policy S to maximize its equity value:[18]

$$P(Y_t, X_t) \ = \ \mathbb{E}_t \left[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)} \left(dU_s - dN_s - dH_s \right) \right], \tag{10}$$

subject to the (sources of funds equal uses of funds) accounting identity (9) and the credit risk pricing equations (3)-(4). Note that we subtract equity issuance costs as $dH_t > 0$ whenever the firm issues equity, $dN_t > 0$. Shareholders only default when they receive zero at time $T^{\mathcal{D}}$. In effect, the firm is engaged in a dynamic game in which it continuously interacts with creditors and external equity investors. In Appendix A, we define the Markov perfect equilibrium (MPE) of this game. In Appendix B, we prove the existence of an MPE and then characterize this equilibrium.

[17] Note that the credit spread η_{t-} internalizes all the benefits and costs of debt issuance for shareholders. Also there are no existing debtholders for shareholders to dilute because all outstanding debt has to be repaid before new financing can be obtained. Therefore, maximizing the firm's total value yields the same outcome as maximizing its equity value.

[18] We can extend our model to allow the firm's shareholders to be more impatient than creditors (see DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007), DeMarzo and Fishman (2007), DeMarzo (2019), and Malenko (2019) among others, for models with this assumption.) See the NBER version of our paper for a model with this assumption. To ease comparisons with standard capital-structure tradeoff models, we leave this extension out.

Let $V_t = V(Y_t, X_t)$ denote firm value. As debt matures instantaneously, the market value of debt equals the face value of debt, so that:

$$V(Y_t, X_t) = P(Y_t, X_t) + X_t \,. \tag{11}$$

Let x_t denote the debt-EBIT ratio: $x_t = X_t/Y_t$, which is the natural state variable in our model.[19] Let p_t denote the equity value-EBIT ratio ($p_t = P_t/Y_t$), and v_t denote the firm value-EBIT ratio ($v_t = V_t/Y_t = p_t + x_t$). Also let ML_t denote market leverage, the ratio of debt value X_t to firm value V_t: $ML_t = X_t/V_t = x_t/v_t$. In our model, $ML_t \in [0, 1]$. When a firm defaults, its market leverage equals one.

3 Solution

In this section, we report model solution and discuss the mechanism. The firm in our model dynamically chooses all decision margins including issuance timing and size decisions for debt and equity, equity payouts, and default. At any time t, depending on its debt-EBIT ratio x_t the firm is in one of the four possible regions.[20]

3.1 Optimal Target $\underline{x}$ and Payout Region ($dU_t > 0$ and $dN_t = 0$): $x < \underline{x}$

A firm with low or no debt (say at $t = 0$) wants to issue some debt because debt can shield taxes for shareholders. How much debt does the firm issue? Because of the model's homogeneity property, it is optimal for the firm to set x to $\underline{x}$, a time-invariant 'optimal target' for x. Because the firm with $x < \underline{x}$ can always increase its debt to $\underline{x}$ at no cost, the following value-matching condition holds:

$$p(x) = p(\underline{x}) + \underline{x} - x \,, \quad \text{for} \;\; x < \underline{x} \,. \tag{12}$$

Taking the limit $x \to \underline{x}$ in (12), we obtain the following smooth-pasting condition for $\underline{x}$:

$$p'(\underline{x}) = -1 \,. \tag{13}$$

[19] This is consistent with Graham (2022), who reports that corporate capital structure appears to "focus on Debt/EBITDA or credit ratings rather than the classic debt/value or debt/assets."

[20] In this section, we analyze the general case in which the solution features four regions. In Appendices E and F, we characterize solutions for special cases with fewer than four regions.

Since the target $\underline{x}$ is optimally chosen by shareholders, we also have the following super-contact condition (Dumas, 1991):

$$p''(\underline{x}) = 0\,. \tag{14}$$

Note that (14), implied by shareholders' equity value maximization, is also the counterpart to the first-order condition that pins down the optimal target leverage in static tradeoff theory (Myers, 1984): Firm value $v(x)$ is highest when x is at its optimal target $\underline{x}$. Unlike in static tradeoff theory, however, it is not optimal for a firm to keep x at its optimal target $\underline{x}$ at all time. This is because x_t is exposed to earnings (diffusion and jump) shocks and issuing external equity to keep x at $\underline{x}$ is too costly. In other words, to always instantaneously cover a loss with a new equity issue is far too costly.

3.2 Business as Usual ($dU_t = dN_t = 0$, implying $dX_t = -E_{t-}dt$): $x \in [\underline{x}, \widehat{x})$

When x_t exceeds $\underline{x}$ but is still moderate, it is suboptimal for the firm to decrease x and bring it back to the 'optimal target' $\underline{x}$. This is because the only way to keep x at the target $\underline{x}$ is to issue external equity, which is costly. Instead, the firm optimally defers its equity issuance until x reaches $\widehat{x}$ where the benefit of issuing external equity (weakly) outweighs the cost of doing so. Also as the debt-EBIT ratio x is already above the optimal target $\underline{x}$, the firm certainly does not want to voluntarily further increase its x by issuing debt to pay dividends. Therefore, $dN_t = dU_t = 0$ in this region.

The accounting identity (9) then implies that in the business-as-usual region, the increase of net debt issuance dX_t must exactly track the decrease of earnings:

$$dX_t = -E_{t-}dt = \big[(r + \eta_{t-})X_{t-} - Y_{t-} + \Theta_{t-}\big]dt\,. \tag{15}$$

That is, the optimal policy is to simply let earnings dictate debt dynamics. When making a profit (loss), i..e, when earnings E_{t-} given in (6) is positive (negative), the firm's debt balance automatically decreases (increases) by the amount $|E_{t-}|$. This is the 'business-as-usual' scenario where the firm spends most of its time.

In effect, the firm behaves as a credit card revolver by letting its debt balance X_t drift up (down) in response to its realized earnings losses (profits). Under this 'credit card revolver' interpretation the firm's debt is a long-term state-contingent liability, as if it was tapping a line of credit whenever needed. Of course, the equilibrium interest rate at which the firm revolves its credit is endogenous and varies with firm leverage.

Next, we describe leverage dynamics in this region. Applying Ito's Lemma to $x_t = X_t/Y_t$ where X_t is given in (8) and Y_t is given in (1), we obtain:

$$dx_t = \mu_x(x_{t-})\, dt - \sigma x_{t-} d\mathcal{B}_t + \left(x_t^{\mathcal{J}} - x_{t-}\right)\, d\mathcal{J}_t\,, \tag{16}$$

where $\mu_x(\,\cdot\,)$ is the drift of x (leaving aside the jump effect) given by:

$$\mu_x(x_{t-}) = [c(x_{t-}) + \theta(c(x_{t-})) - 1] - \left(\mu - \sigma^2\right) x_{t-}\,. \tag{17}$$

The interest payment per unit of EBIT, $c(x_{t-})$, is given by:

$$c(x_{t-}) = (r + \eta(x_{t-}))x_{t-}\,, \tag{18}$$

where $\eta(\,\cdot\,)$ is the equilibrium credit spread given by:

$$\eta(x_{t-}) = \lambda \int_0^{x_{t-}/\overline{x}} \underbrace{\left(1 - \frac{\ell Z}{x_{t-}}\right)}_{\text{fractional LGD}} dF(Z)\,. \tag{19}$$

We obtain (19) by using the homogeneity property to simplify (4).

The intuition for (19) is as follows. First, creditors' stochastic fractional loss given default (LGD) equals $1 - (L_t/X_{t-}) = 1 - \ell Z/x_{t-}$. Second, note that a jump arrival is necessary but not sufficient to trigger default. Only when creditors' LGD is large enough ($Z < x_{t-}/\overline{x}$) will the firm default. These observations together explain (4). As a special case, if default yields no recovery ($\ell = 0$), the credit spread is: $\eta(x_{t-}) = \lambda F(x_{t-}/\overline{x})$.

The $-\sigma x_{t-}$ diffusion term in (16) captures the effect of EBIT diffusion shocks on x_{t-}: A positive shock increases the firm's value and lowers x explaining the negative sign. The last term in (16) describes the effect of a jump shock on x. Upon arrival of a jump loss at t, the debt-EBIT ratio changes from x_{t-} to the post-jump level of $x_t^{\mathcal{J}} = x_{t-}/Z$.

To account for the effect of jumps on the expected change of x, we introduce:

$$g_x(x_{t-}) = \mu_x(x_{t-}) + \lambda \left[\mathbb{E}_{t-}\left(\frac{x_{t-}}{Z}\right) - x_{t-}\right]\,, \tag{20}$$

where $\mu_x(x_{t-})$ is given in (17). As we show later, the drift $g_x(x_{t-})$ is highly nonlinear even under business as usual.

Next, we characterize the equity value $p(x)$. In Appendix B.1, we show that for $x \in (\underline{x}, \widehat{x})$, $p(x)$ satisfies the following integro-differential equation:

$$(r-\mu)\, p(x) = [c(x) + \theta(c(x)) - 1 - \mu x]\, p'(x) + \frac{\sigma^2 x^2}{2} p''(x) + \lambda \mathbb{E}\left[Z p(x/Z) - p(x)\right] . \qquad (21)$$

The first two terms on the right side of (21) capture the drift and volatility effects of x on $p(x)$. The last term in (21) captures the effect of jumps on equity value. A jump causes the debt-EBIT ratio to increase from x_{t-} to $x_t^{\mathcal{J}} = x_{t-}/Z$ and the equity value to decrease from $p(x_{t-})$ to $p(x_t^{\mathcal{J}}) = p(x_{t-}/Z)$.

Although the firm operates its business as usual in this region, by relying on earnings retention and debt rollover, its financial situation is very different depending on the level of x. Near the left payout boundary $\underline{x}$, x_t drifts downward in expectation and the firm has a strong balance sheet. In contrast, near the right boundary $\widehat{x}$, x_t drifts upward at a fast rate and the firm has a vulnerable balance sheet. Importantly, earnings growth jumps not only can cause default but also generate rich leverage dynamics.

Absent jumps, our model only features continuous, diffusive shocks and can be solved using standard finite-difference methods requiring only local information around x. In contrast, in the presence of jump shocks a global solution method is required because $p(x)$ is linked to $p(x/Z)$ for all possible values of the random recovery $Z \in [0, 1]$. In short, the economics of the jump-diffusion model are much richer, but solving this model requires more involved numerical solution methods.

Next, we turn to the region where the firm proactively issues equity to deleverage.

3.3 Recapitalizating and Deleveraging ($dN_t > 0$ and $dU_t = 0$): $x \in [\widehat{x}, \overline{x}]$

When x rises above an endogenous boundary $\widehat{x}$, the firm recapitalizes its balance sheet to deleverage ($dN_t > 0$) even though this involves a costly seasoned equity offering. As long as the cost of issuing external equity is low or moderate, a recapitalization is valuable as it reduces the expected cost of an even worse outcome in the future: a costly default.

Let $m_t = M_t/Y_t \geq 0$ denote the (scaled) net equity issuance proceeds. Let $\widetilde{x}_t = \widetilde{X}_t/Y_t$ denote the *'recapitalization target'*, that is, the debt-EBIT ratio immediately after the firm has delevered through the recapitalization:

$$\widetilde{x}_t = x_t - m_t \,. \tag{22}$$

Given that $p(x)$ must be continuous before and after issuance, the following value-matching condition holds:

$$p(x) = p(\widetilde{x}) - [h_0 + (1+h_1)(x - \widetilde{x})] \quad \text{for } \; x \in [\widehat{x}, \overline{x}] \,. \tag{23}$$

Equation (23) states that the firm's pre-issuance equity value $p(x)$ must equal its post-issuance equity value, $p(\widetilde{x})$, after accounting for: (a.) the fixed equity issuance cost h_0; (b.) the variable issuance cost $h_1 m_t$; and (c.) the net equity issuance amount m_t. Because the firm incurs a proportional cost of h_1 when issuing equity, we have

$$-p'(x) = 1 + h_1 \quad \text{for } \; x \in [\widehat{x}, \overline{x}) \,, \tag{24}$$

which equates the marginal benefit with the marginal cost of issuing equity.

Because the firm optimally reaches the recapitalization target $\widetilde{x}$ after issuing equity, $-p'(\widetilde{x}) = 1 + h_1$ must hold. As (23) and (24) hold at the firm's proactive deleveraging threshold $\widehat{x}$, we summarize the boundary conditions for $\widehat{x}$ and $\widetilde{x}$ as follows:

$$p(\widehat{x}) = p(\widetilde{x}) - [h_0 + (1+h_1)(\widehat{x} - \widetilde{x})] \quad \text{and} \quad -p'(\widehat{x}) = -p'(\widetilde{x}) = 1 + h_1 \,. \tag{25}$$

Importantly, the 'equity recapitalization target' $\widetilde{x}$ is different from the 'optimal leverage target' $\underline{x}$ where firm value $v(x)$ is maximized. The firm deleverages its balance sheet from the $x \geq \widehat{x}$ region to $\widetilde{x}$ in the business-as-usual region by issuing costly equity. In contrast, the firm reaches the 'optimal leverage target', $\underline{x}$, when its marginal cost of issuing debt is zero: $v'(\underline{x}) = 0$ so that at the margin the firm is indifferent between paying a dividend to shareholders or not when it chooses to return to $\underline{x}$. In sum, the 'equity recapitalization target' $\widetilde{x}$ and the 'optimal leverage target' $\underline{x}$ are two different targets and $\underline{x} < \widetilde{x}$ because the marginal cost of debt issuance is lower than the marginal cost of equity issuance ($h_1 > 0$).

Our model highlights that the idea of optimal leverage in a dynamic setting cannot be reduced to the concept of a single target ratio, as the existing literature has done. First of all, the correct concept is a target zone (or interval) in which leverage evolves optimally. And second, when leverage exits this region, there are two distinct return points, one on the left (the optimal leverage target) when leverage exits the target region on the left, and one on the right (the optimal recapitalization target) when leverage exits the recapitalization region on the right.

Because external equity is costly, the firm spends almost no time in the equity-issuance region. Although at a cost, being able to issue costly external equity to proactively deleverage its balance sheet is a valuable option for a firm managing its leverage dynamics. Overall, the equity issuance option generates highly nonlinear and asymmetric leverage dynamics even when the firm is operating in the business-as-usual target zone.

3.4 Equilibrium Debt Limit $\overline{x}$ and Default Region $(t > T^{\mathcal{D}})$: $x > \overline{x}$

The firm only uses default as a last resort. When the firm's debt-EBIT ratio x is so large that it exceeds an endogenously determined level $\overline{x}$, the firm defaults. Because of limited liability and the APR rule, equity value equals zero:

$$p(x) = 0 \quad \text{for} \quad x \geq \overline{x}\,, \tag{26}$$

and creditors collect the firm's liquidation value ℓ per unit of EBIT. As $\overline{x}$ is the lowest value of x above which the firm is unable to refinance its debt and continue as a going concern, we refer to it as the equilibrium debt limit (capacity).

Because the firm is indifferent between defaulting and issuing equity when $x = \overline{x}$, we can pin down the equilibrium debt limit $\overline{x}$ and the equity recapitalization target $\widetilde{x}$ using the following two equations:

$$p(\widetilde{x}) = h_0 + (1 + h_1)(\overline{x} - \widetilde{x}) \quad \text{and} \quad p(\overline{x}) = 0\,. \tag{27}$$

The intuition for the first equation in (27) is as follows. The right side is the sum of (a.) the maximal (net) amount that the firm can raise via external equity: $m = \overline{x} - \widetilde{x}$ and (b.) the associated equity issuance costs $(h_0 + h_1 m)$. Immediately after the firm issues equity when $x = \overline{x}$, old shareholders are completely wiped out so that new shareholders own the entire firm's equity, valued at $p(\widetilde{x})$. This explains the first equation in (27). Together with the zero-equity value condition at $x = \overline{x}$, we can jointly pin down the debt limit $\overline{x}$ and the equity recapitalization target $\widetilde{x}$.

Below we summarize our model solution for the general four-region case.

Summary. The scaled equity value $p(x)$ is continuous in x and takes different functional forms in four regions: i) in the payout region ($x < \underline{x}$), $p(x)$ is linear in x as given in (12); ii) in the business-as-usual region ($\underline{x} \leq x < \widehat{x}$), $p(x)$ satisfies the integro-differential equation (21); iii) in the equity-issuance region ($\widehat{x} \leq x \leq \overline{x}$), $p(x)$ is given by (23); and iv) in the default region ($x > \overline{x}$), $p(x) = 0$. The equilibrium credit spread is given by (19) in the business-as-usual region. The optimal target (payout threshold) $\underline{x}$ satisfies (13)-(14), the recapitalization target $\widetilde{x}$, the equity-issuance boundary $\widehat{x}$ and the equilibrium debt limit $\overline{x}$ jointly satisfy the conditions given in (25) and (27).

In Appendix E, we report solutions for the no-jump ($\lambda = 0$) and zero-jump recovery ($Z = 0$) special cases. The credit spread equals zero in the former and the (risk-neutral) jump arrival rate, respectively, with degenerate default predictions. A key takeaway from these two special cases is that jumps that cause fractional recoveries are key to generate state-contingent default and time-varying equilibrium credit spread. In Appendix F, we analyze four special cases where equity financing costs h_0 and h_1 are too high and/or zero and show that the predictions of these special cases are rather limited. A key takeaway from these four special cases is that both fixed and proportional equity financing costs are important in generating plausible leverage dynamics.

4 Quantitative Predictions

In this section we undertake a quantitative analysis of the baseline model and illustrate the highly non-linear solution using figures. The calibrated model generates empirically plausible predictions about leverage and credit spreads. For example, we show that the predicted market leverage is low, averaging around 23%, consistent with recent evidence (see, e.g., Graham, 2000, 2022; Strebulaev and Whited, 2012). The critical parameter choices are respectively moderate external equity issuance costs and stochastic earnings growth with a realistic risk premium specification for permanent downward jump shocks. We begin by discussing our baseline parameter choices.

Parameter choices. We set the annual risk-free rate r at 6% and the (expected) EBIT growth rate g to 1% per annum under the risk-neutral measure, similar to those used in Leland (1994), Goldstein, Ju, and Leland (2001), and other contingent-claims capital-structure models. We set the corporate tax rate τ at the current US federal rate of 21%. We leave aside personal taxation and other tax code complications for simplicity.

As in the rare-disaster literature (e.g., Barro, 2006, Gabaix, 2012, and Wachter, 2013), we assume that the cumulative distribution function $F(Z)$ for the recovery fraction Z follows a power law (e.g., Luttmer, 2007 and Gabaix, 2009): $F(Z) = Z^{\beta}$ for $0 \le Z \le 1$. The lower is the exponent β, the more fat-tailed is the distribution of $(1 - Z)$.

Following Almeida and Philippon (2007), we interpret our model guided quantitative analysis on a risk-adjusted basis, i.e., under the risk-neutral measure. To do so, we first calibrate the EBIT process under the physical measure and then incorporate risk premia. First, we calibrate the diffusion volatility parameter σ, and the two jump parameters (β and $\lambda^{\mathbb{P}}$) under the physical measure, by targeting the model-implied variance, skewness, and excess kurtosis of the annual EBIT logarithmic growth rate to respectively 0.18, -0.08, and 0.09, using COMPUSTAT.[21] Doing so yields a diffusion volatility (σ) of 41%, a jump arrival rate ($\lambda^{\mathbb{P}}$) of 0.8 per annum, and an expected (percentage) EBIT loss caused by a jump of $(1-\mathbb{E}(Z)) = 1/(\beta+1) = 10\%$ (with $\beta = 9$).

We incorporate a jump risk premium by assuming that the jump arrival rate under the risk-neutral measure, λ, is 1.5 times as large as $\lambda^{\mathbb{P}}$: $\lambda = 1.5\lambda^{\mathbb{P}} = 1.2$ which implies a risk-adjusted expected loss of $\lambda(1 - \mathbb{E}(Z)) = 12\%$, in line with estimates in the credit-risk literature (Huang and Huang, 2012).

For the equity issuance costs, we set $h_0 = 0.62$ and $h_1 = 0.09$, which are in line with estimates and values used in the literature (Hennessy and Whited, 2007; Eckbo, Masulis and Norli, 2007; and Bolton et al., 2011). As in Huang and Huang (2012), we target the recovery value for creditors when the firm defaults to 51% of the debt face value, which implies a liquidation recovery parameter of $\ell = 5.4$, within the range of estimates in the literature (see., e.g., Lian and Ma, 2021).

[21] The dataset is Compustat North American Fundamentals for publicly traded firms in North America, which contains balance sheets reported annually by companies between 1970 and 2020. We drop firms with missing total assets, negative sales, negative cash and short-term investments, negative capital expenditures, negative debt in current liabilities, negative depreciation and amortization, negative R&D expenses, missing SIC, zero common shares outstanding, or negative common equity. We focus on firms incorporated in the United States. Financial firms (SIC code 6000-6999) and utility firms (SIC codes 4900-4999) are excluded from the sample. The final sample comprises 91,657 firm-year observations. We calculate the logarithmic EBIT growth rates and winsorize at 5% on both sides.

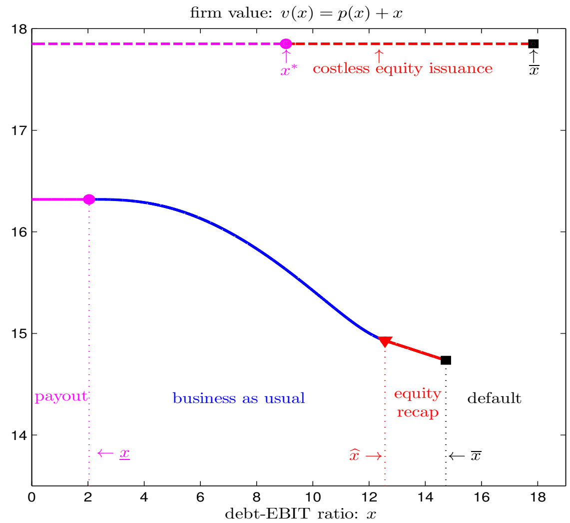

Figure 3: Firm value $v(x)$. This figure plots firm value $v(x) = p(x) + x$ when external equity is costly (the solid line). The optimal target debt-EBIT ratio is $\underline{x} = 2.04$ where market leverage is $ML(\underline{x}) = \underline{x}/v(\underline{x}) = 12\%$. The optimal proactive deleveraging threshold is $\widehat{x} = 12.57$ where market leverage is $ML(\widehat{x}) = \widehat{x}/v(\widehat{x}) = 84\%$. In the business-as-usual region where $x \in [\underline{x}, \widehat{x})$, the firm retains earnings and uses debt as the marginal source of financing. For $x \in [\widehat{x}, \overline{x}]$ where $\overline{x} = 14.74$ is the equilibrium debt capacity, the firm proactively issues equity to deleverage its balance sheet and return to the business-as-usual region. For $x > \overline{x}$, the firm defaults so that $v(x) = \ell = 5.35$. With costless equity issuance, the firm targets its x at $x^* = 9.05$ and defaults if $x > \overline{x} = 17.85$. As a result, firm value is constant: $v(x) = \overline{x}$ for $x \leq \overline{x}$ (the horizontal dashed line), but $v(x) = \ell$ for $x > \overline{x}$.

We report the numerical solution for firm value $v(x)$ and $v'(x)$ rather than $p(x)$ and $p'(x)$. This does not involve any loss of information as $v(x) = p(x) + x$. The figures are slightly more reader friendly when we plot $v(x)$, and this facilitates the comparison with the costless equity issuance case, as we show in Figure 3.

Firm value $v(x)$ and net marginal cost of debt $-v'(x)$. In Figure 3 we plot firm value $v(x) = p(x) + x$, which has three segments defined by three endogenously determined thresholds of x: i.) the optimal target debt-to-EBIT ratio $\underline{x}$ (the magenta dot); ii.) the optimal equity-issuance boundary $\widehat{x}$ (the red triangle); and iii.) the optimal default boundary $\overline{x}$ (the black square). To the left of the magenta dot, firm value is constant: $v(x) = 16.32$ (the magenta horizontal line). This is because shareholders can always increase firm leverage at no cost by moving x to $\underline{x}$, leaving firm value invariant with respect to x for all $x \leq \underline{x}$. Between the magenta dot and the red triangle, the firm is in the business-as-usual region in which firm value $v(x)$ is nonlinear and decreasing in x, reflecting its expected future equity issuance costs. Between the red triangle and the black

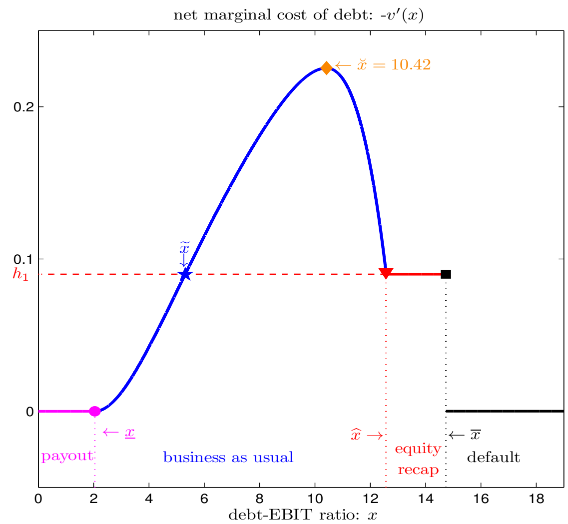

Figure 4: Net marginal cost of debt: $-v'(x)$. This figure shows that $-v'(x)$ is increasing when $x \in (\underline{x}, \breve{x}) = (2.04, 10.42)$ and decreasing when $x \in (\breve{x}, \widehat{x}) = (10.42, 12.57)$, implying that $v(x)$ is concave when $x \in (2.04, 10.42)$ and convex when $x \in (10.42, 12.57)$ region. At the inflection point $\breve{x} = 10.42$ where $v''(x) = 0$, market leverage is $ML(\breve{x}) = \breve{x}/v(\breve{x}) = 67.99\%$. For $x \in (\widehat{x}, \overline{x}) = (12.57, 14.74)$, $-v'(x) = h_1 = 0.09$ as the firm incurs a fixed equity-issuance cost (h_0) and a marginal equity-issuance cost (h_1) to raise $m = x - \widetilde{x}$ in net so that its x returns to the optimal equity recapitalization target: $\widetilde{x} = 5.31$ where market leverage is $ML(\widetilde{x}) = \widetilde{x}/v(\widetilde{x}) = 33\%$. For $x > \overline{x} = 14.74$, the firm defaults.

square, the firm proactively issues equity to deleverage its balance sheet. To the right of the black square, the firm is no longer solvent and defaults.

Note that when external equity issuance is costless, the debt limit is $\overline{x} = 17.85$ and firm value equals 17.85 for $x \leq \overline{x} = 17.85$, both of which are higher than when external equity is costly (see Appendix F.4 for solution details for the costless equity issuance case). The gap between the solid line and the dashed horizontal line measures the firm value gap, which increases with x, as expected future equity issuance costs are higher the higher is x.

In Figure 4, we plot the firm's marginal (opportunity) cost of debt,[22] $-v'(x)$, which shows that $-v'(x)$ is non-monotonic. This non-monotonicity clearly shows that firm value $v(x)$ can be concave, convex, or linear depending on the level of leverage.

In sum, firm value $v(x)$ attains the maximum at $\underline{x} = 2.04$. For all $x > \underline{x} = 2.04$, the firm is financially constrained. If the firm could somehow reduce its debt by a dollar (say, e.g., via

[22] Recall that debt issuance costs are set to zero. The marginal cost of debt here reflects the higher expected future equity issuance costs.

retained earnings), the firm would be strictly better off, as equity value would increase by more than a dollar: $-p'(x) > 1$, which follows from $-v'(x) > 0$ and $p(x) = v(x) - x$.

4.1 Optimal Target Debt-EBIT Ratio $\underline{x}$ and Payouts when $x < \underline{x}$

Because debt issuance is costless and interest payments are tax deductible, the firm starts at $t = 0$ by setting its initial leverage x_0 at $\underline{x} = 2.04$ where market leverage is 12.49%. We refer to $\underline{x}$ as the (optimal) 'target leverage.' With a tax rate of $\tau = 21\%$, the tax benefit is sizeable. Nonetheless, the target market leverage is only 12.49% consistent with the recent evidence on firm capital structures. In Section 5, we show that market leverage remains low even when we increase the tax rate to 30% or an even higher level.

Whenever x_t falls below $\underline{x} = 2.04$ (e.g., due to earnings retention), the firm pays out $\underline{x} - x_t$ (per unit of EBIT) in dividends by increasing x_t to the target $\underline{x} = 2.04$. This strategy is feasible and optimal because debt is costless to issue, so that $v'(x) = 0$ for $x \leq \underline{x}$. But what if firm performance deteriorates so that x_t is larger than $\underline{x}$?

4.2 Business as Usual: $x \in [\underline{x}, \widehat{x})$

In this region, the firm optimally manages its leverage solely by relying on earnings retention and debt financing. Although leverage is, in effect, on auto-pilot, the firm's risk taking behavior can be quite different depending on whether leverage is low or moderate.

4.2.1 Optimality of Passive Leverage Management

When $x > \underline{x} = 2.04$, the firm optimally retains all its earnings and lets x_t drift in response to realized EBIT shocks after making its tax and debt interest payments. This seemingly passive leverage management policy is optimal for a wide range of debt-EBIT ratios: $x \in (\underline{x}, \widehat{x}) = (2.04, 12.57)$, where market leverage ranges from 12% to 84%. That is, any level of market leverage within this wide range of $(12\%, 84\%)$ may be optimal and depending on the firm's past earnings and decisions. Our model predicts a negative relation between positive earnings shocks and leverage, which is related to the leverage-profitability puzzle.[23]

[23] See Titman and Wessels (1988) and Rajan and Zingales (1995) among others, on the leverage-profitability puzzle. Denis and McKeon (2012) find that the evolution of a firm's leverage depends on whether or not it produces a

Our model generates a *target zone* for market leverage and predictions consistent with the evidence in the literature.[24] For example, DeAngelo and Roll (2015) state that "Target-leverage models that place little or no weight on maintaining a particular ratio do a good job replicating the substantial instability of the actual leverage.

These predictions significantly differ from those of other target leverage models, e.g., Goldstein, Ju, and Leland (2001) and Strebulaev (2007). In these models, equity issuance and dividend payouts are a frequently used margin to take the firm to its leverage target. In contrast, in our model leverage is implied by the firm's earnings retention and debt rollover policies. In the target zone, the firm issues no external equity and pays no dividends, because external equity is costly and the firm is financially constrained with x being higher than the 'optimal target' $\underline{x}$. In essence, our mechanism is the opposite of those in dynamic capital structure models without equity issuance costs.

Note that even a small equity issuance cost can induce a large target zone $(\underline{x}, \widehat{x})$, similar to the prediction in menu cost models in macroeconomics (see e.g., Mankiw, 1985). As an example, in Appendix F.2, we show that even with no fixed cost and only a 9% marginal issuance cost, this leverage-target-zone region is wide: $(\underline{x}, \widehat{x}) = (3.11, 10.57)$.

Finally, we note that in the target zone financing via an outstanding credit facility is equivalent to rolling over the firm's debt balance by continuously retiring old and issuing new short-term debt. This is because under these two financing strategies, the credit markets demand the same credit spread for a given level of leverage in equilibrium.

4.2.2 Endogenous Risk Aversion ($x \in (\underline{x}, \breve{x})$) versus Risk Seeking ($x \in (\breve{x}, \widehat{x})$)

Figure 4 shows that the net marginal cost of debt, $-v'(x)$, is not monotonic: it increases with x in the region $(\underline{x}, \breve{x}) = (2.04, 10.42)$, reaching a maximum value of $-v'(\breve{x}) = 0.225$ at $\breve{x} = 10.42$, and then decreases in the region $(\breve{x}, \widehat{x}) = (10.42, 12.57)$. That is, when leverage is low to moderate ($x \in (2.04, 10.42)$), the firm is endogenously risk averse ($v''(x) < 0$). The firm seeks to collect tax

financial surplus. DeAngelo, Gonçalves, and Stulz (2018) show that firms pay down their debt when they receive a positive earnings shock and increase their debt when they have no choice to do otherwise.

[24] See, e.g., Graham and Harvey (2001), Fama and French (2002, 2005), Leary and Roberts (2005), and Lemmon, Roberts, and Zender (2008).

benefits, incurring low expected equity issuance costs, since it has an abundant financial slack (which is measured by the distance between its current x and the debt limit $\overline{x} = 14.74$).

However, when leverage is moderate: $x \in (\breve{x}, \widehat{x}) = (10.42, 12.57)$, the firm becomes a risk seeker ($v''(x) > 0$). That is, the firm would be better off if it could increase the volatility of x even when debt is fairly priced.

This gambling for resurrection when leverage is moderate is caused by the fixed equity issuance cost. The intuition is as follows. Increasing earnings volatility enhances firm value $v(x)$ (and thus equity value $p(x) = v(x) - x$) because the gain from entering into the concave $v(x)$ region sooner outweighs the loss from entering into the deleveraging (or even the default) region sooner. The mechanism for risk seeking in our model is different from that in Jensen and Meckling (1976), which is based on a wealth transfer effect from debtholders to equityholders. In our model, debt is short term and is fairly priced, therefore there is no wealth transfer from risk shifting as in Jensen and Meckling (1976).

4.2.3 Dynamics of x_t and Equilibrium Credit Spread $\eta(x_t)$

Figure 5 plots the equilibrium credit spread and leverage drift.[25] Panel A shows that the credit spread $\eta(x)$ is close to zero when leverage is low (or moderately low). For example, even when $x = 6.12$, three times the level of the 'optimal target' $\underline{x}$, the credit spread is only three basis points (0.03%). But credit spreads are high and become sensitive to changes in leverage when x is moderate (to moderately high), e.g., in the neighborhood of $\breve{x}$, where $v(x)$ inflects and market leverage is 68%. For example, when x increases from $\breve{x} = 10.42$ by 20% to the deleveraging threshold $\widehat{x} = 12.57$, the credit spread increases by more than 5.4 times from an already high level of 3.57% to a whopping 19.32% when market leverage is 84%.

How can credit spreads increase so quickly? The reason is that interest payments and downward jump shocks can make debt quickly outgrow earnings, causing x_t to increase at a fast rate, and posing a significant default risk for the firm. An empirically plausible jump risk premium further enhances the cost of financial distress (Almeida and Philippon, 2007). Together, large earnings

[25] We focus our analysis of the credit spread $\eta(x)$ and the drift $g_x(x)$ in the business-as-usual region, because a firm in the payout or the deleveraging region would immediately leave that region and enter into the business-as-usual region. Mathematically, this is because equity payout is a singular control and equity issuance is an impulse control, both of which involve 'jumpy' decisions.

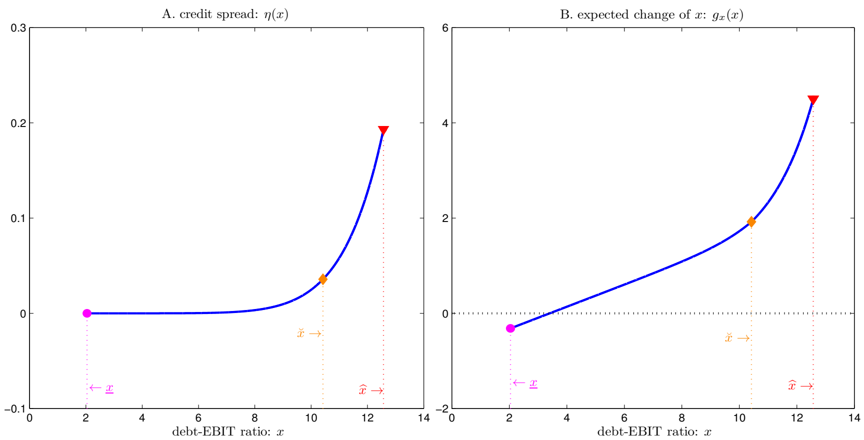

Figure 5: Equilibrium credit spread, $\eta(x)$, and expected change of x, $g_x(x)$, in the business-as-usual region: $x \in [\underline{x}, \widehat{x}) = [2.04, 12.57)$.

shocks and the associated jump risk premium induce a financially constrained firm facing costly external equity to be prudent by targeting low leverage.

Panel B of Figure 5 plots the drift of x, $g_x(x)$, which provides additional insight into the mechanism discussed above. Even under moderate leverage, the firm could find itself in a debt spiral situation. For example, at $\breve{x} = 10.42$ where $v''(\,\cdot\,) = 0$, x_t drifts upward at a fast rate of $g_x(\breve{x}) = 1.92$, and the debt balance increases at the high equilibrium credit spread of 3.57% per annum. Thus the firm is likely to lose control over leverage. Yet, this policy of solely relying on retained earnings and debt rollover to finance its operations is still optimal, because the alternative of tapping external equity to proactively deleverage is too costly. When x is low, the firm retains earnings, expecting x_t to revert to the target $\underline{x}$ where $v(x)$ is maximized. This is why $g_x(x)$ is negative for x near the payout boundary $\underline{x}$ and why $\underline{x}$ is the optimal target.

How high does leverage have to go before the firm taps costly external equity to deleverage for long-term survival? This is the question underlying the determination of the upper boundary of the business-as-usual region: the equity recapitalization threshold $\widehat{x}$. To answer this question, we turn to the third region from the left in Figures 3 and 4.

4.3 Optimal Deleveraging via Equity Recapitalization: From $x \in [\widehat{x}, \overline{x}]$ to $\widetilde{x}$

Figure 3 shows that whenever x exceeds a cutoff $\widehat{x} = 12.57$ where market leverage is 84%, the firm proactively deleverages, if solvent, because the firm is not willing to tolerate a market leverage higher than 84%. This prediction is consistent with the survey evidence of CFOs, who have consistently responded that they attach great importance to keeping leverage within limits to maintain their credit ratings (Graham and Harvey, 2001).[26] In sum, proactively deleveraging the balance sheet before its equity value drops to zero is a unique prediction of our model because of jumps. Absent jumps, the firm always defers costly external equity issuance until it runs out of other options.

What determines the upper boundary for the equity recapitalization region: $\overline{x}$? In other words, how high can x be for a firm to remain solvent? We show that $\overline{x}$ is determined by the firm's ability to issue external equity to bring its debt back to the 'recapitalization target': $\widetilde{x}$. If x exceeds the threshold $\overline{x}$, the firm has to default as it is no longer able to refinance its debt.[27] Intuitively, the most that the firm can offer to new shareholders is the firm's entire equity. This pins down the firm's debt limit $\overline{x}$. In our numerical example, $\overline{x} = 14.74$. That is, the firm's debt limit is just under 15 times expected earnings (before interest and taxes).

In sum, when $x_t \in [\widehat{x}, \overline{x}] = [12.57, 14.74]$, market leverage is in the $[84\%, 100\%]$ range and the firm proactively taps costly external equity to bring its debt-to-EBIT ratio down to the *recapitalization target* $\widetilde{x} = 5.31$ (the blue solid star in Figure 4). Since the firm issues equity at a constant marginal cost of $-v'(x) = h_1 = 0.09$ for $x_t \in [\widehat{x}, \overline{x}] = [12.57, 14.74]$, firm value $v(x)$ is linearly decreasing in this recapitalization region (see the downward sloping line segment in Figure 3) and the recapitalization target $\widetilde{x}$ is constant, independent of the pre-equity-issuance level of x. Note that the 'recapitalization target' of market leverage is only 33%.

4.3.1 Equity Dilution Costs

How much equity does the firm have to issue and how costly is this equity recapitalization for shareholders? We quantify the size of equity issuance and the costs of equity dilution in Appendix D and refer readers there for details. The main takeaway is that the dilution of original shareholders' ownership is substantial.

[26] The proactive deleveraging prediction is consistent with the findings of Fama and French (2005), DeAngelo, DeAngelo, and Stulz (2010), and DeAngelo, Gonçalves, and Stulz (2018): Firms tap equity markets before exhausting their financing capacity. A firm is much less likely to issue debt if doing so increases the likelihood of its credit rating being downgraded (Kisgen, 2006).

[27] Unlike most costly external financing models. e.g., BCW, debt capacity $\overline{x}$ is endogenous in our model.

4.3.2 Two Targets: Optimal Target $\underline{x}$ and Recapitalization Target $\widetilde{x}$

Although the aim of recapitalization is to reduce leverage, it is suboptimal to bring market leverage (ML) all the way down to the optimal target leverage of 12% where $x = \underline{x}$. Instead the firm issues external equity to bring x down to the optimal recapitalization target of $\widetilde{x} = 5.31$ where market leverage is 33% (see Figure 4).

In general, there are two leverage targets in our model: i) optimal target leverage ($\underline{x}$) and ii) the equity recapitalization target ($\widetilde{x}$). This is because the marginal source of financing that the firm uses to reach these two targets is different: debt for the optimal target leverage ($\underline{x}$) and external equity for the recapitalization target ($\widetilde{x}$). These two targets are different as long as the firm incurs a strictly positive proportional equity issuance cost ($h_1 > 0$). Since $p(x)$ is concave between these two targets, it follows from the optimality conditions (13)-(14) for $\underline{x}$ and (24) for $\widetilde{x}$ that the equity recapitalization target $\widetilde{x}$ is higher than the leverage target $\underline{x}$.[28]

5 What Determines Leverage?

Two surveys of CFOs of large US listed companies that are about twenty years apart, Graham and Harvey (2001) and Graham (2022) consistently find that CFOs list financial flexibility, the firm's credit rating, and earnings risk as the top three concerns that influence their debt policy decisions. The predictions of our model line up well with these three factors, as we show in this section through several comparative statics properties of market leverage with respect to: 1) equity issuance costs and, 2) the jump risk (λ and β). Finally, we show that the key parameters in the standard tradeoff theory, tax rate τ and especially the liquidation recovery value parameter ℓ, have limited quantitative effects on leverage, also consistent with findings of the two surveys mentioned above.

5.1 Leverage Decreases with Equity Issuance Costs

A static theory would predict that when the cost of equity increases firms rely more on debt financing. But the opposite is true in a dynamic setting, because by keeping debt low the firm can

[28] In Appendix F.2, we show that the two targets are the same in the special case where $h_1 = 0$: $\widetilde{x} = \underline{x}$.

Table 1: EFFECTS OF EQUITY ISSUANCE COSTS ON MARKET LEVERAGE ML. The last row reports the solution when equity issuance is so costly that the firm never issues equity.

	Mean	Std. dev.	5%	50%	95%
baseline	**0.230**	0.123	0.129	0.182	0.497
$h_0 = 0$	0.341	0.127	0.195	0.305	0.590
$h_0 = 2$	0.161	0.101	0.096	0.122	0.368
$h_1 = 0$	0.306	0.130	0.182	0.261	0.587
$h_1 = 0.3$	0.163	0.107	0.094	0.120	0.396
high h_0 or h_1	0.058	0.021	0.049	0.053	0.083

avoid having to turn to more costly equity markets in the future. That is, in a dynamic setting, preserving financial flexibility is more important when equity issuance becomes more costly. This is achieved by keeping debt low. This fundamental insight is reflected in the comparative statics results reported in Table 1. First, it shows that as we increase equity issuance costs (h_0 or h_1), leverage *decreases*. This seems counterintuitive as a standard substitution argument would suggest that the firm should use more debt if equity becomes more costly. However, this reasoning is incomplete as it misses an important dynamic dimension. As external equity becomes more costly to issue, the firm faces a higher cost of reducing leverage down the road through an equity recapitalization, which in turn discourages its *ex ante* debt issuance. That is, anticipating the high cost of exiting from a high-leverage state in the future makes the firm more prudent with leverage the higher the equity issuance cost.

Table 1 also reveals that our quantitative predictions on leverage are empirically plausible. Namely, average leverage is low, a prediction that has been a key challenge for widely used contingent-claim capital structure models, e.g., Leland (1994). Moreover, our low leverage prediction holds for a wide range of equity issuance cost parameter values. In our baseline ($h_0 = 0.62$ and $h_1 = 9\%$), the average market leverage is 23% and median leverage is 18%. This low leverage prediction is in line with an average leverage of 20-25% for US data for listed companies (see, e.g., Graham, 2001 and Strebulaev and Whited, 2012.) It is important to note that absent equity issuance costs, market leverage would have been constant and much higher at 51%.

Table 1 also illustrates the dynamics of market leverage when the firm is shut out of equity markets (either high h_0 or h_1). In this extreme situation median leverage is close to zero (6%). Since the firm does not have an option to recapitalize, its survival is predicated on leverage staying out of the debt spiral zone. This is achieved by sticking to very conservative leverage and dividend policies.

5.2 Effects of Earnings Jump Risk

Overall, the effects of disaster jump risks on market leverage are significant. The reason is that the firm has few options in adjusting its leverage in response to large downward jump shocks. Echoing our findings for the effect of equity issuance costs, on average market leverage is low for a wide range of jump parameter values. Also consistent with evidence, default is (locally) unpredictable and the credit spread for bonds (with very short maturity) is not zero, both of which are opposite of predictions by both diffusion-based capital structure models following Leland (1994) and dynamic investment models, e.g., Hennessy and Whited (2007). In sum, by combining plausible equity issuance costs and jump risks, our model is able to generate plausible predictions on leverage and equilibrium credit spreads.

5.3 Limited Tax and Financial Distress Costs Effects on Leverage

How much do changes in the corporate tax and bankruptcy costs affect corporate leverage in our model? Overall, we find that changes in the corporate tax rate have some, but very limited effects, on market leverage; a much smaller effect than in the contingent-claims capital structure literature following Leland (1994). For example, when the tax rate τ is increased from 21% to 30%, median market leverage increases from 18% to only 22%. These limited responses of corporate leverage policies to significant changes in corporate tax rates are in line with the evidence (see e.g., Graham, 2000). Changes in the default recovery have an even smaller effect on leverage. In Appendix D, we provide more detailed analyses.

Revisiting Miller (1977)'s famous "rabbit" versus "horse" analogy, we see that neither tax rates nor distress costs are horses in our model once we take external equity financing costs into account.

6 Two Extensions of Baseline Model in Section 2

In this section we analyze two extensions of the baseline model of Section 2. In subsection 6.1, the firm not only borrows but also saves, depending on the history of shocks. In subsection 6.2, the firm can hedge and/or seek additional risk exposure by trading a risky asset. See Appendix G for solution details for these two extensions.

6.1 A Model with both Cash-holding and Debt Regions

In this subsection, we model the firm's (risk-adjusted) EBIT process over dt as

$$Y_{t-}\big(dt + \nu d\mathcal{Z}_t\big), \tag{28}$$

where Y_t is the jump-diffusion process given by (1), $\mathcal{Z}_t$ is a standard Brownian motion, and ν is a volatility parameter for the $d\mathcal{Z}_t$ shock. We add to the locally deterministic EBIT process in Section 2, $Y_{t-}dt$, a shock of $\nu Y_{t-}d\mathcal{Z}_t$, which has the effect of increasing the firm's precautionary savings, inducing it to hold cash (after paying off its debt).

In panel A of Figure 6, we show that the firm has an optimal *target cash-EBIT ratio* of 2.21 (see $\underline{x} = -2.21$ on the solid line) rather than an optimal target debt-EBIT ratio of 2.04 (see Appendix G for parameter values). Now that EBIT is locally risky, the firm delays its payout and indeed is willing to save even given a tax disadvantage of savings. Making earnings locally stochastic increases the firm's precautionary savings demand, lowers firm value, and generates a cash holding region.

6.2 A Model with Dynamic Hedging and Risk Taking

What if the firm can trade a risky asset (e.g., futures) to manage its risk exposure? Under frictionless futures markets, the firm's profits (losses) from its futures position Ω is given by $\Omega_{t-}d\mathcal{Z}_t$, where Ω_{t-} is the futures position and $\mathcal{Z}_t$ is a standard Brownian motion under the risk-neutral measure (Duffie, 2001). Let $\omega_{t-} = \Omega_{t-}/Y_{t-}$. We assume that the constraint on the firm's futures risk exposure is given by $|\Omega_{t-}| \le \overline{\omega}\, Y_{t-}$ or equivalently $|\omega_{t-}| \le \overline{\omega}$ for all $t \ge 0$, where $\overline{\omega} \ge 0$ measures the tightness of this constraint.

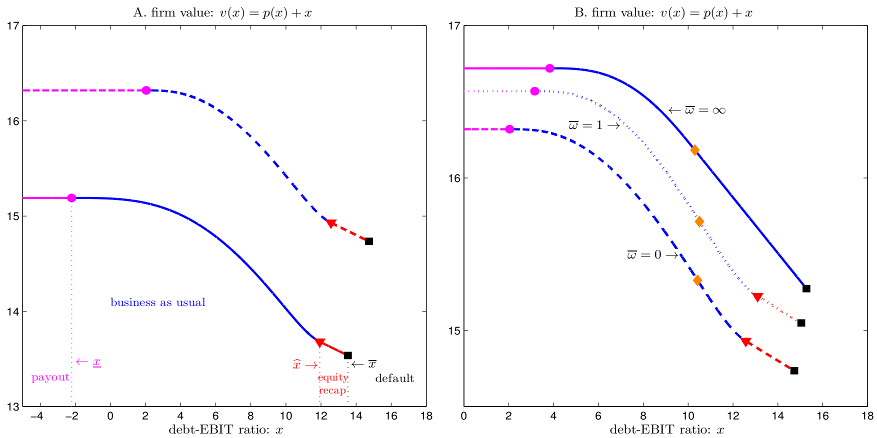

Figure 6: Panels A and B plot firm value $v(x)$ in a model with a cash-holding region (in subsection 6.1) and a model with hedging/risk taking opportunities (in subsection 6.2), respectively.

In panel B of Figure 6, we show that as we relax the constraint $|\omega_{t-}| \le \overline{\omega}$, the firm can manage its risk exposure more effectively, issues more debt, and pays dividends sooner, so that the degree of firm value convexity (in the high-leverage region) decreases. In the limit, as its risk exposure (ω_{t-}) becomes unconstrained ($\overline{\omega} \to \infty$), the firm randomizes over x so much that $v(x)$ becomes (weakly) globally concave. In sum, using the risky asset to randomize is a more efficient way to manage leverage dynamics and allows the firm to do away with costly equity issuance in the limit.

7 Growth Options, Capital Accumulation, and Leverage

Corporate investments feature both small incremental adjustments to capital stock (the intensive margin) and large lumpy 'growth option' increments (the extensive margin).[29] In this section, we model both types of investment decisions by extending our EBIT model of Section 2 to incorporate (1.) a stochastic growth option arrival and (2.) convex capital adjustment costs. This extended model reveals non-monotonic and nonlinear effects of leverage on the exercise of

[29] Models that analyze the effects of financial frictions on growth options include Philippon and Sannikov (2007), Hugonnier, Malamud, and Morellec (2015), Malenko (2019), and Bolton, Wang, and Yang (2019). Philippon and Sannikov (2007) analyze the optimal timing and financing of large, irreversible investments in the presence of moral hazard. Hugonnier, Malamud, and Morellec (2015) analyze the effect of capital supply uncertainty on lumpy growth option exercising. Malenko (2019) analyzes dynamic capital budgeting when the manager with empire-building preferences privately observes the arrival and properties of growth options. Bolton, Wang, and Yang (2019) incorporate costly external equity into a classic real-option setting of McDonald and Siegel (1986) and Dixit and Pindyck (1994).

growth options and capital accumulation, generating new predictions on the joint dynamics of investment, leverage, equity issuance, asset sales, and payouts.

7.1 Model with Capital Accumulation

As in our baseline model, we directly work under the risk-neutral measure. A firm starts off at $t = 0$ with an asset in place (capital). First, to model the intensive margin of day-to-day capital adjustments, we use a q theory of investment. Let K and I denote the capital stock and gross investment, respectively.

Second, we introduce the intensive margin of corporate investment. A growth option arrives at stochastic time $T^{\mathcal{G}}$ at a constant rate of $\lambda_{\mathcal{G}} > 0$ per annum. Upon its arrival, the firm can either exercise the option to permanently enhance its productivity or bypass this option. Exercising the option enhances the firm's productivity but involves a one-time upfront cost: $\phi K_{T^{\mathcal{G}}}$, where $\phi > 0$. By making the investment cost proportional to $K_{T^{\mathcal{G}}}$, we ensure that the value of this growth option does not grow out of this upgrade cost, so that the benefit and cost of exercising the growth option are comparable.[30]

At any time t, the firm is in one of the three phases: $s_t \in \{0, 1, 2\}$. The firm's unlevered free cash flow net of investment in phase s_t is given by:

$$Y_t = A_{s_t} K_t - I_t \,, \tag{29}$$

where A_{s_t} denotes the firm's productivity in phase s_t. Before the growth option arrives ($t < T^{\mathcal{G}}$), it is in the growth phase: $s_t = 0$. After the arrival ($t \geq T^{\mathcal{G}}$), the firm is in one of the two mature phases. If the growth option is exercised at $T^{\mathcal{G}}$, the firm is in the $s_t = 2$ phase with productivity A_2 larger than A_0. If the growth option is bypassed at $T^{\mathcal{G}}$, the firm is in the $s_t = 1$ phase and its productivity is unchanged: $A_1 = A_0$.

In all three phases the capital stock evolves according to:

$$dK_t = \Psi(I_{t-}, K_{t-})dt + \sigma_K K_{t-} d\mathcal{B}_t^K - (1 - Z) K_{t-} d\mathcal{J}_t^K \,, \tag{30}$$

where $\sigma_K > 0$ is the diffusion-volatility parameter, and $\mathcal{B}^K$ is a standard Brownian motion. These continuous shocks can be thought of as stochastic capital depreciation shocks as in Barro (2006).

[30] This is a common assumption in the investment literature, see, e.g., Cooper and Haltiwanger (2006). If the fixed investment cost is independent of firm size, it will not matter when firms become sufficiently large in the long run.

Following Lucas and Prescott (1971), Hayashi (1982), and Jermann (1998), we assume that $\Psi(I,K)$ is homogeneous of degree one in I and K:

$$\Psi(I,K) = \psi(i) \cdot K, \tag{31}$$

where $i = I/K$, $\psi'(i) > 0$, and $\psi''(i) < 0$.

The third term in (30) describes the risk with respect to discrete downward jumps in the level of capital stock K the firm is exposed to, where $\mathcal{J}^K$ is a jump process with a constant arrival rate $\lambda_K > 0$. If a jump occurs at time t, so that $d\mathcal{J}_t^K = 1$, the capital stock drops from K_{t-} to $K_t = ZK_{t-}$. As before, $F(Z)$ denotes a well-behaved cumulative distribution function for $Z \in [0,1]$. At the moment of default $T^{\mathcal{D}}$, creditors receive the firm's liquidation value $L_{T^{\mathcal{D}}} = \ell_K K_{T^{\mathcal{D}}}$, where $\ell_K > 0$ and shareholders are wiped out. With a low recovery value ℓ_K, default generates deadweight losses.

To preserve our generalized model's homogeneity property, we assume that when issuing equity the firm pays both a fixed cost of $h_0^K K_t \geq 0$ and a constant marginal cost $h_1 \geq 0$ so that the total equity issuance cost is $h_0^K K_t + h_1 M_t$ when the firm raises a net amount M_t. The tax schedule and characterization of the equilibrium credit spread are as in Section 2. In Appendix H, we solve the model using backward induction.[31] For brevity, we focus our discussion in this section on the growth ($s_t = 0$) phase.

7.2 Financing Growth Option and Investment: New Predictions

Anticipating that a valuable growth option may stochastically arrive and the prospect of a large expenditure that could entail a costly equity issuance, how should the firm manage its on-going investment and financing? And under what circumstances is the firm able to pay for the large expenditure from exercising the growth option and if so, in which form: debt or equity? In this subsection, let $x_t = X_t/K_t$ denote book leverage.

The model developed in this section answers these questions. In Figure 7, we plot the firm's growth option exercise decision and smooth capital investment policies, in a setting where financing

[31] As the firm's decisions in the growth phase ($s_t = 0$) depend on its decisions in the two mature phases, firm value is higher in the former than in the latter, even though productivity is the same in the growth phase ($s_t = 0$) and the mature phase ($s_t = 1$) in which the firm did not exercise its growth option.

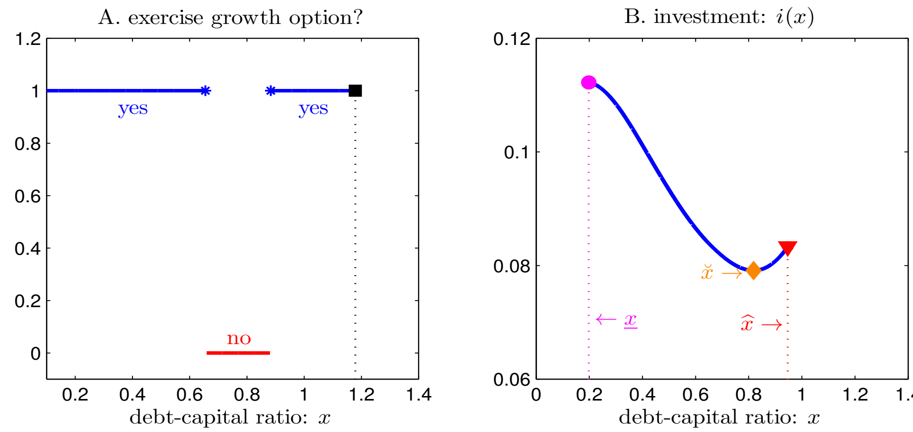

Figure 7: This figure shows that the firm's growth option exercising and investment policies are non-monotonic in book leverage x in the growth phase ($s_t = 0$).

policies critically matter. Both the growth option exercise and investment are *non-monotonic* in leverage.[32] When leverage is high or low, the firm exercises the growth option when it arrives, but bypasses the growth option when leverage is at an intermediate level. What is the intuition for these *non-monotonicity* results?

When leverage is low, the firm can finance the growth option with debt, because the (marginal) cost of debt $-v'(x)$ is low. When leverage is at an intermediate level, the shadow cost of debt $-v'(x)$ is high (see panel B of Figure 8). The firm then bypasses the growth option because financing the growth option expenditure with debt increases leverage too much. The firm cuts its investment more as x increases in this region (see panel B of Figure 7). Bypassing the growth option and underinvesting in capital reflect a significant debt overhang as indicated by a high (shadow) cost of debt: $-v'(x)$.

Finally, when leverage is high, the firm is in a debt-spiral, a dire financial situation. The firm taps costly external equity because there are two benefits of issuing equity: financing the growth option expenditure and at the same time deleveraging the balance sheet, which makes the firm financially more resilient (see the third region in panel A of Figure 7.) Anticipating that its leverage will discretely adjust downward when the firm exercises the growth option, the firm

[32] This non-monotonicity result requires that the growth-option exercising cost and the external equity financing cost are economically comparable. This is the case for Figure 7. Appendix H.6 provides details for the parameter values.)

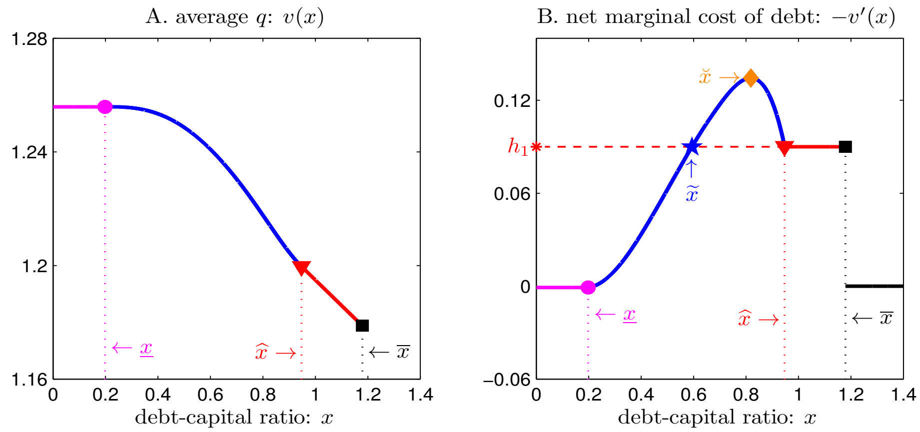

Figure 8: The firm's average q in the growth phase ($s_t = 0$), $v(x)$, is concave in the $x \in (\underline{x}, \widetilde{x})$ sub-region but convex in the $x \in (\widetilde{x}, \widehat{x})$ sub-region of the business-as-usual region. Panel B makes the sign of $v''(x)$ clearly by showing that the marginal cost of debt, $-v'(x)$, is non-monotonic in x.

optimally reduces its underinvestment as x increases in the high x region (see panel B of Figure 7).

In sum, the exercise of growth options and investment have implications on both value creation and prudent debt (and risk) management. When exercising a growth option calls for a lumpy cost, external equity financing can become a key source of financing to avoid excessively high leverage. This is because the exercise of growth options and standard capital accumulation policies are affected by *debt overhang* (Myers, 1977) and *risk shifting* (Jensen and Meckling, 1976) effects, the latter being driven by the interaction between downward jumps and fixed equity issuance costs.

8 Conclusion

Inspired by the concluding remark in Fama and French (2005): *"it is best to regard the (pecking order and tradeoff) models as stable mates, with each having elements of truth that help explain some aspects of financing decisions,"* we develop a tradeoff model for a firm facing equity issuance costs (Myers and Majluf, 1984) and risky cash flows with downward jump risk, to analyze the interconnected dynamics of leverage and credit risk.

The model predicts that default occurs suddenly, and equilibrium credit spreads are stochastic and sizable, even for bonds with very short maturity (consistent with the evidence in Duffie and Singleton, 2003). The model also predicts that the firm begins with a low target leverage and revolves its debt at equilibrium interest rates by letting leverage drift in response to realized earnings. When leverage jumps up in response to an earnings (jump) loss, the firm may proactively deleverage by issuing equity to manage its default risk. This contrasts with the proactive releveraging predictions in the contingent-claim capital structure literature (e.g., Goldstein, Ju, and Leland, 2001 and DeMarzo and He, 2021) and differs significantly from the prediction in dynamic investment models (e.g., Hennessy and Whited, 2007) that firms issue equity only as a last resort.

The main predictions of our model are in line with the criteria summarized in the concluding remark in DeAngelo, Gonçalves, and Stulz (2018) for a credible capital structure theory: (1) most firms proactively deleverage; (2) financial flexibility and the option to borrow are valuable; (3) firms recognize the benefits of internally generated equity through earnings retention; and (4) there are significant interdependencies among leverage, cash balances, and decisions on retention versus payouts, driven by financial flexibility considerations. The model's predictions are also consistent with the survey findings of Graham and Harvey (2001) and Graham (2022).

We also explain why corporate leverage on average is low. Paradoxically, the explanation for the observed low corporate leverage is not that debt is costly, but that equity is costly. One would think that when equity is costly firms would want to rely more on debt. But that is a static intuition. From a dynamic perspective firms seek to delay or avoid costly equity issuance by maintaining financial slack and keeping leverage low today. When equity issuance is very costly, the firm barely takes on any debt.

Appendices

A Markov Perfect Equilibrium (MPE): Definition

We begin by defining the firm's strategy set, which is the set of all Markov strategies. We then proceed to the definition of creditors' beliefs. Finally, we characterize the corporate debt and equity pricing formulae given creditors' beliefs over the firm's Markov strategies.

Definition 1. Markov strategies. *Let S denote a triplet of corporate actions, consisting of the default timing $T^{\mathcal{D}}$, equity issuance N_t, and equity payout U_t:*

$$S = (T^{\mathcal{D}}, \{N_t, U_t; t \in (0, T^{\mathcal{D}})\}).$$

Let $\mathcal{S}$ denote the set of Markov strategies. A strategy $S \in \mathcal{S}$ is Markov if 1.) it is adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ generated by $(\mathcal{B}_t, \mathcal{J}_t)_{t \geq 0}$; 2.) the process X_t, defined by equation (8), has a unique strong solution; and 3.) there exist three sets, $\mathcal{D}$, $\mathcal{N}$, $\mathcal{U}$, in the space of $\Sigma = \{(Y, X) : Y > 0, X \geq 0\}$, with the following properties: a.)

$$T^{\mathcal{D}} = \inf\{t \geq 0 : (Y_t, X_t) \in \mathcal{D}\} \tag{A.1}$$

and b.) for any $t < T^{\mathcal{D}}$,

$$N_t = \int_0^t \mathbf{1}_{(Y_s, X_{s-}) \in \mathcal{N}} dN_s \,, \quad U_t = \int_0^t \mathbf{1}_{(Y_s, X_{s-}) \in \mathcal{U}} dU_s, \tag{A.2}$$

and $\Delta N_t := N_t - N_{t-} \geq 0$ and $\Delta U_t := U_t - U_{t-} \geq 0$ are functions of (Y_t, X_{t-}).

Since dN_t and dU_t only depend on (Y_t, X_{t-}), (8) implies that dX_t also only depends on (Y_t, X_{t-}).

Creditors' beliefs. Let $\widetilde{S}$ denote creditors' beliefs about the firm's strategy, which is also a triplet: $\widetilde{S} = (T^{\widetilde{\mathcal{D}}}, \{\widetilde{N}_t, \widetilde{U}_t; t \in (0, T^{\widetilde{\mathcal{D}}})\})$. Given a Markov strategy S and creditors' belief $\widetilde{S}$, the firm's equity value is given by:

$$\mathcal{P}(Y_t, X_t; S, \widetilde{S}) = \mathbb{E}_t\left[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)} \left(dU_s - dN_s - dH_s\right)\right], \tag{A.3}$$

subject to the accounting identity (8) implied by S, and the credit spreads that depend on $\widetilde{S}$. We next define the set of *feasible* Markov strategies, given creditors' belief $\widetilde{S}$, which is a subset of $\mathcal{S}$, the set of Markov strategies. Let $\Phi(\widetilde{S})$ denote the set of all feasible Markov strategies S for a given belief $\widetilde{S}$.

Definition 2. Feasible Markov strategies. *For a given belief $\widetilde{S}$ with default region $\widetilde{\mathcal{D}}$, a Markov strategy S is feasible, $S \in \Phi(\widetilde{S})$ under the following conditions:*

(i) if there is no jump at t ($d\mathcal{J}_t = 0$ so that $Y_t = Y_{t-}$) and the firm does not default at $t < T^{\mathcal{D}}$, creditors believe that the firm is solvent at t: $(Y_t, X_{t-}) \notin \widetilde{\mathcal{D}}$ and $(Y_t, X_t) \notin \widetilde{\mathcal{D}}$;

(ii) limited liability: $\mathcal{P}(Y, X; S, \widetilde{S}) \geq 0$ for all $(Y, X) \in \Sigma$.

Condition (i) requires that if the firm has been solvent up to time t, creditors expect it to continue to be solvent unless a jump arrives at t ($d\mathcal{J}_t = 1$). Moreover, conditional on being in the $\widetilde{\mathcal{D}}$ set, the firm must either default or issue equity to immediately exit $\widetilde{\mathcal{D}}$.

Next, we pin down the credit spread given creditors' belief $\widetilde{S}$ and the firm's strategy $S \in \Phi(\widetilde{S})$. A necessary condition for creditors to believe that the firm will default is a jump arrival. Therefore, for a small time interval dt, competitive debt pricing implies that the following zero-profit condition for creditors must hold:

$$X_{t-}(1+rdt) = (X_{t-}+C_{t-}dt)\left[1-\lambda\mathbb{E}_{t-}\left(\mathbf{1}_t^{\widetilde{\mathcal{D}}}\right)dt\right] + \mathbb{E}_{t-}\left[\min\{L_t, X_{t-}\}\,\mathbf{1}_t^{\widetilde{\mathcal{D}}}\right]\lambda dt, \tag{A.4}$$

where $\mathbf{1}_t^{\widetilde{\mathcal{D}}} = 1$ if $(Y_t, X_{t-}) \in \widetilde{\mathcal{D}}$ and $\mathbf{1}_t^{\widetilde{\mathcal{D}}} = 0$ otherwise. The first term on the right side of (A.4) equals the product of the total contractual repayment to creditors, $(X_{t-}+C_{t-}dt)$, over dt and the (risk-neutral) probability at time $t-$ that the firm won't default over dt: $\left[1-\lambda\mathbb{E}_{t-}\left(\mathbf{1}_t^{\widetilde{\mathcal{D}}}\right)dt\right]$. The second term gives the creditors' expected payoff if the firm defaults under the risk-neutral measure $\mathbb{Q}$. The left side of (A.4) states that creditors' expected rate of return over dt under $\mathbb{Q}$ must equal the risk-free rate r. Indeed, in equilibrium, the two sides of (A.4) must be equal (Duffie, 2001). Simplifying (A.4) yields:

$$C_{t-} = rX_{t-} + \lambda\mathbb{E}_{t-}\left[\left(X_{t-}-\min\{L_t, X_{t-}\}\right)\mathbf{1}_t^{\widetilde{\mathcal{D}}}\right] = rX_{t-} + \lambda\mathbb{E}_{t-}\left[(X_{t-}-L_t)^+\,\mathbf{1}_t^{\widetilde{\mathcal{D}}}\right], \tag{A.5}$$

where $x^+ = \max\{x, 0\}$. Using $C_{t-} = (r+\eta_{t-})X_{t-}$ given in (3) to rewrite (A.5), we obtain[33]

$$\eta_{t-} = \lambda\mathbb{E}_{t-}\left[\left(1-\frac{L_t}{X_{t-}}\right)^+\mathbf{1}_t^{\widetilde{\mathcal{D}}}\right]. \tag{A.6}$$

Specifically, if $\widetilde{\mathcal{D}} = \{(Y, X) : X/Y > \overline{x}\}$ for some constant $\overline{x} > 0$, then $\eta_{t-} = \eta\big(x_{t-}; \overline{x}\big)$, where

$$\eta\big(x; \overline{x}\big) = \lambda\int_0^{x/\overline{x}}\left(1-\frac{\ell Z}{x}\right)^+ dF(Z), \quad x < \overline{x}. \tag{A.7}$$

Definition 3. Markov perfect equilibrium (MPE). *In an MPE, a feasible Markov strategy $S^* \in \Phi(S^*)$ satisfies the following best-response condition for all $(Y, X) \in \Sigma$:*

$$\mathcal{P}(Y, X; S^*, S^*) \geq \mathcal{P}(Y, X; S, S^*), \quad \forall\, S \in \Phi(S^*). \tag{A.8}$$

Note that, in Definition 3, the credit spreads appearing on both sides of (A.8) are identical and are given by (A.6), where $\widetilde{\mathcal{D}}$ is the default region under the strategy S^*.

B Derivation of the Solution and Summary of Main Results for the Baseline Model in Sections 2 and 3

Here we heuristically solve the baseline model using dynamic programming and then formally state the optimality results. Proofs of these results can be found in Internet Appendix IA.

B.1 Heuristic Derivation of Main Results in Section 3

We first solve the firm's dynamic optimization problem for a given credit spread function $\eta(x)$ and then use the competitive market pricing condition to determine the equilibrium $\eta(x)$ via a fixed-point argument. First, we characterize the optimal debt target $\underline{X}(Y)$ and payout region: $X < \underline{X}(Y)$. Using

[33] Since $\mathcal{D} = \widetilde{\mathcal{D}}$ and $X_{T^{\mathcal{D}}-} \geq L_{T^{\mathcal{D}}}$ in an equilibrium, then we obtain (4).

$P(Y,X)=P(Y,\underline{X}(Y))+\underline{X}(Y)-X$, we have $-P_X(Y,\underline{X}(Y))=1$, which is the smooth-pasting condition.[34] Since $\underline{X}(Y)$ is optimally chosen, the super-contact condition must hold, so that: $P_{XX}(Y,\underline{X}(Y))=0$ (Dumas, 1991). Using the homogeneity property to simplify these three equations, we obtain (12), (13), and (14).

Second, in the business-as-usual region where $\underline{X}(Y)\le X<\widehat{X}(Y)$, the firm retains its earnings and issues debt. Equity value $P(Y,X)$ satisfies:

$$rP(Y,X)=(C+\Theta-Y)\,P_X+\mu YP_Y+\frac{\sigma^2Y^2}{2}P_{YY}+\lambda\mathbb{E}\left[P(ZY,X)-P(Y,X)\right], \tag{B.1}$$

where the $\mathbb{E}[\,\cdot\,]$ operator in (B.1) is evaluated with respect to the distribution $F(\,\cdot\,)$ for the recovery fraction Z. The last term captures the jump effect. When a jump arrives at t: $d\mathcal{J}_t=1$, EBIT falls from Y_{t-} to $Y_t=ZY_{t-}$. Using the homogeneity property, we can simplify (B.1) to the integro-differential equation (21) for $p(x)$ in the business-as-usual region where $\underline{x}\le x<\widehat{x}$.

Third, we analyze the equity-issuance region: $\widehat{X}(Y)\le X\le\overline{X}(Y)$. When $\widehat{X}(Y)<\overline{X}(Y)$, $P(Y,X)=P(Y,\widetilde{X}(Y))-\left[h_0Y+(1+h_1)(X-\widetilde{X}(Y))\right]$ holds. As the net equity issuance $M=X-\widetilde{X}(Y)$ is chosen optimally, $-P_X(Y,X)=-P_X(Y,\widetilde{X}(Y))=(1+h_1)$ must hold.[35] Using the homogeneity property, we obtain (23) and (24). Finally, when $X>\overline{X}(Y)$, the firm defaults and $P(Y,X)=0$, which implies (26). We summarize our four-region solution in the **Summary** paragraph at the end of Section 3.[36]

In next subsection, we state assumptions and establish key results.

B.2 Existence and Characterization of Markov Perfect Equilibrium (MPE)

Assumption 1. *(i) The expected growth rate of the EBIT process under the risk-neutral measure $\mathbb{Q}$, g given in (2), is strictly smaller than the risk-free rate: $g<r$. (ii) Equity value with zero leverage is larger than the liquidation recovery value: $\frac{1-\tau}{r-g}>\ell$.*

Part (i) of Assumption 1 is to ensure that equity value is finite and part (ii) of Assumption 1 requires that the firm's value under all equity financing is larger than its liquidation value.

Let $\mathcal{S}_x$ denote the set of Markov strategies $S=(T^{\mathcal{D}},\{N_t,U_t;t\in(0,T^{\mathcal{D}})\})$ with $T^{\mathcal{D}}=\inf\{t\ge0:X_t/Y_t>\overline{x}\}$ for some $\overline{x}\ge\ell$. We first establish the homogeneity property given creditors' beliefs that the firm's default policy is given by a cutoff threshold $\overline{x}$.

Proposition 1. *Suppose Assumption 1 holds. Given creditors' belief $\widetilde{S}\in\mathcal{S}_x$, let $S^*\in\mathcal{S}_x$ denote a feasible Markov strategy that satisfies $\mathcal{P}(Y,X;S^*,\widetilde{S})\ge\mathcal{P}(Y,X;\widehat{S},\widetilde{S})$ for all $(Y,X)\in\Sigma$ and $\widehat{S}\in\Phi(\widetilde{S})$. Then the firm's equity value, $\widetilde{P}(Y,X)=\mathcal{P}(Y,X;S^*,\widetilde{S})$, satisfies the following homogeneous property:*

$$\widetilde{P}(Y,X)=Y\widetilde{P}(1,X/Y). \tag{B.2}$$

Let $\widetilde{p}(x)=\widetilde{P}(1,x)$ and $v=\inf\{x>0:\widetilde{p}(x)=0\}$. Then $\widetilde{p}(x)=0$ for any $x\ge v$ and $\mathcal{D}=\{(Y,X):X/Y>v\}$ is the default region under the strategy S^*.

[34] Note that the right side of the value-continuity condition $P(Y,X)-P(Y,\underline{X}(Y))=\underline{X}(Y)-X$ is the lump-sum payment to shareholders. Let $\Delta=\underline{X}(Y)-X$. Dividing both sides by Δ and letting $\Delta\to0$, we obtain the smooth-pasting condition.

[35] As $P(Y,X)$ is continuous in X, $P(Y,\widehat{X}(Y)+)=P(Y,\widehat{X}(Y)-)$ where $\widehat{X}(Y)+$ and $\widehat{X}(Y)-$ are the left and right limits of $\widehat{X}(Y)$, respectively. The homogeneity property then implies $p(\widehat{x}+)=p(\widehat{x}-)$.

[36] It is possible that the equity issuance region degenerates to a singleton: $\widehat{x}=\overline{x}$. This happens when equity issuance is sufficiently (but not too) costly. When equity issuance becomes too costly, the firm never issues equity, its debt capacity equals its liquidation value: $\overline{x}=\ell$, and for $x\ge\ell$ equity is worth zero: $p(x)=0$. See Appendix F.1 for details and examples for these two special cases.

Assumption 2. *For any creditors' belief* $\widetilde{S} \in \mathcal{S}_x$*, there exists an optimal Markov strategy* $S \in \Phi(\widetilde{S})$*, such that* $\mathcal{P}(Y, X; S, \widetilde{S}) \geq \mathcal{P}(Y, X; \widehat{S}, \widetilde{S})$ *for all* $(Y, X) \in \Sigma$ *and* $\widehat{S} \in \Phi(\widetilde{S})$*.*

Intuitively speaking, given creditors' beliefs that the firm's default policy is given by a cutoff threshold $\overline{x}$, Assumption 2 requires that the firm's optimization problem has a solution. Under these assumptions, we establish the existence of an MPE that features a cutoff default threshold.

Proposition 2. *Under Assumptions 1-2, there exists an MPE in which the default region is* $\mathcal{D} = \{(Y, X) : X/Y > \overline{x}\}$ *for some* $\overline{x} \geq \ell$*. In equilibrium, the firm's equity value* $P(Y, X)$ *is strictly increasing in* Y *and strictly decreasing in* X *for any* $Y > 0$ *and* $X \in [0, \overline{x}Y]$*.*

Let $p(x; S^*)$ denote the scaled equity value associated with a candidate equilibrium strategy S^*: $p(x; S^*) = \mathcal{P}(1, x; S^*, S^*)$.

Next, we state a verification theorem for an MPE.

Theorem 1. Verification theorem. *Consider a Markov strategy* $S^* \in \mathcal{S}_x$ *that satisfies* (B.2) *and includes a default region* $\mathcal{D}^* = \{(Y, X) : X/Y > \overline{x}\}$ *for some* $\overline{x} \geq \ell$*. Let* $v = \inf\{x > 0 : p(x; S^*) = 0\}$*.* S^* *is an MPE if 1.)* $v = \overline{x}$*, 2.) the function* $P(Y, X) = \mathcal{P}(Y, X; S^*, S^*)$ *satisfies* $P(Y, X) = 0$ *for any* $X \geq \overline{x}Y > 0$*, and 3.) the following variational inequality for* $(Y, X) \in \mathbb{Z} = \{(Y, X) | 0 < Y < \infty,\ 0 \leq X \leq \overline{x}Y\}$ *holds:*

$$\max\{\mathcal{L}P,\ P_X(Y, X) + 1,\ P^N(Y, X) - P(Y, X)\} = 0\,. \tag{B.3}$$

In (B.3)*,* $P^N(Y, X)$ *is defined by*

$$P^N(Y, X) = \sup_{M \geq 0} P(Y, X - M) - h_0 Y - (1 + h_1)M \tag{B.4}$$

and $\mathcal{L}P$ *is given by*

$$\mathcal{L}P = \mu Y P_Y + \big(C + \Theta - Y\big)P_X + \frac{\sigma^2 Y^2}{2} P_{YY} + \lambda \mathbb{E}\left[P(ZY, X) - P(Y, X)\right] - rP, \tag{B.5}$$

where $\Theta = \theta(c)Y$ *is the firm's tax payment (with* $\theta(c)$ *given in (5) and* $c = C/Y$*),* $C = (r + \eta(X/Y; \overline{x}))X$ *is the interest payment, and* $\eta(x; \overline{x})$ *is the credit spread given in* (A.7)*.*

Next, we prove the existence of the best MPE, which is the equilibrium with the highest equity value.

B.3 The Best MPE

Theorem 2. Existence of the best MPE. *Under Assumptions 1-2, there exists an MPE associated with strategy* $S^* \in \mathcal{S}_x$ *that satisfies* $\mathcal{P}(Y, X; S^*, S^*) \geq \mathcal{P}(Y, X; S, S)$ *for all* $(Y, X) \in \Sigma$ *and any MPE associated with strategy* $S \in \mathcal{S}_x$*, where* $\mathcal{S}_x$ *denotes the set of Markov strategies* S *such that* $T^{\mathcal{D}} = \inf\{t \geq 0 : X_t/Y_t > \overline{x}\}$ *for some* $\overline{x} \geq \ell$*.*

Theorem 3. Solving for the best MPE. *Under Assumptions 1-2, for any MPE* $S^* \in \mathcal{S}_x$*, the firm's scaled equity value* $p(x) = p(x; S^*)$ *is the unique bounded viscosity solution to the following variational inequality:*[37]

$$\max\left\{\mathcal{L}^{\overline{x}} p(x),\ 1 + p_x(x),\ p^N(x) - p(x)\right\} = 0, \quad x \in [0, \overline{x}], \tag{B.6}$$

[37] Boundary conditions are $p_x(0) = -1$ and $p(\overline{x}) = 0$. See Barles and Souganidis (1991) and Seydel (2009) for the technique details for the viscosity solution.

where $\overline{x} = \inf\{x > 0 : p(x) = 0\} \geq \ell$ *and*

$$\mathcal{L}^{\overline{x}} p(x) = \left[c(x;\overline{x}) + \theta(c(x;\overline{x})) - 1 - \mu x\right] p_x(x) + \frac{\sigma^2 x^2}{2} p_{xx}(x) + \lambda \mathbb{E}\left[Z p(x/Z) - p(x)\right] - (r - \mu)\, p(x), \tag{B.7}$$

and $p(x) = 0$ *for any* $x \geq \overline{x}$. *In* (B.7), $c(x;\overline{x}) = (r + \eta(x;\overline{x}))x$ *is the interest payment,* $\eta(x;\overline{x})$ *is the credit spread given in* (A.7), *and* $p^N(x) = \max_{m\geq 0}\ p(x-m) - h_0 - (1+h_1)m$. *Denote* $\overline{\mathbb{X}}$ *as the set of default boundaries* $\overline{x}$ *for all MPEs in the set* $\mathcal{S}_x$ *and* $\overline{x}^* = \sup \overline{\mathbb{X}}$. *Then* $\overline{x}^* \leq \frac{1}{r-g}$, *and* $\overline{x}^* \in \overline{\mathbb{X}}$ *is the default boundary of the best equilibrium in the set* $\mathcal{S}_x$.

Using Theorem 3, to solve for the best equilibrium, we only need to find the maximum trigger $\overline{x} \in [\ell, \frac{1}{r-g}]$, such that (B.6) admits a solution $p(x)$ satisfying $p(x) > 0$ for all $x < \overline{x}$.

We provide proofs of Propositions 1-2 and Theorems 1-3 in Internet Appendix IA.

B.4 Change of Measure

Finally, we provide a derivation for the EBIT $\{Y_t\}$ process (1) under the risk-neutral measure $\mathbb{Q}$ starting from the physical measure $\mathbb{P}$ (see Duffie, 2001 for a textbook treatment and Chen, 2010 for an application.) Assume that the EBIT process follows:

$$\frac{dY_t}{Y_{t-}} = \mu^{\mathbb{P}} dt + \sigma d\mathcal{B}_t^{\mathbb{P}} - (1 - Z) d\mathcal{J}_t^{\mathbb{P}}\,, \tag{B.8}$$

where $\mathcal{B}^{\mathbb{P}}$ is a standard Brownian motion and $\mathcal{J}^{\mathbb{P}}$ is a pure jump process with a constant arrival rate $\lambda^{\mathbb{P}}$ under $\mathbb{P}$. Assume further that the SDF $\mathbb{M}_t$ evolves under $\mathbb{P}$ as follows:

$$\frac{d\mathbb{M}_t}{\mathbb{M}_{t-}} = -r dt - \vartheta d\mathcal{Z}_t^{\mathbb{P}} - (1 - e^{\kappa})\left(d\mathcal{J}_t^{\mathbb{P}} - \lambda^{\mathbb{P}} dt\right),\ \mathbb{M}_0 = 1\,, \tag{B.9}$$

where $\mathcal{Z}_t^{\mathbb{P}}$ is a Brownian motion (driving the systematic component of the diffusion shocks), the parameter ϑ measures the market price of this risk, and the parameter κ captures the market price of jump risk. Let ρ denote the (instantaneous) correlation coefficient between the firm's EBIT diffusion shock $d\mathcal{B}_t^{\mathbb{P}}$ and the systematic diffusion shock $d\mathcal{Z}_t^{\mathbb{P}}$.

To ease exposition, we let the recovery distribution Z under the physical measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$ be the same so that the jump risk premium arises from the uncertainty of the jump arrival time. Using the Girsanov theorem, we can show that the firm's risk-adjusted EBIT process under the risk-neutral measure $\mathbb{Q}$ is given by (1) with $\lambda = e^{\kappa}\lambda^{\mathbb{P}}$, $\mu = \mu^{\mathbb{P}} - \rho\vartheta\sigma$, and $d\mathcal{B}_t = d\mathcal{B}_t^{\mathbb{P}} + \rho\vartheta dt$. Note that $\kappa = \ln(\lambda/\lambda^{\mathbb{P}})$ measures the jump risk premium.

C Solution Algorithm

We numerically solve the integro-differential equation for the equity value function, $p(x)$, given in (21) using the following algorithm for the baseline model in which the equity issuance region $x \in (\widehat{x}, \overline{x})$ exists.

1. Start with a sufficiently large region $(\underline{x}, \widehat{x})$ by setting $\underline{x} = 0$ and $\widehat{x} = v^*$, where v^* is the firm's equity value under the costless equity issuance case. We use the superscript (i) to denote the i^{th} iteration value for $p(x)$, $\underline{x}$, $\widehat{x}$: $p^{(i)}(x)$, $\underline{x}^{(i)}$, $\widehat{x}^{(i)}$.

2. For the initial values, we set $\underline{x}^{(1)} = 0$ and $\overline{x}^{(1)} = \widehat{x}^{(1)} = x^*$, as the firm's optimal target leverage $\underline{x}$ is always positive and cannot exceed x^*. We then start with a linear equity value function: $p^{(1)}(x) = p^{(1)}(\underline{x}^{(1)}) + \frac{x-\underline{x}^{(1)}}{\widehat{x}^{(1)}-\underline{x}^{(1)}}\left(p^{(1)}(\widehat{x}^{(1)}) - p^{(1)}(\underline{x}^{(1)})\right)$ for $\underline{x}^{(1)} < x < \widehat{x}^{(1)}$, so that $p^{(1)}(\underline{x}^{(1)}) = v^* - x^*$ and $p^{(1)}(\widehat{x}^{(1)}) = 0$.

3. Use the finite-difference method, e.g., the Matlab function ode45, to solve for $p^{(i)}(x)$ given by the ODE given in (21).

4. Change $p^{(i)}(\underline{x}^{(i)})$ and $p^{(i)}(\widehat{x}^{(i)})$ to repeat step 3 until the boundary conditions (13) and (23) are satisfied.

5. Change $\underline{x}^{(i)}$ and $\widehat{x}^{(i)}$ to repeat step 4 until the boundary conditions (14) and (24) are satisfied.

6. For a given $p^{(i)}(\widehat{x}^{i})$, calculate $p^{(i)}(x)$ in the equity issuance region $x > \widehat{x}$ using (23).

7. Change $\overline{x}^{(i)}$ to repeat steps 3-6 until $p^{(i)}(\overline{x}^{(i)}) = 0$.

D Additional Results of Sections 4 and 5

In this appendix, we provide additional results supporting our analyses in Sections 4 and 5.

D.1 Proactive Deleveraging and Equity Issuance

Figure 9 depicts the leverage dynamics. Consider a firm in the business as usual region, and fix x at 7.9 where market leverage is 50% (the blue solid dot). Suppose that the firm is unexpectedly hit by a jump shock with a fractional loss of $(1 - Z) = 42\%$. Absent recapitalization, market leverage would increase to 92% (the red solid diamond). In response to this disaster earnings shock, the firm immediately issues external equity with a size that is 9.73 times its EBIT, so that it brings x down to the recapitalization target of $\widetilde{x} = 5.31$ where market leverage is only 33%. In Figure 9, the blue upward arrow and the red downward arrow together describe how the firm proactively issues equity to deleverage its balance sheet by bringing x down to the 'recapitalization target' $\widetilde{x} = 5.31$.

Our model's recapitalization prediction is consistent with the evidence in DeAngelo and Roll (2015), who find that "many firms have leverage ratios above 0.500 at some point, but almost no firms keep leverage ratios consistently above 0.500 for long periods of time." They suggest that "something akin to distress costs must encourage rebalancing downward from very high leverage." Our recapitalization prediction sheds new light on their somewhat vague statement: It is optimal for the firm to reduce the likelihood of inefficient liquidation by preemptively recapitalizing its balance sheet as soon as its market leverage exceeds $ML(\widehat{x}) = 84\%$. Shareholders issue new equity out of necessity to deleverage the balance sheet because the alternative is worse: a higher risk of default. This explains why in equilibrium firms rebalance downward from very high leverage even though this is highly dilutive and costly for shareholders.

Let $\delta(x_{t-})$ and $\alpha(x_{t-})$ denote respectively the (dilution) cost $h_0 + h_1 m_t$ and the new shareholders' firm equity ownership, when the firm issues m_t:

$$\delta(x_{t-}) = \frac{h_0 + h_1 m_t}{p(\widetilde{x})} \quad \text{and} \quad \alpha(x_{t-}) = \frac{m_t}{p(\widetilde{x})} + \delta(x_{t-})\,. \tag{D.1}$$

Old shareholders' equity ownership is reduced from 100% to $1 - \alpha(x)$. Note that the denominator of the dilution cost $\delta(x_{t-})$ is the post-issuance equity value $p(\widetilde{x})$.

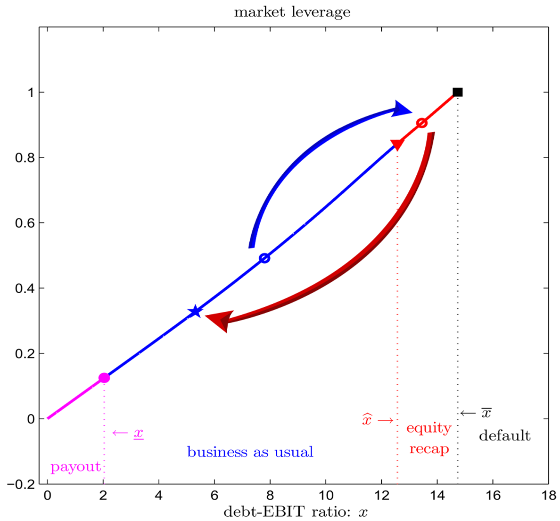

Figure 9: Leverage dynamics, optimal target $\underline{x}$, and recapitalization target $\widehat{x}$. For $x \in [\widehat{x}, \overline{x}) = [12.57, 14.74)$, the firm proactively deleverages its balance sheet by issuing costly external equity. The blue upward arrow and the red downward arrow together describe how the firm after being hit by an earnings jump shock immediately issues equity to deleverage its balance sheet by bringing x down to the 'recapitalization target' $\widetilde{x} = 5.31$ (blue star) in the business-as-usual region.

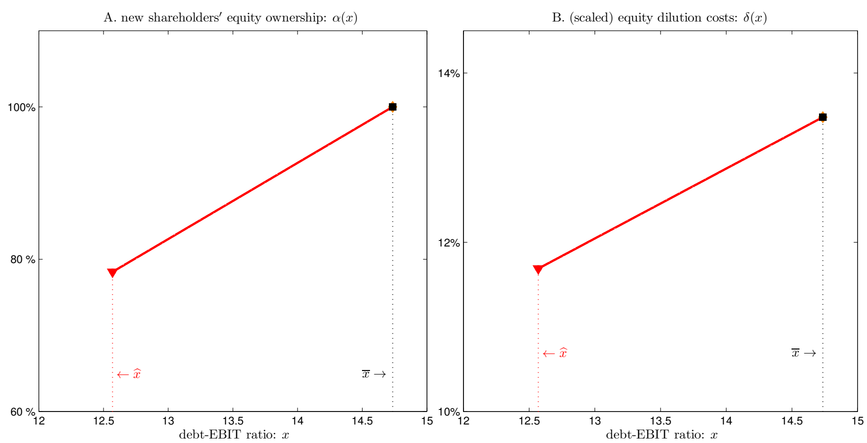

Figure 10: New shareholders' equity ownership $\alpha(x)$ and the cost of equity dilution as a fraction of equity value: $\delta(x)$. External equity issuance significantly lowers (old) shareholders' equity (panel A) and the equity dilution cost is large (panel B).

We plot the new shareholders' ownership, $\alpha(x)$, and the dilution/issuance cost as a fraction of the firm's post-issuance equity value, $\delta(x)$. Panel A shows that old shareholders will give up 78.3% of their equity in order to raise $m = \widehat{x} - \widetilde{x} = 7.26$ at the deleveraging threshold $\widehat{x}$, and are completely wiped out if the firm wants to raise $m = \overline{x} - \widetilde{x} = 9.43$ by issuing equity at its debt limit: $x = \overline{x} = 14.74$. Panel B shows that the equity dilution/issuance cost as a fraction of its equity value, $\delta(x)$, is substantial: $\delta(\cdot)$ increases from 11.7% to 13.5% as x increases from $\widehat{x} = 12.57$ to $\overline{x} = 14.74$.

Returning to the illustrative example that we introduced at the beginning of this subsection, when a firm with $x = 7.9$ is unexpectedly hit by a jump shock with a fractional loss of $(1 - Z) = 42\%$, it gives

Table 2: Effects of a jump's (expected risk-adjsuted) loss $\mathbb{E}(1-Z)$ and (risk-adjusted) arrival rate λ on market leverage ML.

	Mean	Std. dev.	5%	50%	95%
baseline	**0.230**	0.123	0.129	0.182	0.497
20% (jump) loss	0.150	0.105	0.071	0.105	0.383
5% (jump) loss	0.316	0.156	0.161	0.264	0.645
$\lambda = 0.6$ (jump) rate	0.268	0.140	0.144	0.215	0.569
$\lambda = 0.3$ (jump) rate	0.303	0.156	0.156	0.247	0.637

Table 3: Effects of taxes (τ) and liquidation recovery φ (as a fraction of the firm's enterprise value under first-best) on market leverage ML.

	Mean	Std. dev.	5%	50%	95%
baseline	**0.230**	0.123	0.129	0.182	0.497
$\tau = 10\%$	0.169	0.104	0.098	0.126	0.394
$\tau = 30\%$	0.268	0.130	0.151	0.221	0.548
$\varphi = 10\%$	0.224	0.118	0.127	0.178	0.481
$\varphi = 40\%$	0.251	0.135	0.136	0.198	0.545

up $\alpha(7.9/(1-0.42)) = 89.3\%$ of the firm's equity to new shareholders. This substantial equity issuance includes a large Myers-Majluf dilution/issuance cost of $\delta(7.9/(1-0.42)) = 12.5\%$ of firm equity.

D.2 Effects of Jumps, Taxes, and Distress Costs on Leverage in Section 5

Effects of jumps on leverage. Two critical parameters drive jump risk in our model, λ the arrival rate of earnings shocks and β affecting the distribution of earnings losses Z under the risk-neutral measure (Recall that the distribution of recovery Z is the same under $\mathbb{P}$ and $\mathbb{Q}$.) For our comparative static analysis, we fix $\kappa = \ln(\lambda/\lambda^{\mathbb{P}})$ throughout. Therefore, decreasing the arrival rate λ under the risk-neutral measure $\mathbb{Q}$ by a certain fraction is the same as decreasing $\lambda^{\mathbb{P}}$ under the physical measure $\mathbb{P}$ by the same fraction.

Table 2 shows that reducing the expected loss following the arrival of a jump $\mathbb{E}(1-Z) = 1/(\beta+1)$ by a half from 10% to 5% increases average market leverage from 23% to 32%. A reduction in the arrival rate λ from 1.2 (once every 10 months) to 0.6 (once every 20 months) has a somewhat smaller effect. As Table 2 reports, average ML increases from 23% to just 27% as a result of this reduction in the arrival rate.

Effects of tax rates and default recovery on leverage. In Table 3, we explore the effect of changing the tax rate on leverage. For example, when the tax rate τ is increased from 21% to 30% median market leverage increases from 18% to only 22%. When the tax rate τ is lowered from 21% to

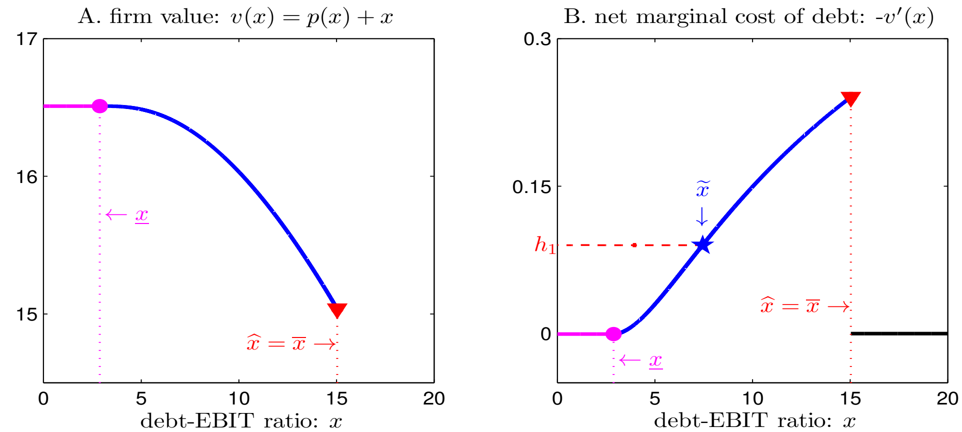

Figure 11: No-jump ($\lambda = 0$). Panels A and B plot firm value, $v(x)$, and the net marginal cost of debt, $-v'(x)$, respectively, for the $\lambda = 0$ case. Equity value $p(x)$ is concave because default does not happen.

10% median market leverage declines from 18% to 13%. Overall, we find some but limited effects of taxes on leverage.

Changes in the default recovery (as a fraction of the enterprise value under first-best) φ have an even smaller effect on leverage as we see from Table 3. When we increase the recovery fraction φ from 10% to 40%, median market leverage only increases from 18% to 20%.[38] The reason why corporate leverage policies are not very sensitive to changes of the default recovery value is that default is a low (even risk-adjusted) probability event, because the firm has an option to issue equity to deleverage.

E Polar Cases with Respectively No Jumps and A Wipe-out Jump

In this appendix, we analyze two special cases on the opposite ends of jump effects: no jump at all ($\lambda = 0$) and 100% EBIT loss if a jump arrives ($Z = 0$).

E.1 No-jump ($\lambda = 0$)

If $\lambda = 0$, the EBIT $\{Y_t\}$ process is a diffusion. Debt is then risk free and the equilibrium credit spread equals zero: $\eta(x) = 0$. The ODE for $p(x)$ in the $x \in (\underline{x}, \widehat{x})$ region then simplifies to:

$$(r - \mu)\, p(x) = [\theta(rx) - 1 + (r - \mu)x]\, p'(x) + \frac{\sigma^2 x^2}{2} p''(x)\,. \tag{E.1}$$

Since external equity is costly, the firm delays its equity issuance as much as possible. As a result, the equity-issuance boundary $\widehat{x}$ equals the default boundary: $\widehat{x} = \overline{x}$ so that $p(\widehat{x}) = 0$. When tapping external equity, the firm issues the net amount m (per unit of EBIT) so that its leverage x reaches the recapitalization target $\widetilde{x}$ where $p'(\widetilde{x}) = -(1 + h_1)$. Equity value in the $x \leq \underline{x}$ region is given by (12) where the payout boundary $\underline{x}$ is characterized by (13) and (14) as in Section 3.[39]

[38] Note that $\varphi = 10\%$ corresponds to $\ell = 2$ and $\varphi = 40\%$ corresponds to $\ell = 8$ in terms of the recovery EBIT multiple ℓ. See footnote 14 in Section 2 for details of this mapping.

[39] Here, we focus on the case where the equity issuance cost is moderate. When equity issuance is too costly, the firm issues no equity. Then, the debt limit equals the firm's liquidation value: $\overline{x} = \ell$, debt is fully collateralized and hence is risk free. See Appendix F for an analysis of this case.

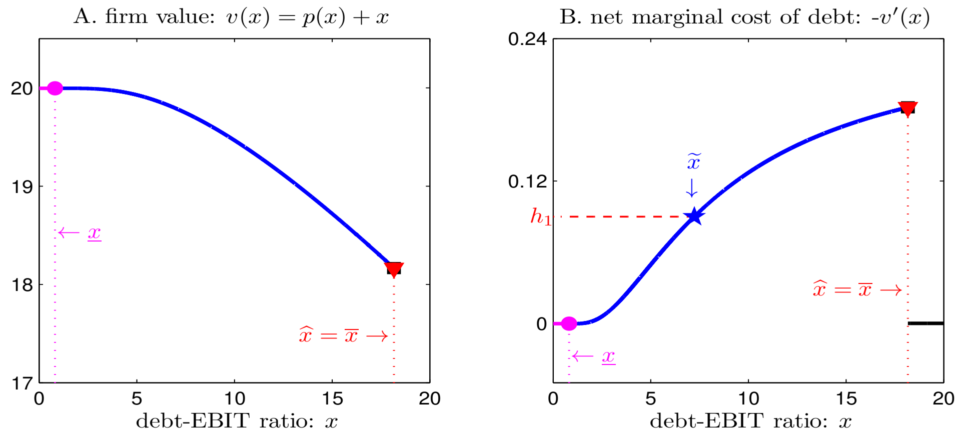

Figure 12: Wipe-out jump ($Z = 0$). Panels A and B plot firm value, $v(x)$, and the net marginal cost of debt, $-v'(x)$, respectively, for the $Z = 0$ case. Equity value $p(x)$ is concave because default is beyond the firm's control.

In Panels A and B of Figure 11, we see that the firm is endogenously risk averse so that $v(x)$ is concave due to costly equity issuance. At $t = 0$, the firm (with no debt) increases its debt-EBIT ratio to $x_0 = \underline{x} = 2.88$ where market leverage is 17% through a lumpy dividend recapitalization with debt issuance of 2.88 per unit of EBIT. When x reaches the equilibrium debt limit: $\overline{x} = 15.04$, the firm's old shareholders are completely wiped out: $p(\overline{x}) = 0$ and the firm issues new equity by raising the net amount $m = 7.6$ while incurring an issuance cost of $h_0 + h_1 m = 1.3$ per unit of EBIT so that the $\{x_t\}$ process continues from $\widetilde{x} = 7.44$, where $p'(\widetilde{x}) = -(1 + h_1) = -1.09$, $v(\widetilde{x}) = 16.34$, and market leverage is 45.5%.

E.2 Wipe-out jump: $Z = 0$

If a jump causes a 100% earnings loss, the firm defaults when the first jump arrives. The equilibrium credit spread then equals the (risk-neutral) jump arrival rate: $\eta(x) = \lambda$, so that $c(x) = (r + \lambda)x$. The firm's equity value, $p(x)$, satisfies the ODE in the $x \in (\underline{x}, \widehat{x})$ region:

$$(r + \lambda - \mu)\, p(x) = [\theta((r + \lambda)x) - 1 + (r + \lambda - \mu)x]\, p'(x) + \frac{\sigma^2 x^2}{2} p''(x)\,. \tag{E.2}$$

As in the no-jump case, the firm delays its equity issuance as much as possible by setting $\widehat{x} = \overline{x}$ so that $p(\widehat{x}) = 0$ and $p'(\widetilde{x}) = -(1 + h_1)$. Other conditions for the payout boundary $\underline{x}$ are the same as in the $\lambda = 0$ case.

In Panels A and B of Figure 12, again we see that the firm is endogenously risk averse so that $v(x)$ is concave due to costly equity issuance. At $t = 0$, the firm starts from $x_0 = \underline{x} = 0.82$ where market leverage is 4.1% by financing a lumpy dividend payment with debt issuance of 0.82 per unit of EBIT. When x reaches the equilibrium debt limit: $\overline{x} = 18.17$, the firm's old shareholders are completely wiped out: $p(\overline{x}) = 0$. The firm issues new equity when x_t reaches the optimal recapitalization threshold $\widehat{x}$, which is also the equilibrium debt limit: $\widehat{x} = \overline{x} = 18.17$, by injecting a net amount $m = 10.94$ while incurring an issuance cost of $h_0 + h_1 m = 1.6$ per unit of EBIT into the firm so that the $\{x_t\}$ process continues from $\widetilde{x} = 7.23$, where $p'(\widetilde{x}) = -(1 + h_1) = -1.09$, $v(\widetilde{x}) = 19.77$, and market leverage is 36.6%.

In sum, equity value $p(x)$ and hence firm value $v(x) = p(x) + x$ are globally concave and the firm is endogenously risk averse for all levels of x in both $\lambda = 0$ and $Z = 0$ cases. This is because in one case it is suboptimal to default when $\lambda = 0$ and in the other case the first jump arrival triggers default with

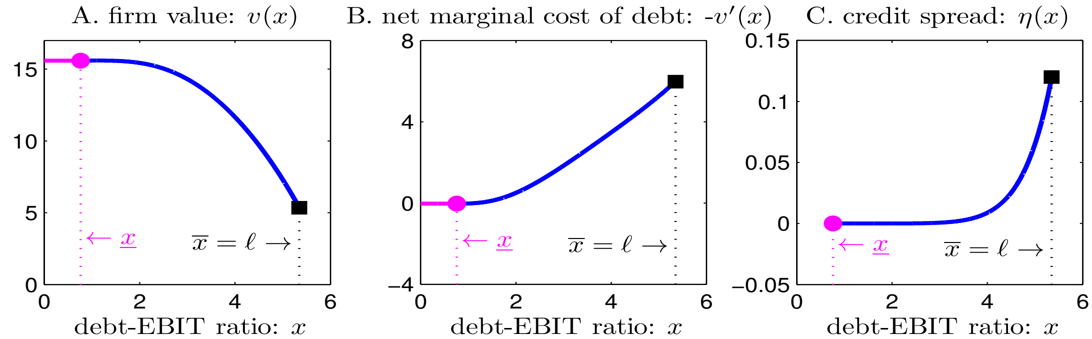

Figure 13: High fixed (h_0) and/or proportional (h_1) external equity issuance costs case. When external equity is too costly, the firm issues no equity and the equilibrium debt capacity equals the firm's liquidation value: $\overline{x} = \ell$. Equity value $p(x)$ is concave.

probability one (that is, default is beyond the firm's control) when $Z = 0$. A key takeaway from these two polar cases is that in order for a firm to have a risk seeking motive so that firm value may be locally convex in x, it is necessary that Y_t features *jumps* that cause a *fractional loss*: $\lambda > 0$ and $Z \in (0, 1)$, so that default is state-contingent.

F When Equity Issuance Costs are High or Zero

In this appendix, we analyze special cases where fixed and proportional equity issuance costs are alternately high or zero, and show that each of these special cases have key limitations in terms of the empirical plausibility of their predictions. We thus conclude that both fixed and proportional equity issuance costs play important roles in generating plausible leverage predictions.

F.1 High Fixed (h_0) and/or Proportional (h_1) Issuance Costs

When either h_0 or h_1 is high enough, it is suboptimal for the firm to issue equity. The firm then uses debt issuance and retained earnings to finance its operations. Unlike our baseline model specification of Section 2, the firm is endogenously risk averse and has no incentive to engage in risk seeking even when leverage is high. Also, the firm's debt capacity equals its liquidation value: $\overline{x} = \ell$.[40] These predictions are restrictive and often counterfactual. For instance, when $\ell = 5.4$, being unable to issue equity at a reasonable cost, means that debt capacity is low. As a result, the firm lowers its target (market) leverage to 4.9% by setting its payout boundary at $\underline{x} = 0.76$ to preserve financial slack. Figure 13 illustrates the solution for this case (for example when $h_0 > 5$.)

When fixed ($h_0 > 0$) and/or proportional (h_1) equity issuance is sufficiently (but not too) costly, the firm still issues equity but only when $x = \widehat{x} = \overline{x}$. That is, the equity-issuance region turns into a singleton. Because the firm can issue equity to continue its operations, its equilibrium debt capacity is larger than the firm's liquidation value: $\overline{x} > \ell$. But old shareholders are completely wiped out after equity issuance. Figure 14 illustrates these results with $h_0 = 5$.

[40] Suppose this were not true. If $\overline{x} < \ell$, the firm can always make itself better off by increasing debt capacity, which is feasible. If $\overline{x} > \ell$, creditors cannot break even in response to diffusion shocks as the firm cannot issue external equity. We solve for $p(x)$ and the payout boundary $\underline{x}$ using (12) in the $x \leq \underline{x}$ region and the integro-differential equation (21) in the $x \in (\underline{x}, \ell)$ region subject to the boundary conditions given in (13)-(14) and the equilibrium credit spread given in (19).

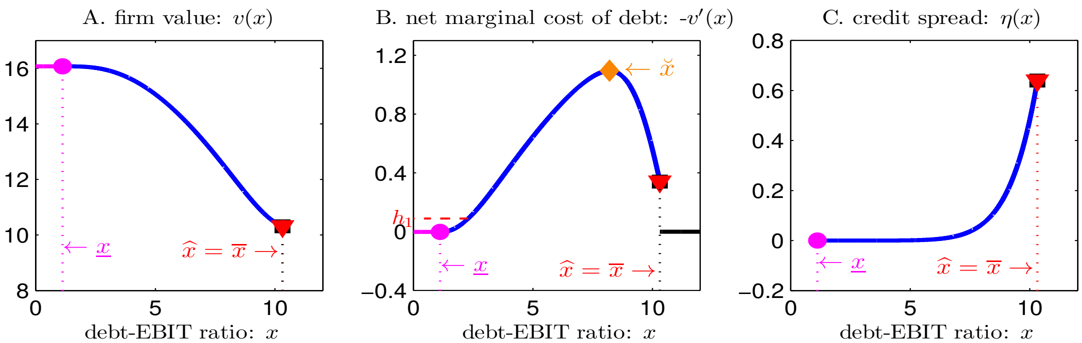

Figure 14: Fixed ($h_0 > 0$) and/or proportional (h_1) external equity issuance is sufficiently (but not too) costly. The firm issues equity only when $x = \widehat{x} = \overline{x}$ and the equilibrium debt capacity is larger the firm's liquidation value: $\overline{x} > \ell$. When $h_0 > 0$, equity value $p(x)$ is convex when x is high.

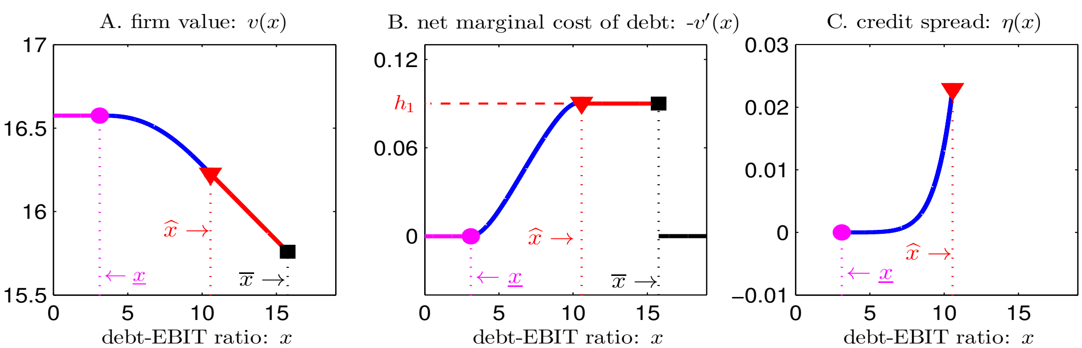

Figure 15: No fixed equity issuance cost ($h_0 = 0$) case. The optimal proactive delveraging threshold and the recapitalization target are equal: $\widehat{x} = \widetilde{x} = 10.57$. Equity value $p(x)$ is concave.

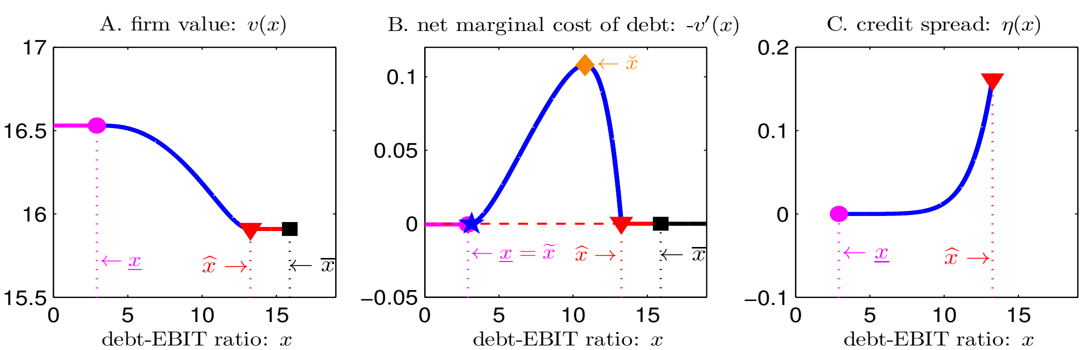

Figure 16: No proportional equity issuance cost ($h_1 = 0$) case. The optimal target and the recapitalization target are equal: $\underline{x} = \widetilde{x} = 2.91$. Equity value $p(x)$ is convex when x is high.

F.2 No Fixed Equity Issuance Cost ($h_0 = 0$) and Moderate Proportional Cost h_1

Absent fixed equity issuance costs, equity value $p(x)$ is concave in x as the firm has no risk seeking incentive. Also, the equity issuance boundary $\widehat{x}$ is equal to the recapitalization target $\widetilde{x}$. Figure 15 plots the solution for the case where $h_1 = 9\%$ and confirms these results.

At $t = 0$, the firm starts with $x_0 = \underline{x} = 3.11$ and an implied market leverage of 18.8% by financing a dividend payment with debt. When x hits $\overline{x} = 15.76$, the firm exhausts its debt capacity and old shareholders are completely wiped out. When a jump results in x entering the region $(\widehat{x}, \overline{x}) = (10.57, 15.76)$, the firm issues an amount of external equity to bring leverage x down to $\widehat{x} = 10.57$. Since $h_1 = 9\%$, the net marginal cost of debt at all x lies inside the $(0, 9\%)$ region. The higher the level of x, the higher the marginal cost of debt: $-v'(x)$.

F.3 No Proportional Equity Issuance Cost ($h_1 = 0$) and Moderate Fixed Cost h_0

What happens if $h_1 = 0$ and h_0 is strictly positive but not high? In this case, firm value is not globally concave due to the fixed equity issuance cost. However, because $h_1 = 0$, whether using external equity or debt as the marginal source of financing to adjust its capital structure, the firm returns to the same target, so that the recapitalization target ($\widetilde{x}$) and the optimal target ($\underline{x}$) are equal. This prediction is implausible as equity is more subject to information sensitivity (Myers and Majluf, 1984). Figure 16 confirms these results.

F.4 Dynamic Tradeoff Theory under Costless Equity: $h_0 = h_1 = 0$

When $h_0 = h_1 = 0$, a solvent firm can switch between debt and equity at no cost. As a result, $p(x) + x$ must be equal for all levels of $x \leq \overline{x}$ where $\overline{x}$ is the firm's equilibrium debt capacity above which the firm defaults. Moreover, we show that the optimal target debt-EBIT ratio (x^*) is constant. We explain how to obtain analytical expressions for firm value $v(x)$, the optimal target x^*, and the debt limit $\overline{x}$ in Internet Appendix IB.

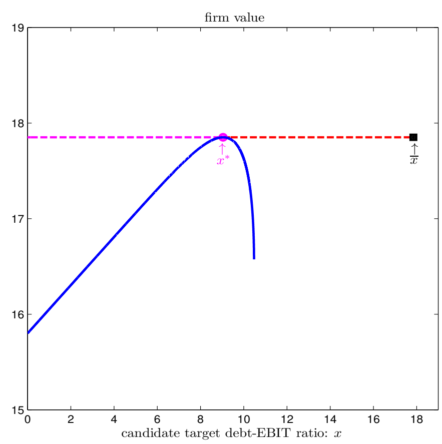

Figure 17: Classic tradeoff theory under costless equity issuance: $h_0 = h_1 = 0$. The optimal target debt-EBIT ratio x^* can be determined by plotting firm value, $v(x) = p(x) + x$ for a range of candidate debt-EBIT ratio target x (see the solid line). The magenta dot where $v(x)$ is maximized gives the optimal target debt-EBIT ratio of $x^* = 9.05$ where firm value is maximized: $v(x^*) = 17.85$ with an implied optimal target market leverage of $ML^* = 51\%$. All parameter values other than $h_0 = h_1 = 0$ are the same as in the baseline numerical example of Section 4.

In Figure 17, we plot firm value, $v(x) = p(x) + x$ for a range of candidate debt-EBIT ratio targets x. At the optimum $x^* = 9.05$ firm value reaches the maximum value of $v(x^*) = 17.85$; the optimal target market leverage is then $ML^* = x^*/v(x^*) = 9.05/17.85 = 51\%$, as indicated by the magenta dot. Figure 17 exhibits a *dynamic version* of the classic static tradeoff theory solution (see, e.g., Figure 1 of Myers (1984), which pits the benefits of debt against the expected costs of default). The costless equity issuance version of our model is fully dynamic and features a realistic EBIT process. Note that the control variable is the debt-EBIT ratio as Y_t is stochastic, rather than the level of debt as in the static tradeoff theory in Myers (1984).[41]

G Derivations of Solutions of the Two Models in Section 6

G.1 Model with both Cash-holding and Debt Regions

When the EBIT process is given by (28), a solvent firm's debt balance is also stochastic over dt:

$$dX_t = \big(C_{t-} + \Theta_{t-}\big)dt - Y_{t-}\big(dt + \nu d\mathcal{Z}_t\big) - dN_t + dU_t\,. \tag{G.1}$$

Let ρ denote the correlation coefficient between the newly introduced Brownian motion $\mathcal{Z}_t$ and the Brownian motion $\mathcal{B}_t$ that drives the expected EBIT (Y) process.[42]

In the business-as-usual region where $dU_t = dN_t = 0$, the following HJB equation for the firm's equity value $P(Y, X)$ holds:

$$\begin{aligned} rP(Y,X) &= \mu Y P_Y(Y,X) + (C + \Theta - Y)\,P_X(Y,X) + \frac{\sigma^2 Y^2}{2} P_{YY}(Y,X) \\ &\quad + \frac{\nu^2 Y^2}{2} P_{XX}(Y,X) - \rho\sigma\nu Y^2 P_{XY}(Y,X) + \lambda \mathbb{E}\left[P(ZY,X) - P(Y,X)\right]. \end{aligned} \tag{G.2}$$

Using the homogeneity property to simplify (G.2), we obtain the ODE

$$\begin{aligned} (r-\mu)\,p(x) &= [c(x) + \theta(c(x)) - 1 - \mu x]\,p'(x) + \frac{(\sigma^2 x^2 + 2\rho\nu\sigma x + \nu^2)}{2} p''(x) \\ &\quad + \lambda \mathbb{E}\left[Zp(x/Z) - p(x)\right] \end{aligned} \tag{G.3}$$

for the scaled equity value $p(x)$. Compared with (21) for the baseline model, (G.3) contains the $\big(2\rho\sigma\nu x + \nu^2\big)p''(x)/2$ term, which induces the firm to save.

The characterization of other parts of the solution is essentially the same as in the baseline model of Section 2. For brevity, we leave out these details. Panel A of Figure 6 provides an illustration. We set $\nu = 2$, $\rho = 0$, and use the same values for all other parameters as in Section 4.

G.2 Model with Dynamic Hedging and Risk Taking

In this subsection, we extend our baseline model of Section 2 by introducing a risky financial asset (futures). Absent default, over dt, the firm's debt balance is stochastic and evolves as:

$$dX_t = (C_{t-} + \Theta_{t-} - Y_{t-})\,dt - dN_t + dU_t + \Omega_{t-} d\mathcal{Z}_t\,, \tag{G.4}$$

where the last term captures the gains (losses) from trading futures under competitive markets (Duffie, 2001). Let ρ denote the correlation between the shock $\mathcal{Z}_t$ and $\mathcal{B}_t$, the standard Brownian motion that drives the EBIT process (1). Without loss of generality, let $\rho \geq 0$.

[41] The solid line in Figure 17 is consistent with the horizontal dashed line in this figure and also in Figure 3, where x is a state variable as in our baseline model of Section 2. This is because absent equity issuance costs, it is optimal for the firm to adjust its x to the optimal target debt-EBIT ratio x^*, for all $x \leq \overline{x}$, implying that $v(x) = \overline{x}$ for all $x \leq \overline{x}$. In effect, the distinction between state and control variables disappears when $h_0 = h_1 = 0$. See Berk and DeMarzo (2020) for a textbook treatment on the classic tradeoff theory.

[42] Mathematically, we introduce a bi-variate Brownian motion $(\mathcal{B}_t, \mathcal{Z}_t)$ under the risk-neutral measure. This newly introduced Brownian motion is idiosyncratic and thus carries no risk premium. The parameter ν measures the volatility of the contemporaneous shock to the EBIT process.

Let $\omega_t = \Omega_t / Y_t$ for all t. The firm chooses ω to solve the following HJB equation:

$$\begin{aligned} rP(Y,X) \;=\; \max_{\omega} \;\; & \mu Y P_Y(Y,X) + (C + \Theta - Y)\, P_X(Y,X) + \frac{\sigma^2 Y^2}{2} P_{YY}(Y,X) \\ & + \frac{\omega^2 Y^2}{2} P_{XX}(Y,X) + \rho\sigma\omega Y^2 P_{XY}(Y,X) + \lambda \mathbb{E}\left[P(ZY,X) - P(Y,X)\right] , \end{aligned} \tag{G.5}$$

subject to the constraint $|\omega| \leq \overline{\omega}$. Using the homogeneity property to simplify (G.5), we obtain:

$$\begin{aligned} rp(x) \;=\; \max_{|\omega| \leq \overline{\omega}} \;\; & \left[c(x) + \theta(c(x)) - 1\right] p'(x) + \mu(p(x) - xp'(x)) + \frac{\left(\sigma^2 x^2 - 2\rho\omega\sigma x + \omega^2\right)}{2} p''(x) \\ & + \lambda \mathbb{E}\left[Zp(x/Z) - p(x)\right] . \end{aligned} \tag{G.6}$$

Note that choosing ω in (G.6) boils down to solving the following constrained (static) problem:

$$\max_{|\omega| \leq \overline{\omega}} \;\; \left(\sigma^2 x^2 - 2\rho\omega\sigma x + \omega^2\right) p''(x) . \tag{G.7}$$

When the firm is endogenously risk averse, so that $p''(x) < 0$, the optimal policy is $\omega = \rho\sigma x$ as $\rho \geq 0$. That is, absent hedging costs it is optimal to eliminate the firm's hedgeable risk exposure and retain unhedgeable (idiosyncratic) risk. However, when the firm is endogenously risk seeking, so that $p''(x) > 0$, the firm seeks maximal risk so that the optimal policy is $\omega = -\overline{\omega}$ (as $\rho \geq 0$). Finally, we verify that the optimal policy ω is consistent with the sign of $p''(x)$.

H Solution for the Investment and Growth Option Model in Section 7

In this appendix, we first solve the extended model with growth option and capital accumulation in Section 7 and report parameter choices for the quantitative analysis used in that section.

H.1 Equity Value under Business as Usual in Mature Phases: $s = 1$ and $s = 2$

The firm must be in one of the two mature phases after the arrival of the growth option at time $T^{\mathcal{G}}$. If it has exercised the growth option at $T^{\mathcal{G}}$, the firm is in phase 2 with productivity A_2. Otherwise, it is in phase 1 with productivity $A_1 < A_2$. The only difference between the two mature phases is productivity. In phase $s = 1, 2$, the firm's equity value $P(K, X; A_s)$ satisfies the following HJB equation in the business-as-usual region:

$$\begin{aligned} rP(K,X;A_s) \;\; = \;\; \max_{I} \; & \left[C + \Theta(C,K) - (A_s K - I)\right] P_X + \Psi\left(I,K\right) P_K + \frac{\sigma_K^2 K^2}{2} P_{KK} \\ & + \lambda_K \mathbb{E}\left[P(ZK,X;A_s) - P(K,X;A_s)\right] , \end{aligned} \tag{H.1}$$

where $\Theta = \Theta(C, K)$ is the tax payment. Let $c_t = C_t / K_t$ and $x_t = X_t / K_t$. As in the baseline model of Section 2, $c(x) = (r + \eta(x))x$ and the equilibrium credit spread $\eta(x)$ is given by

$$\eta(x) = \lambda \int_0^{x/\overline{x}} \left(1 - \frac{\ell_K Z}{x}\right) dF(Z) . \tag{H.2}$$

We write $\Theta(C_t, K_t) = \theta(c_t) K_t$, where $\theta(c) = \tau(A_s - c)\mathbf{1}_{c < A_s}$.

Using the homogeneity property in K, we obtain the following ODE for scaled equity value $p_s(x) = P(K, X; A_s)/K$ in the business-as-usual region where $\underline{x}_s \leq x < \widehat{x}_s$ for $s = 1$ and $s = 2$:

$$\begin{aligned} rp_s(x) \;\; = \;\; \max_{i} \;\; & -\left[A_s - i - c(x) - \theta(c(x))\right] p_s'(x) + \psi(i)(p_s(x) - xp_s'(x)) \\ & + \frac{\sigma_K^2 x^2}{2} p_s''(x) + \lambda_K \mathbb{E}\left[Zp_s(x/Z) - p_s(x)\right] . \end{aligned} \tag{H.3}$$

H.2 Equity Value under Business as Usual in the Growth Phase: $s=0$

In the growth phase ($s=0$), the firm's equity value $P(K,X;A_0)$ satisfies the following HJB equation in the business-as-usual region:

$$\begin{aligned} rP(K,X;A_0) &= \max_{I,\mathbf{1}^A} \left[C+\Theta-(A_0K-I)\right]P_X+\Psi\left(I,K\right)P_K+\frac{\sigma_K^2K^2}{2}P_{KK} \\ &\quad +\lambda_K\mathbb{E}\left[P(ZK,X;A_0)-P(K,X;A_0)\right] \\ &\quad +\lambda_{\mathcal{G}}\left[\mathbf{1}^A P(K,X+\phi K;A_2)+(1-\mathbf{1}^A)P(K,X;A_1)-P(K,X;A_0)\right], \end{aligned} \tag{H.4}$$

The indicator function that $\mathbf{1}^A$ is a control variable that equals one if and only if the firm exercises its growth option at the arrival time $T^{\mathcal{G}}$. If it exercises the option, the firm incurs a one-time cost of ϕK and its productivity permanently increases to A_2. Otherwise, the firm enters into the mature phase with productivity A_1.

Using the homogeneity property, we obtain the following ODE for scaled equity value $p_0(x)=P(K,X;A_0)/K$ in the business-as-usual region where $\underline{x}_0\leq x<\widehat{x}_0$:

$$\begin{aligned} rp_0(x) &= \max_{i,\mathbf{1}^A} \quad -\left[A_0-i-c(x)-\theta(c(x))\right]p_0'(x)+\psi(i)(p_0(x)-xp_0'(x)) \\ &\quad +\frac{\sigma_K^2x^2}{2}p_0''(x)+\lambda_K\mathbb{E}\left[Zp_0(x/Z)-p_0(x)\right] \\ &\quad +\lambda_{\mathcal{G}}\left[\mathbf{1}^A p_2(x+\phi)+\left(1-\mathbf{1}^A\right)p_1(x)-p_0(x)\right]. \end{aligned} \tag{H.5}$$

Equation (H.5) implies that it is optimal to exercise the growth option when the firm obtains a higher equity value $p_2(x+\phi)\geq p_1(x)$. The firm pits the benefit of a higher productivity A_2 going forward against the cost of exercising the growth option: ϕ per unit of capital K.[43]

H.3 Investment in Phase $s=0,1,2$

Since the firm immediately leaves the equity-issuance region and the payout region, it chooses investment I only in the business-as-usual region. In all three phases $s=0,1,2$, investment I is determined by the FOC:

$$\frac{1}{\psi'(i_s(x))}=-\frac{q_s^m(x)}{p_s'(x)}, \tag{H.6}$$

where $-p_s'(x)>1$ is shareholders' marginal cost of servicing debt, $q_s^m(x)$ is the firm's *marginal* q given by $q_s^m=V_K(K,X;A_s)=p_s(x)-xp_s'(x)$. That is, investment is determined by the ratio between marginal q and shareholders' marginal cost of debt: $-p'(x)=-v'(x)-1>0$. Differentiating (H.6) and using the concavity of $\psi(\cdot)$, we see that investment is decreasing in x when $p''(x)<0$ and increasing in x when $p''(x)>0$.[44]

[43] If $x+\phi$ exceeds $\widehat{x}_2$ and if $p_2(x+\phi)\geq p_1(x)$, the firm recapitalizes to $\widetilde{x}_2$ after exercising the option.

[44] Dropping the subscript for phase s to simplify notations, we obtain:

$$i'(x)=-\frac{(\psi'(i(x)))^2}{\psi''(i(x))}\frac{\partial}{\partial x}\left[\frac{q_m(x)}{-p'(x)}\right]=\underbrace{-\frac{(\psi'(i(x)))^2}{\psi''(i(x))}\frac{p(x)}{(p'(x))^2}}_{\text{positive as }\psi(\,\cdot\,)\text{ is concave}}p''(x),$$

which implies that the sign of $i'(x)$ is the same as that of $p''(x)=v''(x)$, as $v(x)=p(x)+x$.

H.4 Other Conditions in the Growth and Mature Phases ($s = 0, 1, 2$)

In the payout region the following condition holds:

$$p_s(x) = p_s(\underline{x}_s) + \underline{x}_s - x \,, \quad \text{for} \quad x < \underline{x}_s \,, \tag{H.7}$$

and the payout boundary $\underline{x}_s$ satisfies

$$p_s'(\underline{x}_s) = -1, \quad \text{and} \quad p_s''(\underline{x}_s) = 0 \,. \tag{H.8}$$

In the equity-issuance region, equity value satisfies:

$$p_s(x) = p_s(\widetilde{x}_s) - \left[h_0^K + (1 + h_1)(x - \widetilde{x}_s)\right] \,, \quad \widehat{x}_s \le x \le \overline{x}_s \,, \tag{H.9}$$

where $\widetilde{x}_s$ is the equity recapitalization target in phase s and

$$-p_s'(x) = -p_s'(\widetilde{x}_s) = 1 + h_1 \,, \quad \widehat{x}_s \le x < \overline{x}_s \,. \tag{H.10}$$

Finally the firm defaults and $p_s(x) = 0$ for $x > \overline{x}_s$.

H.5 Summary

In phase s, equity value $p_s(x)$ takes different forms: i) in the payout region ($x < \underline{x}_s$) where $p_s(x)$ is linear in x as given in (H.7); ii) in the business-as-usual region ($\underline{x}_s \le x < \widehat{x}_s$) where $p_s(x)$ satisfies the integro-differential equation (H.3) for $s = 1, 2$ and $p_0(x)$ satisfies the integro-differential equation (H.5); iii) in the equity-issuance region ($\widehat{x}_s \le x \le \overline{x}_s$) where $p_s(x)$ is given by (H.9); and iv) in the default region ($x > \overline{x}_s$) where $p_s(x) = 0$. The equilibrium credit spread is given by (H.2). In the growth phase, the firm exercises its growth option if and only if $p_2(x + \phi) \ge p_1(x)$. In all phases the optimal i_s satisfies (H.6), the optimal target leverage (payout threshold) $\underline{x}_s$ satisfies (H.8), both the recapitalization target $\widetilde{x}_s$ and the equity-issuance boundary satisfy (H.10), and the debt capacity $\overline{x}_s$ is the lowest value of x where $p_s(x) = 0$.

H.6 Parameter Choices

We use a quadratic function: $\psi(i) = i - \frac{\xi}{2} i^2 - \delta_K$ where $\xi > 0$ measures how costly it is to adjust the capital stock and δ_K is the depreciation rate. As in our EBIT-based model of Section 2, the distribution of $Z \in [0, 1]$ is assumed to be given by $F(Z) = Z^{\beta_K}$, where $\beta_K > 0$. To ease comparison with the EBIT-based model, we use $r = 6\%$, $\lambda_K = 1.2$, $\beta_K = 9$, $\delta_K = 6\%$, $\sigma_K = 40.6\%$, $h_0^K = 0.01$ and $h_1 = 0.09$. As for the other five parameters, in the mature phases without exercising the growth option we set $\xi = 1.82$, $A_1 = 34.1\%$, and $\ell_K = 0.4$ so as to target: i) an average investment-capital ratio i of 10%; ii) a mean average q of 1.24; and iii) a debt recovery upon default of 51% of the debt face value. These parameter values are in line with the values used in dynamic corporate finance and q theory literatures. The quantitative predictions of this extended model can be related to the findings in Lian and Ma (2021), especially their comparison of asset-value-based versus cash-flow-based borrowing constraints.

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Internet Appendices: Additional Results and Proofs

IA Proofs of Main Results

In the following proofs, we use a pair $(S, \widetilde{S})$ to represent the firm's strategy S and creditors' belief $\widetilde{S}$ such that $S \in \Phi(\widetilde{S})$.

Proof of Proposition 1.

Step 1: We prove that (B.2) *holds.*

Starting from any state (Y, X), consider a Markov strategy $S = (T^{\mathcal{D}}, \{N_t, U_t; t \in (0, T^{\mathcal{D}})\})$ such that $S \in \Phi(\widetilde{S})$. Let $(Y_t^S, X_t^S)_{t \geq 0}$ denote the state processes under the strategy pair $(S, \widetilde{S})$. Fix any constant $k > 0$. Starting from state (kY, kX), we denote strategy $kS = (T^{\mathcal{D}^k}, \{N_t^k, U_t^k; t \in (0, T^{\mathcal{D}^k})\})$, where $U_t^k - U_0^k = k(U_t - U_0)$, $N_t^k - N_0^k = k(N_t - N_0)$ and $T^{\mathcal{D}^k} = \{t \geq 0 : (Y_t^S, X_t^S) \in \mathcal{D} \mid (Y_0^S, X_0^S) = (Y, X)\}$. Since $\widetilde{S} \in \mathcal{S}_x$, its default region is given by $\widetilde{\mathcal{D}} = \{(Y, X) : Y/X \in \mathbb{X}^{\widetilde{\mathcal{D}}}\}$, where $\mathbb{X}^{\widetilde{\mathcal{D}}} = (\overline{x}, \infty)$ for some $\overline{x} \geq \ell$. We derive from (A.6) that credit spread at time t with state (Y, X) is given by $\eta(x) = \lambda \mathbb{E}\big[\big(1 - \frac{\ell}{x}\big)^+ \mathbf{1}_{x/Z \in \mathbb{X}^{\widetilde{\mathcal{D}}}}\big]$, where $x = X/Y$. Combining with the homogeneity property of Y, we conclude that the state processes (Y_t^{kS}, X_t^{kS}) under the strategy pair $(kS, \widetilde{S})$ satisfy $(Y_t^{kS}, X_t^{kS}) = (kY_t^S, kX_t^S)$ when $(Y_0^{kS}, X_0^{kS}) = (kY, kX)$ and $(Y_0^S, X_0^S) = (Y, X)$. It follows that $\mathcal{P}(kY, kX; kS, \widetilde{S}) = k\mathcal{P}(Y, X; S, \widetilde{S})$.

In particular, for $S^* = \arg\max_{\widehat{S}} \mathcal{P}(Y, X; \widehat{S}, \widetilde{S})$, we have

$$\mathcal{P}(kY, kX; kS^*, \widetilde{S}) = k\mathcal{P}(Y, X; S^*, \widetilde{S}) \geq k\mathcal{P}(Y, X; 1/k\widehat{S}, \widetilde{S}) = \mathcal{P}(kY, kX; \widehat{S}, \widetilde{S}), \tag{IA.1}$$

for any $\widehat{S} \in \Phi(\widetilde{S})$. It follows that $k\mathcal{P}(Y, X; S^*, \widetilde{S}) = \max_{\widehat{S}} \mathcal{P}(kY, kX; \widehat{S}, \widetilde{S})$. That is, $k\widetilde{P}(Y, X) = \widetilde{P}(kY, kX)$. Let $k = 1/Y$, so that we obtain (B.2).

In the following, we denote $\mathbb{X}^{\mathcal{D}}$ as the scaled default region under strategy S^*.

Step 2: We prove that $\mathbb{X}^{\mathcal{D}} = (v, \infty)$, and $\widetilde{p}(x) = 0$ for any $x \geq v$, where $v = \inf\{x > 0 : \widetilde{p}(x) = 0\}$ and $\widetilde{p}(x) = \widetilde{P}(1, x)$.

Under shareholder limited liability, we have $\widetilde{p}(x) \geq 0$ for any $x > 0$. For the sake of contradiction, assume there exists $x_1 < v$ such that $x_1 \in \mathbb{X}^{\mathcal{D}}$. Since $x \geq \ell$ for any $x \in \mathbb{X}^{\mathcal{D}}$, we have $\widetilde{p}(x_1) = \max\{\ell - x_1, 0\} = 0$. Then we arrive at a contradiction as $v = \inf\{x > 0 : \widetilde{p}(x) = 0\}$. Thus, $\mathbb{X}^{\mathcal{D}} \subseteq [v, \infty)$.

For the sake of contradiction, assume $\widetilde{p}(x) > 0$ for some $x > v$. Let $x_0 = \inf\{x \geq v : \widetilde{p}(x) > 0\}$. Then for any $\varepsilon > 0$, there exists $x_1 < x_2$, such that $|x_i - x_0| \leq \varepsilon$, $i = 1, 2$ and $\widetilde{p}(x_1) = 0$, $\widetilde{p}(x_2) > 0$.

Recall $\mathbb{X}^{\widetilde{\mathcal{D}}} = (\overline{x}, \infty)$, where $\overline{x} \geq \ell$. If $x_1 \geq \overline{x}$, then $x_2 \in (\overline{x}, \infty) = \mathbb{X}^{\widetilde{\mathcal{D}}}$. Since $\widetilde{p}(x_2) > 0 = \max\{\ell - x_2, 0\}$, firm does not default at x_2. According Definition 2, firm must issue equity to immediately exit $\widetilde{\mathcal{D}}$. Thus, there exists $\widetilde{x}_2 \leq \overline{x}$, such that $\widetilde{p}(x_2) = \widetilde{p}(\widetilde{x}_2) - (1 + h_1)(x_2 - \widetilde{x}_2) - h_0$. Starting from $X_t/Y_t = x_1$, the firm can issue equity $x_1 - \widetilde{x}_2$, and receive

$$\widetilde{p}(\widetilde{x}_2) - (1 + h_1)(x_1 - \widetilde{x}_2) - h_0 > \widetilde{p}(\widetilde{x}_2) - (1 + h_1)(x_2 - \widetilde{x}_2) - h_0 = \widetilde{p}(x_2) > 0, \tag{IA.2}$$

where the first inequality uses $x_1 < x_2$. Since $\widetilde{p}(x)$ is the optimal equity value, the left side of (IA.2) is smaller than $\widetilde{p}(x_1)$. It follows from (IA.2) that $\widetilde{p}(x_1) > 0$. Then we arrive at a contradiction as $\widetilde{p}(x_1) = 0$.

If $x_1 < \overline{x}$, then $x_2 < \overline{x}$ for sufficiently small ε. Since $[x_1, x_2] \cap \mathbb{X}^{\widetilde{\mathcal{D}}} = \emptyset$, starting from $X_t/Y_t = x_1$, the firm can make a payout $x_2 - x_1$, and shareholders receive $x_2 - x_1 + \widetilde{p}(x_2) > 0$. Since $\widetilde{p}(x)$ is the optimal equity value, we have $\widetilde{p}(x_1) \geq x_2 - x_1 + \widetilde{p}(x_2) > 0$. Then we arrive at a contradiction as $\widetilde{p}(x_1) = 0$.

Therefore, $\widetilde{p}(x) = 0$ for any $x > v$ and $(v, \infty) \subseteq \mathbb{X}^{\mathcal{D}}$. In addition, it is clear that $\widetilde{p}(v) = 0$. Since we have shown that $\mathbb{X}^{\mathcal{D}} \subseteq [v, \infty)$, we have $\mathbb{X}^{\mathcal{D}} = (v, \infty)$. ■

Proof of Proposition 2. In the following, we denote $S_{\overline{x}}$ as the strategy with default region $\{(Y, X) : Y/X > \overline{x}\}$, and denote $S_{\overline{x}}^*$ as the optimal strategy such that $\mathcal{P}(Y, X; S_{\overline{x}}^*, S_{\overline{x}}) \geq \mathcal{P}(Y, X; \widehat{S}, S_{\overline{x}})$ for all $(Y, X) \in \Sigma$ and $\widehat{S} \in \Phi(S_{\overline{x}})$.

Step 1: We show that for $\mathbb{X}^{\mathcal{D}_1} \supseteq \mathbb{X}^{\mathcal{D}_2}$, strategy $\widetilde{S}_i$ with default region $\{(Y,X) : Y/X \in \mathbb{X}^{\mathcal{D}_i}\}$ and $S_i = \arg\max_S \mathcal{P}(Y,X;S,\widetilde{S}_i)$, $i = 1,2$, we have

$$\mathcal{P}(Y,X;S_1,\widetilde{S}_1) \le \mathcal{P}(Y,X;S_2,\widetilde{S}_2). \tag{IA.3}$$

Let $\eta_i(x) = \lambda \mathbb{E}\big[\big(1 - \frac{\ell}{x}\big)^+ \mathbf{1}_{x/Z \in \mathbb{X}^{\mathcal{D}_i}}\big]$. Since $\mathbb{X}^{\mathcal{D}_1} \supseteq \mathbb{X}^{\mathcal{D}_2}$, we have $\eta_1(x) \ge \eta_2(x)$ for any $x \notin \mathbb{X}^{\mathcal{D}_1}$. Let the sum of (scaled) coupon and tax payments be given by

$$\Lambda(x,\eta) := (r+\eta)x + \tau(1-(r+\eta)x)\mathbf{1}_{(r+\eta)x \le 1}. \tag{IA.4}$$

Since $\tau \in (0,1)$, $\Lambda(x,\eta)$ is nondecreasing in the credit spread η.

Let $S_1 = (T^{\mathcal{D}_1}, \{N_t^1, U_t^1; t \in (0,T^{\mathcal{D}_1})\})$. Consider the strategy $\widehat{S} = (T^{\mathcal{D}_1}, \{N_t^1, \widehat{U}_t^1; t \in (0,T^{\mathcal{D}_1})\})$, where $\widehat{U}_t^1$ is given by

$$\widehat{U}_t^1 = U_t^1 + \int_0^t Y_s \Big[\Lambda(x_s,\eta_1(x_s)) - \Lambda(x_s,\eta_2(x_s))\Big] ds, \tag{IA.5}$$

where $x_s = X_s/Y_s$ and (Y_s,X_s) are states derived from the firm's strategy S_1 and creditors' belief $\widetilde{S}_1$. By definition (IA.5), one can see that $\{(Y_s,X_s)\}_{s\ge 0}$ are the same under strategy pair $(S_1,\widetilde{S}_1)$ and strategy pair $(\widehat{S},\widetilde{S}_2)$. Thus,

$$\begin{aligned}
&\mathcal{P}(Y,X;S_2,\widetilde{S}_2) = \max_S \mathcal{P}(Y,X;S,\widetilde{S}_2) \ge \mathcal{P}(Y,X;\widehat{S},\widetilde{S}_2) \\
=&\mathcal{P}(Y,X;S_1,\widetilde{S}_1) + \mathbb{E}\left[\int_0^{T^{\mathcal{D}_1}} e^{-rs} Y_s \Big[\Lambda(x_s,\eta_1(x_s)) - \Lambda(x_s,\eta_2(x_s))\Big] ds\right] \\
\ge&\mathcal{P}(Y,X;S_1,\widetilde{S}_1),
\end{aligned} \tag{IA.6}$$

where the second inequality uses $\eta_1(x) \ge \eta_2(x)$ and the property that $\Lambda(x,\eta)$ is nondecreasing in η.

Using (B.2), we note that for $x = X/Y$, and $p(x;\overline{x}) := \mathcal{P}(1,x;S^*_{\overline{x}},S_{\overline{x}})$,

$$p(x;\overline{x}) = \mathcal{P}(Y,X;S^*_{\overline{x}},S_{\overline{x}})/Y. \tag{IA.7}$$

Define

$$\mathcal{A}(\overline{x}) = \inf\{x > 0 : p(x;\overline{x}) = 0\}. \tag{IA.8}$$

Using the results proved in step 1, we can see that $\mathcal{A}(\overline{x})$ is nondecreasing in $\overline{x} \ge \ell$.

Step 2: We prove that $\mathcal{A}(\overline{x}) < \overline{x}$ for all $\overline{x} > \frac{1}{r-g}$.

Recall that strategy $S^*_{\overline{x}}$ is the optimal strategy satisfying (IA.7). Since the firm can also use the strategy $S^*_{\overline{x}} = (T^{\mathcal{D}}, \{N_t, U_t; t \in (0,T^{\mathcal{D}})\})$ when creditors' belief is $S_{\overline{x}}$ and equity issuance is costless, and obtain the same state processes, we have

$$\begin{aligned}
\mathcal{P}^{\text{costless}}(Y,X;S^*_{\overline{x}},S_{\overline{x}}) &= \mathbb{E}_t\left[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)}\,(dU_s - dN_s)\right] \\
&\ge \mathbb{E}_t\left[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)}\,(dU_s - dN_s - dH_s)\right] = \mathcal{P}(Y,X;S^*_{\overline{x}},S_{\overline{x}}),
\end{aligned} \tag{IA.9}$$

where $\mathcal{P}^{\text{costless}}$ is the equity value with costless equity issuance.

Let

$$P^{\text{costless}}(Y,X;S_{\overline{x}}) = \max_{\widehat{S}} \mathcal{P}^{\text{costless}}(Y,X;\widehat{S},S_{\overline{x}}), \quad \text{and} \quad p^{\text{costless}}(x;\overline{x}) = P^{\text{costless}}(1,x;S_{\overline{x}}).$$

Then we have $P^{\text{costless}}(Y, X; S_{\overline{x}}) \geq \mathcal{P}^{\text{costless}}(Y, X; S^*_{\overline{x}}, S_{\overline{x}})$. Combining with (IA.9), and using $p(x; \overline{x}) = \mathcal{P}(Y, X; S^*, S_{\overline{x}})/Y$, we have $p^{\text{costless}}(x; \overline{x}) \geq p(x; \overline{x})$. Let $v^{\text{costless}}(\overline{x}) = \inf\{x > 0 : p^{\text{costless}}(x; \overline{x}) = 0\}$ and recall that $\mathcal{A}(\overline{x}) = \inf\{x > 0 : p(x; \overline{x}) = 0\}$. Then we have $v^{\text{costless}}(\overline{x}) \geq \mathcal{A}(\overline{x})$.

Using Lemma 2, we have $P^{\text{costless}}(1, 0; S_{\overline{x}}) = v^{\text{costless}}(\overline{x}) < \overline{x}$ when $\overline{x} > \frac{1}{r-g}$. Using Lemma 3, $v^{\text{costless}}(\overline{x})$ is increasing in $\overline{x} \geq \ell$. Then for any $\epsilon > 0$,

$$v^{\text{costless}}\Big(\frac{1}{r-g}\Big) \leq v^{\text{costless}}\Big(\frac{1}{r-g} + \epsilon\Big) < \frac{1}{r-g} + \epsilon. \tag{IA.10}$$

Sending ϵ to 0 in (IA.10), we obtain $v^{\text{costless}}\Big(\frac{1}{r-g}\Big) \leq \frac{1}{r-g}$. Consequently,

$$\mathcal{A}(\overline{x}) \leq v^{\text{costless}}(\overline{x}) \leq \overline{x}, \quad \forall\, \overline{x} \geq \frac{1}{r-g}, \tag{IA.11}$$

$$\mathcal{A}(\overline{x}) \leq v^{\text{costless}}(\overline{x}) < \overline{x}, \quad \forall\, \overline{x} > \frac{1}{r-g}. \tag{IA.12}$$

Using the results proved in step 1, for any $\overline{x} \geq \ell$ and $x \geq 0$, we have

$$p(x; \overline{x}) \leq p(x; \overline{x}_1) \leq p^{\text{costless}}(x; \overline{x}_1) \leq v^{\text{costless}}(\overline{x}_1) \leq \overline{x}_1, \tag{IA.13}$$

where $\overline{x}_1 = \max\{\overline{x}, \frac{1}{r-g}\}$, the last inequality uses (IA.11).

Step 3: We prove that there exists $\overline{x} \in [\overline{x}_0, \overline{x}_1)$, such that $\mathcal{A}(\overline{x}) = \overline{x}$ if $\mathcal{A}(\overline{x}_0) \geq \overline{x}_0$ and $\mathcal{A}(\overline{x}_1) < \overline{x}_1$ for some $\overline{x}_1 > \overline{x}_0 \geq \ell$.

Let $\mathbb{A} = \{x \in [\overline{x}_0, \overline{x}_1] : \mathcal{A}(x) \geq x\}$. Since $\mathcal{A}(\overline{x}_0) \geq \overline{x}_0$, $\mathbb{A}$ is nonempty.

Let $\overline{x} = \sup \mathbb{A}$. We have $\overline{x}_0 \leq \overline{x} \leq \overline{x}_1$. If $\overline{x} \notin \mathbb{A}$, then $\mathcal{A}(\overline{x}) < \overline{x}$. By the definition of $\overline{x} = \sup \mathbb{A}$ and $\overline{x} \notin \mathbb{A}$, there exists an increasing sequence $x_n \in \mathbb{A}$ such that $\lim_{n\to\infty} x_n = \overline{x}$. Thus $\mathcal{A}(x_n) \geq x_n$. Because $\mathcal{A}(x)$ is nondecreasing in x, we have

$$\mathcal{A}(\overline{x}) \geq \mathcal{A}(x_n) \geq x_n. \tag{IA.14}$$

Sending n to infinity in the above inequality, we obtain $\mathcal{A}(\overline{x}) \geq \lim_{n\to\infty} x_n = \overline{x}$. Then we arrive at a contradiction.

Thus $\overline{x} \in \mathbb{A}$, and then $\mathcal{A}(\overline{x}) \geq \overline{x}$. Since $\mathcal{A}(\overline{x}_1) < \overline{x}_1$, we have $\overline{x} < \overline{x}_1$. By the definition of $\overline{x} = \sup \mathbb{A}$, for any small $\varepsilon \in (0, \overline{x}_1 - \overline{x})$, we have $\overline{x} + \varepsilon \notin \mathbb{A}$ and then $\mathcal{A}(\overline{x} + \varepsilon) < \overline{x} + \varepsilon$. Because $\mathcal{A}(x)$ is nondecreasing in x, we have

$$\overline{x} \leq \mathcal{A}(\overline{x}) \leq \mathcal{A}(\overline{x} + \varepsilon) < \overline{x} + \varepsilon. \tag{IA.15}$$

Sending ε to 0 in the above inequality, we obtain $\mathcal{A}(\overline{x}) = \overline{x}$, and $\overline{x} < \overline{x}_1$.

Since $p(x; \overline{x}) \geq \max\{\ell - x, 0\} > 0$ for any $x < \ell$, we have $\mathcal{A}(\ell) \geq \ell$, where $p(x; \overline{x})$ is given in (IA.7). Using (IA.12) and the result in step 3, for any $\overline{x}_1 > \frac{1}{r-g}$, there exists $\overline{x} \in [\ell, \overline{x}_1)$, such that $\mathcal{A}(\overline{x}) = \overline{x}$, which implies that $\overline{x} = \inf\{x > 0 : p(x; \overline{x}) = 0\}$. It follows that $\overline{x}$ is the default boundary of an equilibrium strategy and $p(x; \overline{x})$ is the equilibrium (scaled) equity value.

Step 4: We show that the firm's equilibrium equity value $P(Y, X)$ is strictly increasing in Y and strictly decreasing in X for any $Y > 0$ and $X \in [0, \overline{x}Y]$.

Let $\overline{x}$ denote the default boundary of an equilibrium strategy. Using Proposition 1, equilibrium value $P(Y, X) > 0$ for any $Y > 0$ and $X \in [0, \overline{x}Y)$. For any $Y > 0$ and $X \in [0, \overline{x}Y)$, there exists a small $\varepsilon_0 > 0$, such that $(X + \varepsilon_0)/Y < \overline{x}$ and $P(Y, X + \varepsilon) > 0$ for any $\varepsilon \in [0, \varepsilon_0]$. Since paying dividend ε is a feasible suboptimal strategy, we derive from the optimality of the strategy S^* that

$$P(Y, X) \geq P(Y, X + \varepsilon) + \varepsilon > P(Y, X + \varepsilon).$$

Thus, $P(Y, X)$ is strictly decreasing in X for any $Y > 0$ and $X \in [0, \overline{x}Y]$.

Using Proposition 1, we have $P(Y,X) = YP(1, X/Y)$. For any $Y_1 > Y_2 > 0$ and $X \in [0, \overline{x}Y_2]$, we have

$$P(Y_1, X) = Y_1 P(1, X/Y_1) > Y_2 P(1, X/Y_1) > Y_2 P(1, X/Y_2) = P(Y_2, X).$$

Thus, $P(Y,X)$ is strictly increasing in Y. ■

Proof of Theorem 1. We only need to show that for any Markov strategy $S \in \Phi(S^*)$, we have

$$P(Y,X) \geq \mathcal{P}(Y,X;S,S^*), \quad \forall\ Y > 0,\ X \geq 0. \tag{IA.16}$$

Let $S = (T^{\mathcal{D}}, \{N_t, U_t; t \in (0, T^{\mathcal{D}})\})$. Since U_t and N_t are nondecreasing processes, we have the following decomposition:

$$U_t = U_t^{\mathrm{c}} + \sum_{s \in [0,t]} \Delta U_s, \qquad N_t = N_t^{\mathrm{c}} + \sum_{s \in [0,t]} \Delta N_s,$$

where U_t^{c} denotes the continuous component of U_t, $\Delta U_s := U_s - U_{s-}$ denotes the jump of U_s, N_t^{c} denotes the continuous component of N_t, and $\Delta N_s := N_s - N_{s-}$ denotes the jump of N_s. We normalize $U_{0-} = U_{0-}^{\mathrm{c}} = N_{0-} = N_{0-}^{\mathrm{c}} = 0$. According to (9), we have the following decomposition for X:

$$X_t = X_t^{\mathrm{c}} + \sum_{s \in [0,t]} \Delta X_s,$$

where $X_t^{\mathrm{c}} = X_{0-} + \int_0^t (C_s + \Theta_s - Y_s) ds + U_t^{\mathrm{c}} - N_t^{\mathrm{c}}$ denotes the continuous component of X_t, $\Delta X_t := X_t - X_{t-} = \Delta U_t - \Delta N_t$ denotes the jump of X_t, X_{0-} is the firm's initial debt balance, C_s is the interest payment, and Θ_s is the tax payment.

First, when equity issuance is costly, it is suboptimal for the firm to make a payout and issue equity at the same time. Thus, we only consider the strategy in which $\Delta U_t \Delta N_t = 0$ for all t.[45] Second, at the default time $t = T^{\mathcal{D}}$, the firm is not allowed to make a payout as creditors have absolute priority over equity holders. Thus, we only consider the strategy that satisfies $\Delta U_{T^{\mathcal{D}}} = 0$. Third, the firm is unable to issue equity at the default time $t = T^{\mathcal{D}}$ due to equity holders' limited liability. Thus, we only consider the strategy that satisfies $\Delta N_{T^{\mathcal{D}}} = 0$.

Since $S \in \Phi(S^*)$, $(Y_t, X_t) \notin \mathcal{D}^*$ for almost every $t \in [0, T^{\mathcal{D}})$. Then $(Y_t, X_t) \in \mathbb{Z} = \{(Y,X) | 0 < Y < \infty,\ 0 \leq X \leq \overline{x}Y\}$, and (9) holds for almost every $t \in [0, T^{\mathcal{D}})$. Moreover, the credit spread (4) is given by $\eta_{t-} = \eta\big(X_{t-}/Y_{t-}; \overline{x}\big)$, where $\eta\big(x; \overline{x}\big)$ is given in (A.7). Recall that $p(x; S^*) = \mathcal{P}(1, x; S^*, S^*) = P(1, x)$ and note that $p(x; S^*)$ and $p_x(x; S^*)$ are bounded in the compact set $x \in [0, \overline{x}]$. Since $P(Y,X) = Yp(X/Y; S^*)$, we have that $P_X(Y,X) = p_x(X/Y; S^*)$ and $P_Y(Y,X) = p(X/Y; S^*) - X/Y p_x(X/Y; S^*)$ are bounded in $\mathbb{Z} = \{(Y,X) | 0 < Y < \infty,\ 0 < X \leq \overline{x}Y\}$.

Using the generalized Ito's formula and omitting the subscripts t and $t-$ on the state processes (Y_t, X_t), we can write:

$$\begin{aligned}
& dP(Y,X) \\
&= \left[\mu Y P_Y(Y,X) + \frac{\sigma^2 Y^2}{2} P_{YY}(Y,X)\right] dt + P_Y(Y,X) \sigma Y d\mathcal{B}_t \\
&\quad + [P(ZY,X) - P(Y,X)]\, d\mathcal{J}_t + P_X(Y,X) dX_t^{\mathrm{c}} + P(Y, X + \Delta X_t) - P(Y,X) \\
&= \left[\mu Y P_Y(Y,X) + \frac{\sigma^2 Y^2}{2} P_{YY}(Y,X) + (C + \Theta - Y)\, P_X(Y,X) + \lambda \left[P(ZY,X) - P(Y,X)\right]\right] dt \\
&\quad + P_Y(Y,X) \sigma Y d\mathcal{B}_t + [P(ZY,X) - P(Y,X)]\, (d\mathcal{J}_t - \lambda dt) \\
&\quad + P_X(Y,X) dX_t^{\mathrm{c}} - (C + \Theta - Y)\, P_X(Y,X) dt + P(Y, X + \Delta X_t) - P(Y,X),
\end{aligned}$$

[45] If $\Delta U_t \geq \Delta N_t$, then it is better for the firm to payout $\Delta U_t - \Delta N_t$ without issuing equity. If $\Delta U_t < \Delta N_t$, then it is better for the firm to issue equity $\Delta N_t - \Delta U_t$ without making a payout. When equity issuance is costless, we use $X_t = U_t - N_t$ as control variable, and set $dU_t dN_t = 0$ without loss of generality. See Internet Appendix IB for the details.

where $\Theta = \theta(c)Y$ is the firm's tax payment (with $\theta(c)$ given in (5) and $c = C/Y$), $C = (r + \eta(X/Y; \overline{x}))X$ is the interest payment, and $\eta(x; \overline{x})$ is the credit spread given in (A.7).

Using $dX_t^c = (C_t + \Theta_t - Y_t)dt + dU_t^c - dN_t^c$, we derive from above that

$$\begin{aligned}
&d\left(e^{-rt}P(Y,X)\right) \\
=&e^{-rt}\widetilde{\mathcal{L}}P(Y,X)dt + e^{-rt}P_Y(Y,X)\sigma Y d\mathcal{B}_t + e^{-rt}\left[P(ZY,X) - P(Y,X)\right](d\mathcal{J}_t - \lambda dt) \\
&+ e^{-rt}\Big[P_X(Y,X)(dU_t^c - dN_t^c) + P(Y, X + \Delta X_t) - P(Y,X)\Big],
\end{aligned} \tag{IA.17}$$

where

$$\widetilde{\mathcal{L}}P = \mu Y P_Y + \tfrac{\sigma^2 Y^2}{2}P_{YY} + \big(C + \Theta - Y\big)P_X + \lambda\left[P(ZY,X) - P(Y,X)\right] - rP. \tag{IA.18}$$

Integrating equation (IA.17) from t to $T^{\mathcal{D}}$ and using $\Delta X = \Delta U$ or ΔN, $\Delta U_{T^{\mathcal{D}}} = 0$, $\Delta N_{T^{\mathcal{D}}} = 0$, we obtain

$$\begin{aligned}
&e^{-r(T^{\mathcal{D}}-t)}P(Y_{T^{\mathcal{D}}}, X_{T^{\mathcal{D}}-}) \\
=&P(Y_t, X_{t-}) + \int_t^{T^{\mathcal{D}}} e^{-r(s-t)}\widetilde{\mathcal{L}}P(Y_s, X_s)ds \\
&+ \int_t^{T^{\mathcal{D}}} e^{-r(s-t)}P_Y(Y_s, X_s)\sigma Y_s d\mathcal{B}_s + \int_t^{T^{\mathcal{D}}} e^{-r(s-t)}\left[P(ZY_s, X_s) - P(Y_s, X_s)\right](d\mathcal{J}_s - \lambda ds) \\
&+ \int_t^{T^{\mathcal{D}}} e^{-r(s-t)}P_X(Y_s, X_s)dU_s^c + \sum_{t \le s < T^{\mathcal{D}}} e^{-r(s-t)}\big(P(Y_s, X_{s-} + \Delta U_s) - P(Y_s, X_{s-})\big) \\
&+ \int_t^{T^{\mathcal{D}}} e^{-r(s-t)}P_X(Y_s, X_s)(-dN_s^c) + \sum_{t \le s < T^{\mathcal{D}}} e^{-r(s-t)}\big(P(Y_s, X_{s-} - \Delta N_s) - P(Y_s, X_{s-})\big).
\end{aligned} \tag{IA.19}$$

Since $P_Y(Y,X)$ is bounded in the region $\mathbb{Z}$ and $(Y_s, X_s) \in \mathbb{Z}$, the third term in (IA.19), the integral with respect $d\mathcal{B}_s$, is a square integrable martingale. Similarly, the fourth term in (IA.19), the integral with respect $(d\mathcal{J}_s - \lambda ds)$, is also a square integrable martingale. Taking expectations in (IA.19), we obtain:

$$\begin{aligned}
P(Y_t, X_{t-}) =& \mathbb{E}_t\left[e^{-r(T^{\mathcal{D}}-t)}P(Y_{T^{\mathcal{D}}}, X_{T^{\mathcal{D}}-})\right] - \mathbb{E}_t\left[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)}\widetilde{\mathcal{L}}P(Y_s, X_s)ds\right] \\
&+ \mathbb{E}_t\left[-\int_t^{T^{\mathcal{D}}} e^{-r(s-t)}P_X(Y_s, X_s)dU_s^c - \sum_{t \le s < T^{\mathcal{D}}} e^{-r(s-t)}\big(P(Y_s, X_{s-} + \Delta U_s) - P(Y_s, X_{s-})\big)\right] \\
&+ \mathbb{E}_t\left[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)}P_X(Y_s, X_s)dN_s^c + \sum_{t \le s < T^{\mathcal{D}}} e^{-r(s-t)}\big(P(Y_s, X_{s-}) - P(Y_s, X_{s-} - \Delta N_s)\big)\right].
\end{aligned} \tag{IA.20}$$

Using $P_X(Y,X) \le -1$ for any $X \le \overline{x}Y$, and $P(Y,X) = 0$ for any $X \ge \overline{x}Y > 0$, we have $P(Y,X) \ge \overline{x}Y - X$ for $X \le \overline{x}Y$. Since $\overline{x} \ge \ell$, we have

$$P(Y,X) \ge \max\{\ell Y - X, 0\} \tag{IA.21}$$

for any $(Y,X) \in \Sigma = \{(Y,X) : Y > 0, X \ge 0\}$.

Since $S \in \Phi(S^*)$, we observe from Definition 2-(i) that for $t < T^{\mathcal{D}}$ such that $\Delta U_t > 0$, we have $X_{t-} \leq X_{t-} + \Delta U_t \leq \overline{x} Y_t$. Combining with $P_X(Y,X) \leq -1$ for any $X \leq \overline{x}Y$, we conclude that

$$\begin{aligned}
&\mathbb{E}_t\left[-\int_t^{T^{\mathcal{D}}} e^{-r(s-t)} P_X(Y_s,X_s)dU_s^{\mathrm{c}} - \sum_{t\leq s<T^{\mathcal{D}}} e^{-r(s-t)}\big(P(Y_s,X_{s-}+\Delta U_s) - P(Y_s,X_{s-})\big)\right]\\
\geq&\mathbb{E}_t\left[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)}dU_s^{\mathrm{c}} + \sum_{t\leq s<T^{\mathcal{D}}} e^{-r(s-t)}\Delta U_s\right] = \mathbb{E}_t\left[\int_{t\leq s<T^{\mathcal{D}}} e^{-r(s-t)}dU_s\right]. \qquad \text{(IA.22)}
\end{aligned}$$

Since $P_X(Y,X) \leq -1$ for any $X < \overline{x}Y$ and $P(Y,X) = 0$ for any $X \geq \overline{x}Y$, $P(Y,X)$ is non-increasing in X, which implies that $P^N(Y,X) = \sup_{M\geq 0} P(Y, X-M) - h_0 Y - (1+h_1)M$ is non-increasing in X. For any $X \geq \overline{x}Y$, we have

$$P^N(Y,X) \leq P^N(Y,\overline{x}Y) \leq P(Y,\overline{x}Y) = 0,$$

where the second inequality uses $P(Y,X) \geq P^N(Y,X)$ for any $(Y,X) \in \mathbb{Z}$. Thus, $P(Y,X) \geq P^N(Y,X)$ for any $Y > 0$ and $X \geq 0$.

Using $P(Y,X) \geq P^N(Y,X)$, we have $P(Y_s,X_{s-}) \geq P(Y_s, X_{s-} - \Delta N_s) - h_0 Y_s - (1+h_1)\Delta N_s = P(Y_s, X_{s-} - \Delta N_s) - \Delta N_s - \Delta H_s$. It follows that

$$P(Y_s,X_{s-}) - P(Y_s, X_{s-} - \Delta N_s) \geq -(\Delta N_s + \Delta H_s). \qquad \text{(IA.23)}$$

In the case $h_0 > 0$, we must have $dN_s^{\mathrm{c}} = 0$ for almost every s. Otherwise, cumulative equity issuance costs are infinitely large, which implies a negatively infinite equity value. In the case $h_0 = 0$, we have $P(Y,X) \geq P^N(Y,X) \geq P(Y, X-\varepsilon) - (1+h_1)\varepsilon$ for any small $\varepsilon > 0$, which yields that $P_X(Y,X) \geq -1-h_1$ for $(Y,X) \in \mathbb{Z}$. Hence, we have

$$\begin{aligned}
&\mathbb{E}_t\left[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)} P_X(Y_s,X_s)dN_s^{\mathrm{c}} + \sum_{t\leq s<T^{\mathcal{D}}} e^{-r(s-t)}\big(P(Y_s,X_{s-}) - P(Y_s,X_{s-}-\Delta N_s)\big)\right]\\
\geq&\mathbb{E}_t\left[\mathbf{1}_{h_0=0}\int_t^{T^{\mathcal{D}}} e^{-r(s-t)}(-1-h_1)dN_s^{\mathrm{c}} - \sum_{t\leq s<T^{\mathcal{D}}} e^{-r(s-t)}\big(\Delta N_s + \Delta H_s\big)\right]\\
=&\mathbb{E}_t\left[\mathbf{1}_{h_0=0}\int_t^{T^{\mathcal{D}}} e^{-r(s-t)}(-dN_s^{\mathrm{c}} - dH_s^{\mathrm{c}}) - \sum_{t\leq s<T^{\mathcal{D}}} e^{-r(s-t)}\big(\Delta N_s + \Delta H_s\big)\right]\\
=&-\mathbb{E}_t\left[\int_{t\leq s<T^{\mathcal{D}}} e^{-r(s-t)}(dN_s + dH_s)\right], \qquad \text{(IA.24)}
\end{aligned}$$

where the inequality uses (IA.23), $P_X(Y,X) \geq -1-h_1$ for $(Y,X) \in \mathbb{Z}$ and $(Y_s,X_s) \in \mathbb{Z}$ for almost every $s \in [0, T^{\mathcal{D}})$.

Substituting (IA.21)-(IA.24) into (IA.20), and using $\mathbb{E}_t[\widetilde{\mathcal{L}}P(Y_s,X_s)] = \mathcal{L}P(Y_s,X_s) \leq 0$ for almost every $s < T^{\mathcal{D}}$, we obtain

$$\begin{aligned}
P(Y_t,X_{t-}) \geq&\mathbb{E}_t\left[e^{-r(T^{\mathcal{D}}-t)}\max\{\ell Y_{T^{\mathcal{D}}} - X_{T^{\mathcal{D}}-}, 0\}\right] + \mathbb{E}_t\left[\int_{t\leq s<T^{\mathcal{D}}} e^{-r(s-t)}(dU_s - dN_s - dH_s)\right]\\
=&\mathcal{P}(Y_t, X_{t-}; S, S^*). \qquad \text{(IA.25)}
\end{aligned}$$

This proves (IA.16). ■

Proof of Theorem 2. Recall the definition of $\mathcal{A}(\overline{x})$ in (IA.8), and denote $\overline{\mathbb{X}} = \{\overline{x} \geq \ell : \mathcal{A}(\overline{x}) = \overline{x}\}$. For any MPE $S \in \mathcal{S}_x$, we have $S = \arg\max_{\widehat{S}} \mathcal{P}(Y,X;\widehat{S},S)$. Moreover, from Proposition 1, it follows that the default region in the strategy S is given by $\{(Y,X) : Y/X > \overline{x}\}$, where $\overline{x} = \inf\{x > 0 : p(x;\overline{x}) = 0\}$, and $p(x;\overline{x}) = \mathcal{P}(1,x;S,S)$.

We shall denote the equilibrium strategy $S_{\overline{x}}$ with default region $\{(Y,X) : Y/X > \overline{x}\}$ as $S_{\overline{x}}$, and the corresponding equity value as $p(x;\overline{x})$. Then, all equilibria in $\mathcal{S}_x$ can be represented by $\{S_{\overline{x}} : \overline{x} \in \overline{\mathbb{X}}\}$. For any $\overline{x}_1, \overline{x}_2 \in \overline{\mathbb{X}}$ with $\overline{x}_1 \leq \overline{x}_2$, we derive from the result in step 1 in the proof of Proposition 2 that $p(x;\overline{x}_1) \leq p(x;\overline{x}_2)$ for all x. Thus, we only need to show that there exists $\overline{x}^* \in \overline{\mathbb{X}}$, such that $\overline{x}^* \geq \overline{x}$ for any $\overline{x} \in \overline{\mathbb{X}}$.

According to the result in step 2 in the proof of Proposition 2, we have $\mathcal{A}(\overline{x}) < \overline{x}$ for all $\overline{x} > \frac{1}{r-g}$. Thus, $\mathcal{A}(\overline{x}) = \overline{x} \leq \frac{1}{r-g}$ for any $\overline{x} \in \overline{\mathbb{X}}$. Let $\overline{x}^* = \sup \overline{\mathbb{X}}$. We have $\overline{x}^* \leq \frac{1}{r-g}$. We only need to prove that $\overline{x}^* \in \overline{\mathbb{X}}$. For the sake of contradiction, assume that $\overline{x}^* \notin \overline{\mathbb{X}}$. There exists an increasing sequence $x_n \in \overline{\mathbb{X}}$ such that $\lim_{n\to\infty} x_n = \overline{x}$. Thus $\mathcal{A}(x_n) = x_n$. Because $\mathcal{A}(x)$ is nondecreasing in x, we have

$$\mathcal{A}(\overline{x}^*) \geq \mathcal{A}(x_n) = x_n. \tag{IA.26}$$

Sending n to infinity in the above inequality, we obtain $\mathcal{A}(\overline{x}^*) \geq \lim_{n\to\infty} x_n = \overline{x}^*$. Since $\overline{x}^* \notin \overline{\mathbb{X}}$, we have $\mathcal{A}(\overline{x}^*) > \overline{x}^*$. Using the result in step 2 in the proof of Proposition 2, we have $\mathcal{A}(\overline{x}_1) < \overline{x}_1$ for $\overline{x}_1 > \max\{\frac{1}{r-g}, \overline{x}^*\}$. Therefore, we conclude from the result in step 3 in the proof of Proposition 2 that there exists $\overline{x}_2 \in (\overline{x}^*, \overline{x}_1)$, such that $\mathcal{A}(\overline{x}_2) = \overline{x}_2$. It follows that $\overline{x}_2 \in \overline{\mathbb{X}}$. By the definition of $\overline{x}^* = \sup \overline{\mathbb{X}}$, we arrive at a contradiction as $\overline{x}_2 > \overline{x}^*$. Therefore, $\overline{x}^* \in \overline{\mathbb{X}}$. ■

In order to prove Theorem 3, we first prove the following lemma, which gives the variational inequality for the optimal equity value given debtholder's conjectured default region $\{(Y,X) : X/Y > \overline{x}\}$.

Lemma 1. *Consider a Markov strategy $\widetilde{S} \in \mathcal{S}_x$ with default region $\widetilde{\mathcal{D}} = \{(Y,X) : X/Y > \overline{x}\}$ for some $\overline{x} \geq \ell$. Suppose a Markov strategy $S^* \in \mathcal{S}_x$ satisfies $\mathcal{P}(Y,X;S^*,\widetilde{S}) = \max_{\widehat{S}} \mathcal{P}(Y,X;\widehat{S},\widetilde{S})$ for any $(Y,X) \in \Sigma$. Denote $\widetilde{p}(x) = \mathcal{P}(1,x;S^*,\widetilde{S})$ and $v = \inf\{x > 0 : \widetilde{p}(x) = 0\}$. We have*

$$\max\{\widetilde{p}^N(x) - \widetilde{p}(x),\ \max\{\ell - x, 0\} - \widetilde{p}(x)\} = 0\,, \quad x > \min\{v, \overline{x}\}. \tag{IA.27}$$

Moreover, if $v \geq \overline{x}$, then $\widetilde{p}(x)$ is the unique bounded viscosity solution to the following equation for $x \in [0,\overline{x}]$:[46]

$$\max\{\mathcal{L}^{\overline{x}}\widetilde{p}(x),\ 1 + \widetilde{p}_x(x),\ \widetilde{p}^N(x) - \widetilde{p}(x)\} = 0\,, \tag{IA.28}$$

where $\mathcal{L}^{\overline{x}}$ is given in (B.7)*, and $\widetilde{p}^N(x) = \max_{m\geq 0} \widetilde{p}(x-m) - h_0 - (1+h_1)m$.*

Proof of Lemma 1. In the following, we denote equity value $\widetilde{P}(Y,X) = \mathcal{P}(Y,X;S^*,\widetilde{S})$.

Because default and equity financing are feasible suboptimal strategies, we derive from the optimality of strategy S^* that

$$\widetilde{P}(Y,X) \geq \max\{\ell Y - X, 0\}, \tag{IA.29}$$
$$\widetilde{P}(Y,X) \geq \widetilde{P}^N(Y,X), \tag{IA.30}$$

for any $(Y,X) \in \Sigma$, where $\widetilde{P}^N(Y,X) = \sup_{M\geq 0} \widetilde{P}(Y,X-M) - h_0 Y - (1+h_1)M$. Using (B.2) and $\widetilde{p}(x) = \widetilde{P}(1,x)$, we have $\widetilde{P}(Y,X) = Y\widetilde{p}(X/Y)$. Then we derive from (IA.29)-(IA.30) that

$$\widetilde{p}(x) \geq \max\{\ell - x, 0\}, \tag{IA.31}$$
$$\widetilde{p}(x) \geq \widetilde{p}^N(x), \tag{IA.32}$$

[46] Boundary conditions are $\widetilde{p}_x(0) = -1$ and (IA.27) for $x = \overline{x}$.

for any $x \geq 0$. For $x > \overline{x}$, firm can only default or purchase debt back such that firm leaves default region immediately. Therefore, we can see that (IA.27) holds for $x > \overline{x}$. Using Proposition 1, for any $x > v$, we have $\widetilde{p}(x) = 0 \geq \max\{\ell - x, 0\}$ and thus $\ell - v \leq 0$. Therefore, we can see that (IA.27) holds for $x \geq v$ with $\widetilde{p}(x) = 0 = \max\{\ell - x, 0\}$.

According to (IA.13), we can see that $\widetilde{p}(x) \leq \max\{\overline{x}, \frac{1}{r-g}\}$. According to (IA.31), we have $\widetilde{p}(x) \geq 0$. Thus, $\widetilde{p}(x)$ is bounded.

Equation (IA.28) is derived using dynamic programming and (B.2); for technical details, see Fleming and Soner (2006) and Seydel (2009). Moreover, the uniqueness of the viscosity solution to equation (IA.28) follows from the proof provided in Seydel (2009). ■

Proof of Theorem 3. From the homogeneity property (B.2), for any MPE $S^* \in \mathcal{S}_x$, the scaled equity value $p(x) = p(x; S^*)$ and equity value $P(Y, X) = \mathcal{P}(Y, X; S^*, S^*)$ satisfies $P(Y, X) = Yp(X/Y)$.

Since S^* is an MPE, then $\mathcal{P}(Y, X; S^*, S^*) \geq \mathcal{P}(Y, X; \widehat{S}, S^*)$ for all $(Y, X) \in \Sigma$ and $\widehat{S} \in \Phi(S^*)$. Then it follows from Proposition 1 that $p(x) = 0$ for any $x \geq \overline{x}$, and $\mathcal{D} = \{(Y, X) : X/Y > \overline{x}\}$ is the default region in the strategy S^*, where $\overline{x} = \inf\{x > 0 : p(x) = 0\} \geq \ell$. Combining this with Lemma 1, we conclude that $p(x)$ is the unique bounded viscosity solution to the equation (B.6).

From the proof of Theorem 2, we have $\overline{x}^* = \sup \overline{\mathbb{X}} \leq \frac{1}{r-g}$ and $\overline{x}^* \in \overline{\mathbb{X}}$, where $\overline{\mathbb{X}} = \{\overline{x} \geq \ell : \mathcal{A}(\overline{x}) = \overline{x}\}$ is the set of default boundaries $\overline{x}$ for all MPEs in the set $\mathcal{S}_x$, and $\mathcal{A}(\overline{x})$ is given in (IA.8).

Let $p(x; \overline{x})$ denote the scaled equity value for an MPE in the set $\mathcal{S}_x$ with the default boundary $\overline{x} \geq \ell$. For any $\overline{x}_1, \overline{x}_2 \in \overline{\mathbb{X}}$ with $\overline{x}_1 \leq \overline{x}_2$, it follows from the result in step 1 in the proof of Proposition 2 that $p(x; \overline{x}_1) \leq p(x; \overline{x}_2)$ for all x. Hence, the MPE with default boundary $\overline{x}^*$ is the best MPE. ■

IB Solution for the Costless Equity Issue Case: $h_0 = h_1 = 0$

First, as in our baseline model of Section 2, given shareholders' limited liability, there exists a cutoff level $\overline{x}$ above which the firm defaults so that equity value is zero: $p(x) = 0$ for $x > \overline{x}$. Second, because $h_0 = h_1 = 0$, the payout region, the business-as-usual region, and the equity issuance region are all combined into one region where $p'(x) = -1$ and $p''(x) = 0$ for $x \leq \overline{x}$. Substituting these two conditions into the integro-differential equation (21) for $p(x)$ at $x = \widetilde{x}$, we obtain:

$$(r - \mu)\, p(\widetilde{x}) = -\left[c(\widetilde{x}; \overline{x}) + \theta(c(\widetilde{x}; \overline{x})) - 1 - \mu\widetilde{x}\right] + \lambda \mathbb{E}\left[Zp(\widetilde{x}/Z) - p(\widetilde{x})\right], \tag{IB.1}$$

where $\widetilde{x} \in [0, \overline{x}]$ denotes the target debt-EBIT ratio and $c(x; \overline{x}) = (r + \eta(x; \overline{x}))x$. Equation (IB.1) shows that cash-flow volatility has no impact on $p(x)$. This is because firms do not value financial flexibility in the absence of equity issuance costs.

Using $c(x; \overline{x}) = (r + \eta(x; \overline{x}))x$, (A.7) for credit spread $\eta(x; \overline{x})$, and the decomposition $v(x) = p(x) + x$, we can rewrite (IB.1) and obtain the following equation for $v(x)$ at $x = \widetilde{x}$:

$$\begin{aligned} (r - \mu)\, v(\widetilde{x}) &= (r - \mu)\widetilde{x} - \left[c(\widetilde{x}; \overline{x}) + \theta(c(\widetilde{x}; \overline{x})) - 1 - \mu\widetilde{x}\right] + \lambda \mathbb{E}\left[Zp(\widetilde{x}/Z) - p(\widetilde{x})\right] \\ &= 1 - \theta(c(\widetilde{x}; \overline{x})) - \eta(\widetilde{x}; \overline{x})\widetilde{x} + \lambda \mathbb{E}\left[Zp(\widetilde{x}/Z) - p(\widetilde{x})\right] \\ &= 1 - \theta(c(\widetilde{x}; \overline{x})) + \lambda \mathbb{E}\left[Zv(\widetilde{x}/Z) - v(\widetilde{x})\right]. \end{aligned} \tag{IB.2}$$

Since the firm defaults if and only if x exceeds $\overline{x}$, we have $v(x) = \ell$ for $x \geq \overline{x}$. Also note that $v(x) = v(\overline{x}) = \overline{x}$ for all $x < \overline{x}$ because equity issuance is costless. Using these results and $g = \mu - \lambda\left[1 - \mathbb{E}(Z)\right]$ to rewrite (IB.2), we obtain:

$$v(x) = \overline{x} = \frac{1}{r - g}\Big[1 - \underbrace{\theta(c(\widetilde{x}; \overline{x}))}_{\text{taxes}} - \underbrace{\lambda(\overline{x} - \ell)\Big(\int_0^{\widetilde{x}/\overline{x}} ZdF(Z)\Big)}_{\text{expected firm loss given default}}\Big], \quad x < \overline{x}, \tag{IB.3}$$

which defines an implicit function for $\overline{x} = \overline{x}(\widetilde{x})$; see Proposition 3 in Internet Appendix IB.2 for further details. Finally, equityholders choose the optimal target debt-EBIT ratio $\widetilde{x}^*$ to maximize $v(x) = \overline{x}$

given in (IB.3). Firm value associated with this optimal policy is $v(x) = \overline{x}(\widetilde{x}^*)$ and equity value is $p(x) = \overline{x}(\widetilde{x}^*) - x$ for any $x < \overline{x}(\widetilde{x}^*)$.

Next, we provide closed-form solutions for two special cases of our costless equity issuance benchmark: (1.) the no-jump case ($\lambda = 0$) and (2.) the jump case with no recovery ($Z = 0$).

For the costless equity issuance no-jump ($\lambda = 0$) case, shareholders set the target of x at

$$x^* = \frac{1-\tau}{(1-\tau)r - \mu}\,, \tag{IB.4}$$

at all time and never default. In effect, shareholders sell the firm to creditors by setting a 100% market leverage: $v(x^*) = x^*$ and $p(x^*) = 0$. The firm avoids inefficient liquidation by issuing equity at no cost.

For the costless equity issuance jump case with no recovery ($Z = 0$), as in Hu, Varas, and Ying (2023), the firm defaults when a jump arrives. The equilibrium credit spread is $\eta = \lambda$. Shareholders effectively sell the firm to creditors by setting the target of x at

$$x^* = \frac{1-\tau}{(1-\tau)(r+\lambda) - \mu}\,. \tag{IB.5}$$

For both special cases, debt has tax benefits and no cost. Therefore, the firm chooses 100% market leverage by delaying equity issuance as much as possible and by setting the optimal target x^* to the equilibrium debt capacity $\overline{x}$. If $\lambda = 0$, the firm has no default risk as diffusion shocks cannot take a firm's x to the right of its equilibrium debt capacity $\overline{x}$. If $\lambda > 0$ but $Z = 0$, default is purely exogenous. In this case, the firm also defers its equity issuance as much as it can, i.e., until $x = \overline{x}$.

Finally, we provide proofs for the case where $\lambda > 0$ and $F'(z) > 0$, $z \in [0, 1]$. In Appendix IB.1, we prove the existence of an MPE and the best equilibrium. In Appendix IB.2, we prove that the best equilibrium can be solved by choosing the optimal target debt-EBIT ratio $\widetilde{x}$ to maximize $\overline{x}$ given in (IB.3). In Appendix IB.3, we prove the uniqueness of the equilibrium strategy under Assumption 3.

IB.1 Existence of an MPE and the Best Equilibrium

In this section, we prove the existence of a Markov perfect equilibrium, and moreover, prove that there exists a best equilibrium with the highest firm value among all equilibria in the set $\mathcal{S}_x$.

Lemma 2. *Under Assumption 1, given creditors' belief $\widetilde{S}$ with a default region $\{(Y, X) : X/Y > \overline{x}\}$ for some constant $\overline{x} \geq \ell$, the optimal equity value is given by*

$$P(Y, X) = (\mathcal{A}(\overline{x})Y - X)^+, \tag{IB.6}$$

and the firm's optimal strategy is to set X_t/Y_t as a constant for any $t < T^{\mathcal{D}}$, where $\mathcal{A}(\overline{x}) = P(1, 0) > \ell$. Additionally, we have $\mathcal{A}(\overline{x}) < \overline{x}$ if $\overline{x} > \frac{1}{r-g}$.

Lemma 3. *Under Assumption 1, given creditors' belief $\widetilde{S}$ with a default region $\{(Y, X) : X/Y > \overline{x}\}$ for some constant $\overline{x} \geq \ell$, setting the target of x_t at $\widetilde{x}$ is a feasible strategy if and only if $\widetilde{x} \in \mathbb{U}(\overline{x})$, where*

$$\mathbb{U}(\overline{x}) = \{\widetilde{x} \in [0, \overline{x}] : 1 + \mu\widetilde{x} - \theta(c(\widetilde{x}; \overline{x})) - c(\widetilde{x}; \overline{x}) \geq 0\}, \tag{IB.7}$$

and $c(x; \overline{x}) = rx + \lambda \int_0^{x/\overline{x}} (x - \ell Z)\, dF(Z)$. We have $\mathbb{U}(\overline{x}) \subseteq \mathbb{U}(\overline{x} + \Delta x)$ for any $\Delta x > 0$. Furthermore, given the above pair $(\widetilde{x}, \overline{x})$, the corresponding firm value is the unique solution $\widetilde{v} \geq \widetilde{x}$ to the following equation

$$(r-g)\widetilde{v} - \lambda \int_0^{\widetilde{x}/\widetilde{v}} (\widetilde{x} - Z\widetilde{v}) dF(Z) = r\widetilde{x} + 1 - \theta(c(\widetilde{x}; \overline{x})) - c(\widetilde{x}; \overline{x}). \tag{IB.8}$$

The solution $\widetilde{v}(\widetilde{x}, \overline{x})$ to (IB.8) *is nondecreasing in $\overline{x}$, and $\widetilde{v}(\widetilde{x}, \overline{x}) < \overline{x}$ for any $\overline{x} > \frac{1}{r-g}$, $\widetilde{x} \in \mathbb{U}(\overline{x})$, and*

$$\mathcal{A}(\overline{x}) = \max_{\widetilde{x} \in \mathbb{U}(\overline{x})} \widetilde{v}(\widetilde{x}, \overline{x}), \quad \overline{x} \geq \ell, \tag{IB.9}$$

where $\mathcal{A}(\overline{x})$ is given in Lemma 2. Moreover, $\mathcal{A}(\overline{x})$ is increasing in $\overline{x} \geq \ell$.

The existence of equilibrium boils down to finding the fixed point of $\mathcal{A}(\cdot)$. The next theorem proves the existence of an MPE.

Theorem 4. *Suppose Assumption 1 holds. There exists $\overline{x}^* > \ell$ and $\widetilde{x}^* \in \mathbb{U}(\overline{x}^*)$, such that $\overline{x}^* = \mathcal{A}(\overline{x}^*) = \widetilde{v}(\widetilde{x}^*, \overline{x}^*)$, and the strategy S with a default region $\{(Y, X) : X/Y > \overline{x}^*\}$ and target $\widetilde{x}^*$ for X_t/Y_t is an MPE. Furthermore, there exists a best equilibrium which yields the highest firm value in the set $\mathcal{S}_x$.*

IB.2 Solving for the best MPE

In this section, we show that the best MPE can be solved by choosing the optimal recapitalization target debt-EBIT ratio to maximize firm value given by (IB.3).

Define the set of feasible recapitalization target debt-EBIT ratios as follows:

$$\widetilde{\mathbb{U}} = \{\widetilde{x} \geq 0 : \exists\ \overline{x} \geq \widetilde{x},\ \text{s.t. (IB.3) holds }\}. \tag{IB.10}$$

Proposition 3. *Suppose Assumption 1 holds. We have*

$$\widetilde{\mathbb{U}} = [0, \widetilde{x}_{max}], \tag{IB.11}$$

for some $\widetilde{x}_{max} = \max_{z\in[0,1]} z f_0(z) > 0$. For any $\widetilde{x} \in \widetilde{\mathbb{U}}$ and $\overline{x} \geq \widetilde{x}$ satisfying (IB.3)*, we have*

$$\widetilde{x} = z f_0(z), \quad \overline{x} = f_0(z), \tag{IB.12}$$

where $z = \widetilde{x}/\overline{x} \in [0, 1]$, and $f_0(z)$ is defined by

$$\begin{aligned} &f_0(z) = f_1(z) \qquad \text{if } \mu \leq 0, \\ &f_0(z) = f_1(z)\mathbf{1}_{z\leq z_0} + f_2(z)\mathbf{1}_{z>z_0} \qquad \text{if } \mu > 0. \end{aligned} \tag{IB.13}$$

In (IB.13)*, $f_1(z)$ and $f_2(z)$ are given by*

$$f_1(z) = \frac{(1-\tau)\Big[1 + \lambda\ell\int_0^z ZdF(Z)\Big]}{r - g - \tau z[r + \lambda F(z)] + \lambda\int_0^z ZdF(Z)} \quad \text{and} \quad f_2(z) = \frac{1 + \lambda\ell\int_0^z ZdF(Z)}{r - g + \lambda\int_0^z ZdF(Z)},$$

respectively, and $z_0 \in (0, 1)$ is determined by the equation $r - g = rz_0 + \lambda\int_0^{z_0}(z_0 - Z)dF(Z)$ and satisfies $f_1(z_0) = f_2(z_0)$.

Theorem 5. *Under Assumption 1, the best equilibrium in Theorem 4 is characterized by the default boundary $\overline{x}_* > \ell$ and the target $\widetilde{x}_*$, where*

$$\overline{x}_* = \max_{z\in[0,1]} f_0(z) \quad \text{and} \quad \widetilde{x}_* = \overline{x}_* z_*. \tag{IB.14}$$

Here, $f_0(z)$ is given in Proposition 3 and z_* is any maximum point of $f_0(z)$ in the region $[0, 1]$.

IB.3 Uniqueness of the Equilibrium Strategy

In this section, we prove the uniqueness of the equilibrium under the following assumption[47].

Assumption 3. *$zF'(z)/F(z)$ is nondecreasing in $z \in [0, 1]$.*

Proposition 4. *Under Assumptions 1 and 3,*

(i) there exists a unique $z_* \in [0, 1]$, such that $f_0(z_*) = \max_{z\in[0,1]} f_0(z)$ in (IB.14)*;*

(ii) the unique $\overline{x}$ satisfying $\mathcal{A}(\overline{x}) = \overline{x}$ is given by $\overline{x} = \overline{x}_*$ defined in (IB.14)*, where $\mathcal{A}(\overline{x})$ is given in* (IB.9)*.*

As a result, there exists a unique equilibrium in the set $\mathcal{S}_x$.

Proposition 4-(i) implies that there exists a unique target debt-EBIT ratio $\widetilde{x}_* = \overline{x}_* z_*$ achieving the best equilibrium value $\overline{x}_*$; Proposition 4-(ii) proves that $\overline{x}_*$ is the unique equilibrium firm value.

[47] For example, Assumption 3 holds for the power law $F(z) = z^\beta$, where $\beta > 0$.

IB.4 Proofs for the Costless Equity Issuance Case

Proof of Lemma 2. Substituting (9) into (10), we obtain the following expression for the equity value

$$P(Y_t, X_{t-}) = \mathbb{E}_t\Big[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)}\Big[(Y_s - (r+\eta_s)X_s - \Theta_s)\, ds + dX_s\Big]\Big], \tag{IB.15}$$

Using $d(e^{-rs}X_s) = e^{-rs}[-rX_s ds + dX_s]$ and the fact that the firm neither makes a payout nor issues equity when it defaults, we have

$$\mathbb{E}_t\Big[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)}\Big[-rX_s ds + dX_s\Big]\Big] = \mathbb{E}_t\Big[e^{-r(T^{\mathcal{D}}-t)}X_{T^{\mathcal{D}}-} - X_{t-}\Big]. \tag{IB.16}$$

Substituting the above into (IB.15), we obtain

$$P(Y_t, X_{t-}) = \mathbb{E}_t\Big[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)}\Big[(Y_s - \eta_s X_s - \Theta_s)\, ds\Big] + e^{-r(T^{\mathcal{D}}-t)}X_{T^{\mathcal{D}}-}\Big] - X_{t-}. \tag{IB.17}$$

It follows that equity value $P(Y_t, X_{t-}) = P(Y_t, 0) - X_{t-}$ for any $X_{t-} < P(Y_t, 0)$ and $P(Y_t, X_{t-}) = 0$ for any $X_{t-} \geq P(Y_t, 0)$. Thus, firm value $V(Y_t, X_{t-}) = P(Y_t, X_{t-}) + X_{t-} = P(Y_t, 0)$ for any $X_{t-} < P(Y_t, 0)$.

Denote $X_s/Y_s = x_s$. We have

$$P(Y_t, 0) = \mathbb{E}_t\Big[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)}\Big[\big(1 - \eta\big(x_s;\overline{x}\big)x_s - \theta(c_s)\big)\, Y_s ds\Big] + e^{-r(T^{\mathcal{D}}-t)}x_{T^{\mathcal{D}}-}Y_{T^{\mathcal{D}}-}\Big]. \tag{IB.18}$$

where $\eta\big(x;\overline{x}\big)$ is given in (A.7), $\theta(c)$ is given in (5) and $c_s = rx_s + \eta\big(x_s;\overline{x}\big)x_s$.

Let $G_T = \Pi_{s\in\{s\in[t,T]:d\mathcal{J}_s=1\}}Z_s$ and $\widetilde{G}_T = e^{-\frac{1}{2}\sigma^2(T-t)+\sigma(\mathcal{B}_T-\mathcal{B}_t)}$, so that $Y_T = Y_t e^{\mu(T-t)}G_T\widetilde{G}_T$. We define the new probability measure $\widetilde{\mathbb{P}}(D) = \mathbb{E}_t[\mathbf{1}_D\widetilde{G}_T]$ for any event $D \in \mathcal{F}_T$, $T \geq t$, and denote by $\widetilde{\mathbb{E}}$ the expectation under the new probability measure $\widetilde{\mathbb{P}}$. Then we can derive from (IB.18) that

$$P(Y_t, 0) = Y_t\widetilde{\mathbb{E}}_t\Big[\int_t^{T^{\mathcal{D}}} e^{-(r-\mu)(s-t)}\Big[\big(1 - \eta\big(x_s;\overline{x}\big)x_s - \theta(c_s)\big)\, G_s ds\Big] + e^{-(r-\mu)(T^{\mathcal{D}}-t)}x_{T^{\mathcal{D}}-}G_{T^{\mathcal{D}}-}\Big]. \tag{IB.19}$$

By the homogeneity of (IB.19), we conclude that $P(Y_t, 0) = Y_t P(1, 0)$. Moreover, since $P(1, 0)$ is the equity value under the optimal strategy, $\{x_s\}_{s<T^{\mathcal{D}}}$ is the optimal solution to the following problem:

$$\begin{aligned} &P(1,0) \\ &= \max_{\{\widetilde{x}_s\}_{s<T^{\mathcal{D}}}} \widetilde{\mathbb{E}}_t\Big[\int_t^{T^{\mathcal{D}}} e^{-(r-\mu)(s-t)}\Big[\big(1 - \eta\big(\widetilde{x}_s;\overline{x}\big)\widetilde{x}_s - \theta(c_s)\big)\, G_s ds\Big] + e^{-(r-\mu)(T^{\mathcal{D}}-t)}\widetilde{x}_{T^{\mathcal{D}}-}G_{T^{\mathcal{D}}-}\Big]. \end{aligned} \tag{IB.20}$$

Since $\mathcal{J}_s$ is a Poisson process with a constant arrival rate, it is optimal to keep x_s constant for all $s < T^{\mathcal{D}}$. Let $\mathcal{A}(\overline{x}) = P(1, 0)$. We have $P(Y, X) = \mathcal{A}(\overline{x})Y - X$ for $X < \mathcal{A}(\overline{x})Y$, and $P(Y, X) = 0$ for $X \geq \mathcal{A}(\overline{x})Y$. Then (IB.6) holds.

Because $X_s = 0$ for any $s \geq t$ and $T^{\mathcal{D}} = \infty$ is a suboptimal strategy starting from $X_t = 0$, we have

$$P(Y_t, 0) \geq \mathbb{E}_t\Big[\int_t^{\infty} e^{-r(s-t)}(1-\tau)Y_s\Big] = \frac{1-\tau}{r-g}Y_t > \ell Y_t. \tag{IB.21}$$

It follows that $\mathcal{A}(\overline{x}) > \ell$.

In what follows below, if a jump arrives at time s ($d\mathcal{J}_s = 1$), we denote the jump size as Z_s, so that the firm's EBIT decreases from Y_{s-} to $Y_s = Z_s Y_{s-}$. Let $T_1 = \inf\{s \geq t : d\mathcal{J}_s = 1\}$. For the sake of contradiction, assume that $\mathcal{A}(\overline{x}) \geq \overline{x}$ when $\overline{x} > \frac{1}{r-g}$. When $x_t < \mathcal{A}(\overline{x})$, we have $x_s < \mathcal{A}(\overline{x})$ for any $s \in (t, T_1)$ and the firm won't default before time T_1. Thus, for any $X_{t-} < P(Y_t, 0)$,

$$\begin{aligned}
P(Y_t, X_{t-}) =& \mathbb{E}_t\Big[\int_t^{T_1-} e^{-r(s-t)}\Big[(Y_s - (r+\eta_s)X_s - \Theta_s)\,ds + dX_s\Big]\Big] \\
&+\mathbb{E}_t\big[e^{-r(T_1-t)}P(Z_{T_1}Y_{T_1-}, X_{T_1-})\big] \\
=& \mathbb{E}_t\Big[\int_t^{T_1-} e^{-r(s-t)}\,(Y_s - \eta_s X_s - \Theta_s)\,ds\Big] - X_{t-} \\
&+\mathbb{E}_t\Big[e^{-r(T_1-t)}(P(Z_{T_1}Y_{T_1-}, X_{T_1-}) + X_{T_1-})\Big],
\end{aligned} \tag{IB.22}$$

where the second equality uses $\mathbb{E}_t\Big[\int_t^{T_1-} e^{-r(s-t)}\Big[-rX_s ds + dX_s\Big]\Big] = \mathbb{E}_t\Big[e^{-r(T_1-t)}X_{T_1-} - X_{t-}\Big]$. Adding X_{t-} to both sides of (IB.22) and using $P(Y_t, X_{t-}) + X_{t-} = P(Y_t, 0)$ for $X_{t-} < P(Y_t, 0)$, we obtain

$$\begin{aligned}
P(Y_t, 0) =& \mathbb{E}_t\Big[\int_t^{T_1-} e^{-r(s-t)}\,(Y_s - \eta_s X_s - \Theta_s)\,ds\Big] + \mathbb{E}_t\Big[e^{-r(T_1-t)}(P(Z_{T_1}Y_{T_1-}, X_{T_1-}) + X_{T_1-})\Big] \\
\leq& \mathbb{E}_t\Big[\int_t^{T_1-} e^{-r(s-t)}(Y_s - \eta_s X_s)ds\Big] \\
&+\mathbb{E}_t\big[e^{-r(T_1-t)}(P(Z_{T_1}Y_{T_1-}, X_{T_1-}) + X_{T_1-})\mathbf{1}_{P(Z_{T_1}Y_{T_1-},0) > X_{T_1-}}\big] \\
&+\mathbb{E}_t\big[e^{-r(T_1-t)}(P(Z_{T_1}Y_{T_1-}, X_{T_1-}) + X_{T_1-})\mathbf{1}_{P(Z_{T_1}Y_{T_1-},0) \leq X_{T_1-}}\big] \\
=& \mathbb{E}_t\Big[\int_t^{T_1-} e^{-r(s-t)}(Y_s - \eta_s X_s)ds\Big] + \mathbb{E}_t\big[e^{-r(T_1-t)}P(Z_{T_1}Y_{T_1-}, 0)\mathbf{1}_{P(Z_{T_1}Y_{T_1-},0) > X_{T_1-}}\big] \\
&+\mathbb{E}_t\big[e^{-r(T_1-t)}X_{T_1-}\mathbf{1}_{P(Z_{T_1}Y_{T_1-},0) \leq X_{T_1-}}\big] \\
=& \mathbb{E}_t\Big[\int_t^{T_1-} e^{-r(s-t)}(Y_s - \eta_s X_s)ds\Big] + \mathbb{E}_t\big[e^{-r(T_1-t)}P(Z_{T_1}Y_{T_1-}, 0)\big] \\
&+\mathbb{E}_t\Big[e^{-r(T_1-t)}\big(X_{T_1-} - P(Z_{T_1}Y_{T_1-}, 0)\big)\mathbf{1}_{P(Z_{T_1}Y_{T_1-},0) \leq X_{T_1-}}\Big],
\end{aligned} \tag{IB.23}$$

where the second equality uses $P(Y, X) = P(Y, 0) - X$ for any $X < P(Y, 0)$ and $P(Y, X) = 0$ for any $X \geq P(Y, 0)$. Using (A.6), we have

$$\begin{aligned}
&\mathbb{E}_t\Big[\int_t^{T_1-} e^{-r(s-t)}\eta_s X_s ds\Big] = \mathbb{E}_t\Big[\int_t^{T_1} e^{-r(s-t)}\eta_{s-}X_{s-}ds\Big] \\
=& \mathbb{E}_t\Big[\int_t^{T_1} e^{-r(s-t)}\lambda\mathbb{E}_{s-}\Big[(X_{s-} - \ell Z_s Y_{s-})\mathbf{1}_{\overline{x}Z_sY_{s-} \leq X_{s-}}\Big]ds\Big] \\
\geq& \lambda\mathbb{E}_t\Big[\int_t^{T_1} e^{-r(s-t)}(X_{s-} - \ell Z_s Y_{s-})\mathbf{1}_{P(Z_sY_{s-},0) \leq X_{s-}}ds\Big] \\
=& \lambda\mathbb{E}_t\Big[\int_t^{\infty}\int_t^{T} e^{-r(s-t)}\lambda e^{-\lambda(T-t)}(X_{s-} - \ell Z_s Y_{s-})\mathbf{1}_{P(Z_sY_{s-},0) \leq X_{s-}}dsdT\Big] \\
=& \lambda\mathbb{E}_t\Big[\int_t^{\infty}\int_s^{\infty} e^{-r(s-t)}\lambda e^{-\lambda(T-t)}dT(X_{s-} - \ell Z_s Y_{s-})\mathbf{1}_{P(Z_sY_{s-},0) \leq X_{s-}}ds\Big] \\
=& \lambda\mathbb{E}_t\Big[\int_t^{\infty} e^{-(r+\lambda)(s-t)}(X_{s-} - \ell Z_s Y_{s-})\mathbf{1}_{P(Z_sY_{s-},0) \leq X_{s-}}ds\Big] \\
=& \mathbb{E}_t\Big[e^{-r(T_1-t)}\big(X_{T_1-} - \ell Z_{T_1}Y_{T_1-}\big)\mathbf{1}_{P(Z_{T_1}Y_{T_1-},0) \leq X_{T_1-}}\Big],
\end{aligned} \tag{IB.24}$$

where the inequality uses $\mathcal{A}(\overline{x}) \geq \overline{x} > \frac{1}{r-g} > \ell$ and $P(Z_s Y_{s-}, 0) = \mathcal{A}(\overline{x}) Z_s Y_{s-}$. Using (IB.24), we have

$$\begin{aligned}
&\mathbb{E}_t\left[\int_t^{T_1-} e^{-r(s-t)}(-\eta_s X_s)ds\right] + \mathbb{E}_t\left[e^{-r(T_1-t)}(X_{T_1-} - P(Z_{T_1}Y_{T_1-},0))\mathbf{1}_{P(Z_{T_1}Y_{T_1-},0)\leq X_{T_1-}}\right] \\
=&\mathbb{E}_t\left[e^{-r(T_1-t)}\left(\ell Z_{T_1}Y_{T_1-} - P(Z_{T_1}Y_{T_1-},0)\right)\mathbf{1}_{P(Z_{T_1}Y_{T_1-},0)\leq X_{T_1-}}\right] \leq 0,
\end{aligned} \tag{IB.25}$$

where the inequality uses (IB.21).

Substituting $P(Y,0) = \mathcal{A}(\overline{x})Y$ and (IB.25) into (IB.23), we obtain

$$\mathcal{A}(\overline{x})Y_t \leq \mathbb{E}_t\left[\int_t^{T_1-} e^{-r(s-t)}Y_s ds\right] + \mathbb{E}_t\left[e^{-r(T_1-t)}\mathcal{A}(\overline{x})Z_{T_1}Y_{T_1-}\right]. \tag{IB.26}$$

We have

$$\begin{aligned}
&\mathbb{E}_t\left[\int_t^{T_1-} e^{-r(s-t)}Y_s ds\right] \\
=&\int_t^\infty \int_t^T \lambda e^{\lambda(T-t)} e^{-(r-\mu)(s-t)} Y_t ds dT \\
=&\int_t^\infty \int_s^\infty \lambda e^{-\lambda(T-t)} dT e^{-(r-\mu)(s-t)} Y_t ds \\
=&\int_t^\infty e^{-\lambda(s-t)} e^{-(r-\mu)(s-t)} Y_t ds = \frac{Y_t}{r+\lambda-\mu},
\end{aligned} \tag{IB.27}$$

where $r + \lambda - \mu = r - g + \lambda\mathbb{E}[Z] > 0$. In addition, we have

$$\begin{aligned}
&\mathbb{E}_t\left[e^{-r(T_1-t)}\mathcal{A}(\overline{x})Z_{T_1}Y_{T_1-}\right] \\
=&\mathcal{A}(\overline{x})\mathbb{E}[Z]Y_t \int_t^\infty \lambda e^{-\lambda(T-t)} e^{-(r-\mu)(T-t)} dT = \mathcal{A}(\overline{x})\lambda\mathbb{E}[Z]\frac{Y_t}{r+\lambda-\mu}.
\end{aligned} \tag{IB.28}$$

Substituting (IB.27) and (IB.28) into (IB.26), we obtain

$$\mathcal{A}(\overline{x}) \leq \frac{1}{r+\lambda-\mu} + \mathcal{A}(\overline{x})\lambda\mathbb{E}[Z]\frac{1}{r+\lambda-\mu}. \tag{IB.29}$$

It follows that $\mathcal{A}(\overline{x}) \leq \frac{1}{r-g}$. Using $\mathcal{A}(\overline{x}) \geq \overline{x}$ when $\overline{x} > \frac{1}{r-g}$, we arrive at a contradiction. Thus, $\mathcal{A}(\overline{x}) < \overline{x}$ when $\overline{x} > \frac{1}{r-g}$. ■

Proof of Lemma 3. Let $\widetilde{p}(x;\widetilde{x},\overline{x})$ denote the equity value under a target debt-EBIT ratio $\widetilde{x}$ and a conjectured default boundary $\overline{x}$ by creditors, where $\overline{x} \geq \widetilde{x} \geq 0$ and $x \geq 0$. Let $\widetilde{v}(\widetilde{x},\overline{x}) = \inf\{x > 0 : \widetilde{p}(x;\widetilde{x},\overline{x}) = 0\}$. Since $h_0 = h_1 = 0$, we have $\widetilde{p}_x(x;\widetilde{x},\overline{x}) = -1$ for $x \leq \widetilde{v}(\widetilde{x},\overline{x})$, and firm value $\widetilde{p}(x;\widetilde{x},\overline{x}) + x$ is independent of x. As a result, $\widetilde{p}(x;\widetilde{x},\overline{x}) + x = [\widetilde{p}(x;\widetilde{x},\overline{x}) + x]|_{x=v(\widetilde{x},\overline{x})} = \widetilde{v}(\widetilde{x},\overline{x})$ for any $x \leq \widetilde{v}(\widetilde{x},\overline{x})$.

In what follows, we simplify the notation by writing $\widetilde{p}(x;\widetilde{x},\overline{x})$ as $\widetilde{p}(x)$, and $\widetilde{v}(\widetilde{x},\overline{x})$ as $\widetilde{v}$. Substituting $\widetilde{p}_x(x) = -1$ and $\widetilde{p}_{xx}(x) = 0$ into the integro-differential equation (21) which holds at $x = \widetilde{x}$ for $\widetilde{p}(x)$, we obtain:

$$(r-\mu)\,\widetilde{p}(\widetilde{x}) = -\left[c(\widetilde{x};\overline{x}) + \theta(c(\widetilde{x};\overline{x})) - 1 - \mu\widetilde{x}\right] + \lambda\mathbb{E}\left[Z\widetilde{p}(\widetilde{x}/Z) - \widetilde{p}(\widetilde{x})\right], \tag{IB.30}$$

where $c(x;\overline{x}) = rx + \lambda \int_0^{x/\overline{x}} (x - \ell Z)\, dF(Z)$ represents the expected interest payment given the conjectured default boundary $\overline{x}$ of creditors. For any $x \leq \widetilde{v}$,

$$
\begin{aligned}
&\mathbb{E}\left[Z\widetilde{p}(x/Z) - \widetilde{p}(x)\right] \\
=&\int_{x/\widetilde{v}}^{1} [Z\widetilde{p}(x/Z) - \widetilde{p}(x)]dF(Z) + \int_0^{x/\widetilde{v}} (-\widetilde{p}(x))dF(Z) \\
=&\int_{x/\widetilde{v}}^{1} [Z\widetilde{v} - \widetilde{v}\,]dF(Z) + \int_0^{x/\widetilde{v}} (x - \widetilde{v})dF(Z) \\
=&\int_0^{1} [Z-1]\widetilde{v}dF(Z) - \int_0^{x/\widetilde{v}} [Z-1]\widetilde{v}dF(Z) + \int_0^{x/\widetilde{v}} (x - \widetilde{v})dF(Z) \\
=&\mathbb{E}[Z-1]\widetilde{v} + \int_0^{x/\widetilde{v}} (x - Z\widetilde{v})dF(Z),
\end{aligned}
\tag{IB.31}
$$

where the first and second equalities use $\widetilde{p}(x) + x = \widetilde{v}$ for any $x \leq \widetilde{v}$ and $\widetilde{p}(x) = 0$ for $x > \widetilde{v}$. Substituting (IB.31) into (IB.30), and using $\widetilde{p}(\widetilde{x}) = \widetilde{v} - \widetilde{x}$, we have

$$
(r-g)\widetilde{v} = r\widetilde{x} + 1 - \theta(c(\widetilde{x};\overline{x})) - c(\widetilde{x};\overline{x}) + \lambda \int_0^{\widetilde{x}/\widetilde{v}} (\widetilde{x} - Z\widetilde{v})dF(Z). \tag{IB.32}
$$

Then we obtain equation (IB.8) about $\widetilde{v} \geq \widetilde{x}$.

Denote the left side of (IB.8) as $\mathcal{R}(\widetilde{x},\widetilde{v})$, and the right side of (IB.8) as $\mathcal{H}(\widetilde{x},\overline{x})$. For any $\widetilde{v} \geq \widetilde{x} \geq 0$, we have the following partial derivatives:[48]

$$
\mathcal{R}_{\widetilde{x}}(\widetilde{x},\widetilde{v}) = -\lambda F\Big(\frac{\widetilde{x}}{\widetilde{v}}\Big), \tag{IB.33}
$$

$$
\mathcal{R}_{\widetilde{v}}(\widetilde{x},\widetilde{v}) = r - g + \lambda \int_0^{\widetilde{x}/\widetilde{v}} Z dF(Z), \tag{IB.34}
$$

$$
\mathcal{H}_{\widetilde{x}}(\widetilde{x},\overline{x}) = r - \Big[\mathbf{1}_{c(\widetilde{x};\overline{x})\leq 1}(1-\tau) + \mathbf{1}_{c(\widetilde{x};\overline{x})>1}\Big]\Big[r + \lambda\frac{\widetilde{x}}{\overline{x}}\Big(1 - \frac{\ell}{\overline{x}}\Big)F'\Big(\frac{\widetilde{x}}{\overline{x}}\Big) + \lambda F\Big(\frac{\widetilde{x}}{\overline{x}}\Big)\Big], \tag{IB.35}
$$

$$
\mathcal{H}_{\overline{x}}(\widetilde{x},\overline{x}) = \Big[\mathbf{1}_{c(\widetilde{x};\overline{x})\leq 1}(1-\tau) + \mathbf{1}_{c(\widetilde{x};\overline{x})>1}\Big]\lambda\frac{\widetilde{x}^2}{\overline{x}^2}\Big(1 - \frac{\ell}{\overline{x}}\Big)F'\Big(\frac{\widetilde{x}}{\overline{x}}\Big). \tag{IB.36}
$$

Since $\lim_{\widetilde{v}\to\infty} \mathcal{R}(\widetilde{x},\widetilde{v}) = \infty$ and $\mathcal{R}_{\widetilde{v}}(\widetilde{x},\widetilde{v}) \geq r - g > 0$, equation (IB.8) admits a unique solution $\widetilde{v} \geq \widetilde{x}$ if and only if $\mathcal{R}(\widetilde{x},\widetilde{x}) \leq \mathcal{H}(\widetilde{x},\overline{x})$. We define the feasible set of target debt-EBIT ratios $\widetilde{x}$ as follows:

$$
\mathbb{U}(\overline{x}) = \{\widetilde{x} \in [0,\overline{x}] : \mathcal{R}(\widetilde{x},\widetilde{x}) \leq \mathcal{H}(\widetilde{x},\overline{x})\}. \tag{IB.37}
$$

Since $\mathcal{R}(\widetilde{x},\widetilde{x}) \leq \mathcal{H}(\widetilde{x},\overline{x})$ is equivalent to $1 + \mu\widetilde{x} - \theta(c(\widetilde{x};\overline{x})) - c(\widetilde{x};\overline{x}) \geq 0$, we obtain (IB.7).

The set $\mathbb{U}(\widetilde{x})$ is nonempty because $[0,\widetilde{x}_1] \subset \mathbb{U}(\widetilde{x})$, where $\widetilde{x}_1 = \inf\{\widetilde{x} \in [0,\overline{x}] : 1 + \mu\widetilde{x} - \theta(c(\widetilde{x};\overline{x})) - c(\widetilde{x};\overline{x}) \leq 0\} > 0$ since $c(0;\overline{x}) = 0$, and $\theta(0) = \tau < 1$. Moreover, since $c(\widetilde{x};\overline{x})$ is decreasing in $\overline{x}$, $\theta(c(\widetilde{x};\overline{x})) + c(\widetilde{x};\overline{x}) = \mathbf{1}_{c(\widetilde{x};\overline{x})\leq 1}[\tau + c(\widetilde{x};\overline{x})(1-\tau)] + c(\widetilde{x};\overline{x})\mathbf{1}_{c(\widetilde{x};\overline{x})>1}$ is also decreasing in $\overline{x}$. It follows that $\mathbb{U}(\overline{x})$ is larger for larger $\overline{x}$: $\mathbb{U}(\overline{x}) \subseteq \mathbb{U}(\overline{x} + \Delta x)$ for $\Delta x > 0$.

In sum, for any $\overline{x} \geq \ell$ and $\widetilde{x} \in \mathbb{U}(\overline{x})$, firm value $\widetilde{v}(\widetilde{x},\overline{x})$ is uniquely determined by

$$
\mathcal{R}(\widetilde{x},\widetilde{v}(\widetilde{x},\overline{x})) = \mathcal{H}(\widetilde{x},\overline{x}). \tag{IB.38}
$$

For any $\widetilde{x} \in [0,\overline{x}] \setminus \mathbb{U}(\overline{x})$ and $\widetilde{v} \geq \widetilde{x}$, $\mathcal{R}(\widetilde{x},\widetilde{v}) \geq \mathcal{R}(\widetilde{x},\widetilde{x}) > \mathcal{H}(\widetilde{x},\overline{x})$, and thus (IB.8) has no solution $\widetilde{v} \geq \widetilde{x}$. It follows that the strategy S which maintains $X_s/Y_s = \widetilde{x}$ is not feasible given creditor's belief $\widetilde{S}$ with a default region $\{(Y,X) : X/Y > \overline{x}\}$.

[48] For $\widetilde{x}$ satisfying $c(\widetilde{x};\overline{x}) = 1$, we have $c(x;\overline{x}) < 1$ for $x < \widetilde{x}$, and $\mathcal{H}_{\widetilde{x}}(\widetilde{x},\overline{x})$ refers the left partial derivative $\mathcal{H}_{\widetilde{x}}(\widetilde{x}-,\overline{x})$. Similarly, we have $c(\widetilde{x};x) < 1$ for $x > \overline{x}$, and $\mathcal{H}_{\overline{x}}(\widetilde{x},\overline{x})$ refers the right partial derivative $\mathcal{H}_{\overline{x}}(\widetilde{x},\overline{x}+)$.

Using (IB.34) and (IB.36), we have $\mathcal{R}_{\widetilde{v}}(\widetilde{x}, \widetilde{v}) > 0$ and $\mathcal{H}_{\overline{x}}(\widetilde{x}, \overline{x}) > 0$. Then it follows from (IB.38) that $\widetilde{v}(\widetilde{x}, \overline{x})$ is increasing in $\overline{x} \geq \ell$. For any $\Delta x > 0$, and $\widetilde{x} \in \mathbb{U}(\overline{x}) \subseteq \mathbb{U}(\overline{x} + \Delta x)$, $\mathcal{A}(\overline{x} + \Delta x) \geq \widetilde{v}(\widetilde{x}, \overline{x} + \Delta x) > \widetilde{v}(\widetilde{x}, \overline{x})$, which implies that $\mathcal{A}(\overline{x} + \Delta x) > \max_{\widetilde{x} \in \mathbb{U}(\overline{x})} \widetilde{v}(\widetilde{x}, \overline{x}) = \mathcal{A}(\overline{x})$. That is, $\mathcal{A}(\overline{x})$ is increasing in $\overline{x} \geq \ell$.

According to Lemma 2, the firm's optimal strategy is to maintain X_t/Y_t at a constant level. Since $\widetilde{v}(\widetilde{x}, \overline{x})$ denotes the firm value under a target debt-EBIT ratio $\widetilde{x}$ and the conjectured default boundary $\overline{x}$ by creditors, we then obtain (IB.9).

For each $\overline{x} \geq \widetilde{x} \geq 0$, since $\widetilde{v} = \widetilde{v}(\widetilde{x}, \overline{x})$ satisfies equation (IB.8), when $\widetilde{v} = \widetilde{v}(\widetilde{x}, \overline{x}) \geq \overline{x} \geq \ell$, we have

$$\begin{aligned}
(r-g)\widetilde{v} =& \lambda \int_0^{\widetilde{x}/\widetilde{v}} (\widetilde{x} - Z\widetilde{v}) dF(Z) + \mathcal{H}(\widetilde{x}, \overline{x}) \\
\leq& \lambda \int_0^{\widetilde{x}/\widetilde{v}} (\widetilde{x} - Z\widetilde{v}) dF(Z) + \mathcal{H}(\widetilde{x}, \widetilde{v}) \\
=& \lambda \int_0^{\widetilde{x}/\widetilde{v}} (\widetilde{x} - Z\widetilde{v}) dF(Z) + 1 - \theta(c(\widetilde{x}; \widetilde{v})) - \lambda \int_0^{\widetilde{x}/\widetilde{v}} (\widetilde{x} - \ell Z) \, dF(Z) \\
=& 1 - \theta(c(\widetilde{x}; \widetilde{v})) + \lambda \int_0^{\widetilde{x}/\widetilde{v}} (\ell Z - Z\widetilde{v}) dF(Z) \leq 1,
\end{aligned} \tag{IB.39}$$

where the first inequality uses $\mathcal{H}_{\overline{x}}(\widetilde{x}, \overline{x}) \geq 0$ for $\overline{x} \geq \ell$, the second inequality uses $\widetilde{v} \geq \ell$.

Thus, we derive from (IB.39) that

$$\widetilde{v}(\widetilde{x}, \overline{x}) \leq \frac{1}{r-g} \quad \text{if } \widetilde{v}(\widetilde{x}, \overline{x}) \geq \overline{x}. \tag{IB.40}$$

Using (IB.40), we can see that $\widetilde{v}(\widetilde{x}, \overline{x}) < \overline{x}$ for any $\overline{x} > \frac{1}{r-g}$, $\widetilde{x} \in \mathbb{U}(\overline{x})$. ■

Proof of Theorem 4.

Step 1: We prove that there exists $\overline{x} \in (\overline{x}_0, \overline{x}_1)$, such that $\mathcal{A}(\overline{x}) = \overline{x}$ if $\mathcal{A}(\overline{x}_0) > \overline{x}_0$ and $\mathcal{A}(\overline{x}_1) < \overline{x}_1$.

Let $\mathbb{A} = \{x \in [\overline{x}_0, \overline{x}_1] : \mathcal{A}(x) \geq x\}$. Since $\mathcal{A}(\overline{x}_0) > \overline{x}_0$, $\mathbb{A}$ is nonempty.

Let $\overline{x} = \sup \mathbb{A}$. We have $\overline{x}_0 \leq \overline{x} \leq \overline{x}_1$. If $\overline{x} \notin \mathbb{A}$, then $\mathcal{A}(\overline{x}) < \overline{x}$. By the definition of $\overline{x} = \sup \mathbb{A}$ and $\overline{x} \notin \mathbb{A}$, there exists an increasing sequence $x_n \in \mathbb{A}$ such that $\lim_{n\to\infty} x_n = \overline{x}$. Thus $\mathcal{A}(x_n) \geq x_n$. Recall that we have shown that $\mathcal{A}(x)$ is increasing in x in the proof of Lemma 3. Then we have

$$\mathcal{A}(\overline{x}) \geq \mathcal{A}(x_n) \geq x_n. \tag{IB.41}$$

Sending n to infinity in the above inequality, we obtain $\mathcal{A}(\overline{x}) \geq \lim_{n\to\infty} x_n = \overline{x}$. Then we arrive at a contradiction.

Thus $\overline{x} \in \mathbb{A}$, and then $\mathcal{A}(\overline{x}) \geq \overline{x}$. Since $\mathcal{A}(\overline{x}_1) < \overline{x}_1$, we have $\overline{x} < \overline{x}_1$. By the definition of $\overline{x} = \sup \mathbb{A}$, for any small $\varepsilon \in (0, \overline{x}_1 - \overline{x})$, we have $\overline{x} + \varepsilon \notin \mathbb{A}$ and then $\mathcal{A}(\overline{x} + \varepsilon) < \overline{x} + \varepsilon$. Since $\mathcal{A}(x)$ is increasing in x, we have

$$\overline{x} \leq \mathcal{A}(\overline{x}) \leq \mathcal{A}(\overline{x} + \varepsilon) < \overline{x} + \varepsilon. \tag{IB.42}$$

Sending ε to 0 in the above inequality, we obtain $\mathcal{A}(\overline{x}) = \overline{x}$.

Step 2: We show that there exist $\overline{x}^* > \ell$ and $\widetilde{x}^* \in \mathbb{U}(\overline{x}^*)$, such that $\widetilde{v}(\widetilde{x}^*, \overline{x}^*) = \overline{x}^* \geq \widetilde{v}(\widetilde{x}, \overline{x}^*)$ for any $\widetilde{x} \in \mathbb{U}(\overline{x}^*)$.

Using Lemma 2, we have $\mathcal{A}(\overline{x}) < \overline{x}$ for any $\overline{x} > \frac{1}{r-g}$. In addition, since $\mathcal{R}(0, \widetilde{v}) = (r-g)\widetilde{v}$ and $\mathcal{H}(0, \overline{x}) = 1 - \tau$, we derive from (IB.38) that $\widetilde{v}(0, \overline{x}) = \frac{1-\tau}{r-g} > \ell$ under Assumption 1. It follows that $\mathcal{A}(\overline{x}) \geq \widetilde{v}(0, \overline{x}) > \ell$ for any $\overline{x} \geq \ell$. Using $\mathcal{A}(\ell) - \ell > 0$, $\mathcal{A}(\overline{x}) - \overline{x} < 0$ for any $\overline{x} > \frac{1}{r-g}$, and the result proved in step 1, we conclude that there exists $\overline{x}^* \in (\ell, \frac{1}{r-g}]$, such that $\mathcal{A}(\overline{x}^*) = \overline{x}^*$. Let $\widetilde{x}^* = \arg\max_{\widetilde{x} \in \mathbb{U}(\overline{x}^*)} \widetilde{v}(\widetilde{x}, \overline{x}^*)$. Then we have $\widetilde{v}(\widetilde{x}^*, \overline{x}^*) = \mathcal{A}(\overline{x}^*) = \overline{x}^* \geq \widetilde{v}(\widetilde{x}, \overline{x}^*)$ for any $\widetilde{x} \in \mathbb{U}(\overline{x}^*)$.

Step 3: We prove that $\mathcal{L}^{\overline{x}^*}\widetilde{p}(x) \leq 0$ for any $x \in [0, \overline{x}^*]$, where $\widetilde{p}(x) = \widetilde{p}(x; \widetilde{x}^*, \overline{x}^*)$ is the equity value under a target debt-EBIT ratio $\widetilde{x}^*$ and a conjectured default boundary $\overline{x}^*$ by creditors, and $\mathcal{L}^{\overline{x}}$ is defined in (B.7).

Since $\widetilde{p}(x) + x = \widetilde{v}(\widetilde{x}^*, \overline{x}^*) = \overline{x}^*$ for any $x \in [0, \overline{x}^*]$, we have

$$\begin{aligned}
&\mathcal{L}^{\overline{x}^*}\widetilde{p}(x)\\
=&(r-\mu)x - (r-\mu)\,\overline{x}^* - [c(x;\overline{x}^*) + \theta(c(x;\overline{x}^*)) - 1 - \mu x] + \lambda\mathbb{E}\left[Z\widetilde{p}(x/Z) - \widetilde{p}(x)\right]\\
=&rx + 1 - c(x;\overline{x}^*) - \theta(c(x;\overline{x}^*)) - (r-\mu)\,\overline{x}^* + \lambda\Big(\mathbb{E}[Z-1]\overline{x}^* + \int_0^{x/\overline{x}^*}(x - Z\overline{x}^*)dF(Z)\Big)\\
=&\mathcal{H}(x, \overline{x}^*) - \mathcal{R}(x, \overline{x}^*),
\end{aligned} \tag{IB.43}$$

where the second equality uses (IB.31) and $\widetilde{v}(\widetilde{x}^*, \overline{x}^*) = \overline{x}^*$, and the third equality uses $g = \mu - \lambda(1 - \mathbb{E}(Z))$. Using (IB.37), we conclude that for any $x \in [0, \overline{x}^*] \setminus \mathbb{U}(\overline{x}^*)$ and $\widetilde{v} \geq x$, $\mathcal{R}(x, \widetilde{v}) \geq \mathcal{R}(x, x) > \mathcal{H}(x, \overline{x}^*)$. In particular, $\mathcal{R}(x, \overline{x}^*) > \mathcal{H}(x, \overline{x}^*)$ for any $x \in [0, \overline{x}^*] \setminus \mathbb{U}(\overline{x}^*)$.

Using the result proved in step 2, for any $x \in \mathbb{U}(\overline{x}^*)$, we have $\widetilde{v}(x, \overline{x}^*) \leq \mathcal{A}(\overline{x}^*) = \overline{x}^*$. Since $\mathcal{R}_v(x, v) > 0$, we have $\mathcal{R}(x, \overline{x}^*) \geq \mathcal{R}(x, \widetilde{v}(x, \overline{x}^*)) = \mathcal{H}(x, \overline{x}^*)$ for any $x \in \mathbb{U}(\overline{x}^*)$. Then it follows from (IB.43) that $\mathcal{L}^{\overline{x}^*}\widetilde{p}(x) = \mathcal{H}(x, \overline{x}^*) - \mathcal{R}(x, \overline{x}^*) \leq 0$ for any $x \in [0, \overline{x}^*]$.

Since $\widetilde{p}(x) = \overline{x}^* - x$, for $x \in [0, \overline{x}^*]$, and $\overline{x}^* > \ell$, we have $\widetilde{p}(x) \geq \max\{\ell - x, 0\}$, $\widetilde{p}_x(x) = -1$ for $x \in [0, \overline{x}^*]$. Combining these with $\mathcal{L}^{\overline{x}^*}\widetilde{p}(x) \leq 0$ for any $x \in [0, \overline{x}^*]$, we can see that $\widetilde{p}(x)$ satisfies (IA.28) for $x \in [0, \overline{x}^*]$. In addition, note that $\widetilde{p}(x) = 0$ for $x \geq \overline{x}^*$ and $\widetilde{p}(x) > 0$ for $x \in [0, \overline{x}^*)$.

Let $\widetilde{P}(Y, X) = Y\widetilde{p}(X/Y)$, and let S denote the strategy with target debt-EBIT ratio $\widetilde{x}^*$, and default boundary $\overline{x}^*$. We can see that $\widetilde{P}(Y, X) = \mathcal{P}(Y, X; S, S)$ and $\widetilde{P}(Y, X)$ satisfies all conditions in Theorem 1. Hence, S is an MPE.

Step 4: We prove the existence of the best equilibrium.

For any MPE $S \in \mathcal{S}_x$, its default region is given by $\mathcal{D} = \{(Y, X) : X/Y > \overline{x}\}$ for some $\overline{x} \geq \ell$. Combining with Lemma 2, we can see any MPE $S \in \mathcal{S}_x$ is characterized by a default boundary $\overline{x} > \ell$ and a recapitalization target $\widetilde{x}$. As a result, the set of default boundaries in all equilibria $\in \mathcal{S}_x$ can be represented by

$$\mathbb{S}^* = \{\overline{x} > \ell : \mathcal{A}(\overline{x}) = \overline{x}\}. \tag{IB.44}$$

By applying Lemma 2, we obtain $\mathcal{A}(\overline{x}) < \overline{x}$ for any $\overline{x} > \frac{1}{r-g}$. It follows that $\overline{x} \leq \frac{1}{r-g}$ for any $\overline{x} \in \mathbb{S}^*$, and then $\overline{x}^* = \sup \mathbb{S}^* \leq \frac{1}{r-g}$.

Next, we prove that $\overline{x}^* \in \mathbb{S}^*$. For the sake of contradiction, assume that $\overline{x}^* \notin \mathbb{S}^*$ and then $\mathcal{A}(\overline{x}^*) \neq \overline{x}^*$. Then there exists a sequence $\{\overline{x}_n\}_{n\geq 1}$ in the set $\mathbb{S}^*$, such that $\overline{x}_n < \overline{x}^*$ and $\lim_{n\to\infty} \overline{x}_n = \overline{x}^*$. Because $\mathcal{A}(\cdot)$ is a nondecreasing function, we have $\mathcal{A}(\overline{x}^*) \geq \mathcal{A}(\overline{x}_n) = \overline{x}_n$. Sending n to infinity, we have $\mathcal{A}(\overline{x}^*) \geq \overline{x}^*$. Using $\mathcal{A}(\overline{x}^*) \neq \overline{x}^*$, we have $\mathcal{A}(\overline{x}^*) > \overline{x}^*$. Recalling Lemma 2 implies that $\mathcal{A}(\overline{x}) < \overline{x}$ for any $\overline{x} > \frac{1}{r-g}$, and using the result proved in step 1, we conclude that for $\overline{x}_2 > \frac{1}{r-g}$, there exists $\overline{x}_1 \in (\overline{x}^*, \overline{x}_2)$, such that $\mathcal{A}(\overline{x}_1) = \overline{x}_1$ so that $\overline{x}_1 \in \mathbb{S}^*$. Then we arrive at a contradiction as $\overline{x}^* = \sup \mathbb{S}^*$ and $\overline{x}_1 > \overline{x}^*$.

Therefore, we have $\overline{x}^* \in \mathbb{S}^*$. Let $\widetilde{x}^* = \arg\max_{\widetilde{x} \in \mathbb{U}(\overline{x}^*)} \widetilde{v}(\widetilde{x}, \overline{x}^*)$. Then the strategy S^* with target debt-EBIT ratio $\widetilde{x}^*$, and default boundary $\overline{x}^*$ is the MPE with the highest firm value $\overline{x}^*$. ■

Proof of Proposition 3.

Step 1: We characterize the firm value given by (IB.3).

Note that (IB.8) with $\widetilde{v} = \overline{x}$ is equivalent to (IB.3). Hence, $\widetilde{\mathbb{U}}$ given in (IB.10) is equivalent to

$$\widetilde{\mathbb{U}} = \{\widetilde{x} \geq 0 : \exists\ \overline{x} \geq \widetilde{x},\ \text{s.t. } \widetilde{v}(\widetilde{x}, \overline{x}) = \overline{x}\}, \tag{IB.45}$$

where $\widetilde{v}(\widetilde{x}, \overline{x})$ is defined in (IB.38). From Theorem 4, we know that $\widetilde{\mathbb{U}}$ is nonempty.

Substituting $\theta(c(\widetilde{x};\overline{x})) \leq \tau$ into (IB.3), we have

$$\begin{aligned}\overline{x} \geq & \frac{1}{r-g}\Big[1-\tau-\lambda(\overline{x}-\ell)\Big(\int_0^{\widetilde{x}/\overline{x}} ZdF(Z)\Big)\Big] \\ > & \ell+\frac{1}{r-g}\Big[-\lambda(\overline{x}-\ell)\Big(\int_0^{\widetilde{x}/\overline{x}} ZdF(Z)\Big)\Big],\end{aligned} \tag{IB.46}$$

where the second inequality uses Assumption 1. It follows that $\overline{x} > \ell$. Since $\overline{x} > \ell$, it follows from (IB.3) that $\overline{x} \leq \frac{1}{r-g}$. Thus, for any $\widetilde{x} \in \widetilde{\mathbb{U}}$ and $\overline{x} \geq \widetilde{x}$ satisfying (IB.3), we have

$$\ell < \overline{x} \leq \frac{1}{r-g}. \tag{IB.47}$$

Next, we give the closed-form characterization of any pair $(\widetilde{x}, \overline{x})$ satisfying $\widetilde{x} \in \widetilde{\mathbb{U}}$, $\overline{x} \geq \widetilde{x}$, and (IB.3).

We first consider the case $c(\widetilde{x};\overline{x}) \leq 1$. Using $c(\widetilde{x};\overline{x}) = r\widetilde{x}+\lambda\int_0^{\widetilde{x}/\overline{x}}(\widetilde{x}-\ell Z)\,dF(Z) = \widetilde{x}[r+\lambda F(\widetilde{x}/\overline{x})]-\ell\lambda\int_0^{\widetilde{x}/\overline{x}} Z\,dF(Z)$, and $\theta(c(\widetilde{x};\overline{x})) = \tau-\tau\Big[\widetilde{x}[r+\lambda F(\widetilde{x}/\overline{x})]-\ell\lambda\int_0^{\widetilde{x}/\overline{x}} Z\,dF(Z)\Big]$, we derive from (IB.3) that

$$\begin{aligned}(r-g)\overline{x} = & 1-\tau+\tau\Big[\widetilde{x}[r+\lambda F(\widetilde{x}/\overline{x})]-\ell\lambda\int_0^{\widetilde{x}/\overline{x}} Z\,dF(Z)\Big]-\lambda(\overline{x}-\ell)\Big(\int_0^{\widetilde{x}/\overline{x}} ZdF(Z)\Big) \\ = & (1-\tau)\Big[1+\lambda\ell\int_0^{\widetilde{x}/\overline{x}} ZdF(Z)\Big]+\tau\widetilde{x}[r+\lambda F(\widetilde{x}/\overline{x})]-\lambda\overline{x}\Big(\int_0^{\widetilde{x}/\overline{x}} ZdF(Z)\Big).\end{aligned}$$

Dividing both sides by $\overline{x}$ and letting $z = \widetilde{x}/\overline{x}$, we obtain

$$r-g = \frac{1-\tau}{\overline{x}}\Big[1+\lambda\ell\int_0^z ZdF(Z)\Big]+\tau z[r+\lambda F(z)]-\lambda\int_0^z ZdF(Z). \tag{IB.48}$$

Therefore $\overline{x} = f_1(z)$, where

$$f_1(z) = \frac{(1-\tau)\Big[1+\lambda\ell\int_0^z ZdF(Z)\Big]}{r-g-\tau z[r+\lambda F(z)]+\lambda\int_0^z ZdF(Z)}, \tag{IB.49}$$

where the denominator is positive as $f_1(z) = \overline{x} > \ell \geq 0$ (see (IB.47)). In sum, for any pair $(\widetilde{x}, \overline{x})$ satisfying $\widetilde{x} \in \widetilde{\mathbb{U}}$, $\overline{x} \geq \widetilde{x}$, and $c(\widetilde{x};\overline{x}) \leq 1$, (IB.3) holds if and only if $\overline{x} = f_1(z)$ and $\widetilde{x} = zf_1(z)$, where $z = \widetilde{x}/\overline{x} \in [0,1]$.

Next, we consider the case $c(\widetilde{x};\overline{x}) \geq 1$. Substituting $c(\widetilde{x};\overline{x}) \geq 1$ into (IB.3), we have $\theta(c(\widetilde{x};\overline{x})) = 0$ and thus

$$\begin{aligned}(r-g)\overline{x} = & 1-\lambda(\overline{x}-\ell)\Big(\int_0^{\widetilde{x}/\overline{x}} ZdF(Z)\Big) \\ = & 1+\lambda\ell\int_0^{\widetilde{x}/\overline{x}} ZdF(Z)-\lambda\overline{x}\int_0^{\widetilde{x}/\overline{x}} ZdF(Z).\end{aligned} \tag{IB.50}$$

Dividing $\overline{x}$ in both sides of the above equation and letting $z = \widetilde{x}/\overline{x}$, we obtain

$$r-g = \frac{1}{\overline{x}}\Big[1+\lambda\ell\int_0^z ZdF(Z)\Big]-\lambda\int_0^z ZdF(Z). \tag{IB.51}$$

So that $\overline{x} = f_2(z)$, where

$$f_2(z) = \frac{1+\lambda\ell\int_0^z ZdF(Z)}{r-g+\lambda\int_0^z ZdF(Z)}. \tag{IB.52}$$

In sum, for any pair $(\widetilde{x}, \overline{x})$ satisfying $\widetilde{x} \in \widetilde{\mathbb{U}}$, $\overline{x} \geq \widetilde{x}$, and $c(\widetilde{x}; \overline{x}) \geq 1$, (IB.3) holds if and only if $\overline{x} = f_2(z)$ and $\widetilde{x} = zf_2(z)$, where $z = \widetilde{x}/\overline{x} \in [0,1]$.

Step 2: We prove that the following results for $i = 1, 2$.

(i) *When $\mu < 0$, $c(zf_i(z); f_i(z)) < 1$ for any $z \in [0,1]$.*

(ii) *When $\mu = 0$, $c(zf_i(z); f_i(z)) < 1$ for any $z \in [0,1)$, $c(f_i(1); f_i(1)) = 1$.*

(iii) *When $\mu > 0$, there exists $z_0 \in (0,1)$, such that $f_1(z_0) = f_2(z_0)$, $c(z_0 f_i(z_0); f_i(z_0)) = 1$. Moreover, $c(zf_i(z); f_i(z)) < 1$ for any $z \in [0, z_0)$, and $c(zf_i(z); f_i(z)) > 1$ for any $z \in (z_0, 1]$.*

If $\widetilde{x} \in \widetilde{\mathbb{U}}$ and $\overline{x} \geq \widetilde{x}$ satisfy (IB.3) and $c(\widetilde{x}; \overline{x}) = 1$, then for $z_0 = \frac{\widetilde{x}}{\overline{x}}$, we have $\overline{x} = f_1(z_0) = f_2(z_0)$, and $\widetilde{x} = z_0 f_1(z_0) = z_0 f_2(z_0)$. Combining (IB.49) and (IB.52), we have

$$\begin{aligned} \frac{f_2(z)}{f_1(z)} - 1 =& \frac{r - g - \tau z[r + \lambda F(z)] + \lambda \int_0^z Z dF(Z)}{(1-\tau)\Big(r - g + \lambda \int_0^z Z dF(Z)\Big)} - 1 \\ =& \frac{\tau}{1-\tau} \frac{r - g - \Big(rz + \lambda \int_0^z (z - Z) dF(Z)\Big)}{r - g + \lambda \int_0^z Z dF(Z)}. \end{aligned} \tag{IB.53}$$

Since $rz + \lambda \int_0^z (z - Z) dF(Z)$ is an increasing function, z_0 is uniquely determined by the following equation:

$$r - g = rz_0 + \lambda \int_0^{z_0} (z_0 - Z) dF(Z). \tag{IB.54}$$

We derive from (IB.54) that $z_0 \leq 1$ if and only if $r - g \leq \Big(rz + \lambda \int_0^z (z - Z) dF(Z)\Big) |_{z=1} = r + \lambda(1 - \mathbb{E}[Z])$, which is equivalent to $\mu \geq 0$.

When $\mu < 0$, $c(zf_1(z); f_1(z)) \neq 1$, $c(zf_2(z); f_2(z)) \neq 1$ for any $z \in [0,1]$. Since $c(0; f_1(0)) = c(0; f_2(0)) = 0$, we have $c(zf_1(z); f_1(z)) < 1$, $c(zf_2(z); f_2(z)) < 1$ for any $z \in [0,1]$. So that the result in case (i) holds.

When $\mu = 0$, for $i = 1, 2$, $c(zf_i(z); f_i(z)) = 1$ if and only if $z = z_0 = 1$. Since $c(0; f_1(0)) = c(0; f_2(0)) = 0$, we have $c(zf_1(z); f_1(z)) < 1$, $c(zf_2(z); f_2(z)) < 1$ for any $z \in [0,1)$. So that the result in case (ii) holds.

When $\mu > 0$, for $i = 1, 2$, $c(zf_i(z); f_i(z)) = 1$ if and only if $z = z_0 < 1$, where z_0 is uniquely determined by (IB.54). For the case $z_0 < 1$, we derive from (IB.53) and (IB.54) that

$$\begin{aligned} &f_1(z) < f_2(z), \quad \text{if } 0 \leq z < z_0, \\ &f_1(z) = f_2(z), \quad \text{if } z = z_0, \\ &f_1(z) > f_2(z), \quad \text{if } z_0 < z \leq 1. \end{aligned} \tag{IB.55}$$

Furthermore, we have

$$\begin{aligned} c(f_2(1); f_2(1)) - 1 &= (r + \lambda) f_2(1) - \ell\lambda\mathbb{E}[Z] - 1 \\ &= (r + \lambda) \frac{1 + \ell\lambda\mathbb{E}[Z]}{r - g + \lambda\mathbb{E}[Z]} - \ell\lambda\mathbb{E}[Z] - 1 \\ &= (1 + \ell\lambda\mathbb{E}[Z]) \Big[\frac{r + \lambda}{r - g + \lambda\mathbb{E}[Z]} - 1\Big] = \frac{(1 + \ell\lambda\mathbb{E}[Z])\mu}{r - g + \lambda\mathbb{E}[Z]}. \end{aligned} \tag{IB.56}$$

Thus $c(f_2(1); f_2(1)) > 1$. Combining it with $c(0; f_2(0)) = 0$, and $c(zf_2(z); f_2(z)) = 1$ if and only if $z = z_0 < 1$, we have $c(zf_2(z); f_2(z)) < 1$ for any $z \in [0, z_0)$, and $c(zf_2(z); f_2(z)) > 1$ for any $z \in (z_0, 1]$.

Since $c(zf_1(z); f_1(z))$ is continuous in $z \in [0,1]$, $c(0; f_1(0)) = 0$, and $c(zf_1(z); f_1(z)) = 1$ if and only if $z = z_0$, we have $c(zf_1(z); f_1(z)) < 1$ for $z \in [0, z_0)$. For the sake of contradiction, assume $c(zf_1(z); f_1(z)) <$

1 for some $z \in (z_0, 1]$. Since $c(zf_1(z); f_1(z))$ is continuous in $z \in [0, 1]$, and $c(zf_1(z); f_1(z)) = 1$ if and only if $z = z_0$, we have $c(zf_1(z); f_1(z)) < 1$ for any $z \in (z_0, 1]$. In particular, $c(f_1(1); f_1(1)) < 1$. Since $c(x; x) = (r+\lambda)x - \ell\lambda\mathbb{E}[Z]$ is increasing in x, and $c(f_2(1); f_2(1)) > 1$, and $f_1(1) > f_2(1)$ using (IB.55), we have $c(f_1(1); f_1(1)) > c(f_2(1); f_2(1)) > 1$. Then we arrive at a contradiction. Thus, $c(zf_1(z); f_1(z)) \geq 1$ for any $z \in (z_0, 1]$. Since $c(zf_1(z); f_1(z)) = 1$ if and only if $z = z_0$, we have $c(zf_1(z); f_1(z)) > 1$ for any $z \in (z_0, 1]$. Then the result in case (iii) holds.

Combining the results proved in steps 1-2, we conclude that for any $\widetilde{x} \in \widetilde{\mathbb{U}}$ and $\overline{x} \geq \widetilde{x}$ satisfying (IB.3), we have (IB.12).

Furthermore, for the case $\mu \leq 0$, we have $c(\widetilde{x}; \overline{x}) \leq 1$ for any $\widetilde{x} \in \widetilde{\mathbb{U}}$ and $\overline{x} \geq \widetilde{x}$ satisfying (IB.3), and

$$\widetilde{\mathbb{U}} = \{zf_1(z) : z \in [0, 1]\} = [0, \widetilde{x}_{max}], \tag{IB.57}$$

where $\widetilde{x}_{max} = \max_{z\in[0,1]} zf_1(z)$. For the case $\mu > 0$, we have

$$\widetilde{\mathbb{U}} = \{zf_1(z) : z \in [0, z_0]\} \cup \{zf_2(z) : z \in [z_0, 1]\} = [0, \widetilde{x}_{max}], \tag{IB.58}$$

where $\widetilde{x}_{max} = \max\left\{ \max_{z\in[0,z_0]} zf_1(z), \max_{z\in[z_0,1]} zf_2(z)\right\}$. Combining (IB.57) and (IB.58), we conclude (IB.11) holds. ■

Proof of Theorem 5. Since $f_0(z)$, as given in Proposition 3, is continuous in $z \in [0, 1]$, there exists $z_* \in [0, 1]$, such that (IB.14) holds with $\overline{x}_* = f_0(z_*)$.

Using (IB.47), we have $\ell < \overline{x}_* \leq \frac{1}{r-g}$. Recall that in step 4 of the proof of Theorem 4, we have shown that the strategy with target debt-EBIT ratio $\widetilde{x}^*$, and default boundary $\overline{x}^*$ is the MPE with highest firm value $\overline{x}^*$. Moreover, we have shown that $\overline{x}^* \in \mathbb{S}^*$ given in (IB.44). Thus, $\overline{x}^* = \mathcal{A}(\overline{x}^*) = \widetilde{v}(\widetilde{x}^*, \overline{x}^*)$. Then $\widetilde{x}^* \in \widetilde{\mathbb{U}}$ and thus

$$\overline{x}^* = f_0(\widetilde{x}^*/\overline{x}^*) \leq \overline{x}_*, \tag{IB.59}$$

where the equality uses Proposition 3, and the inequality uses $\overline{x}_* = \max_{z\in[0,1]} f_0(z)$.

By the definition of $\overline{x}_*$ and Proposition 3, we have $\widetilde{v}(\widetilde{x}_*, \overline{x}_*) = \overline{x}_*$. For the sake of contradiction, assume there exists $\widetilde{x}_1 \in \mathbb{U}(\overline{x}_*)$ such that $\widetilde{v}(\widetilde{x}_1, \overline{x}_*) > \overline{x}_*$. Using (IB.40), we can see that $\widetilde{v}(\widetilde{x}_1, \overline{x}) < \overline{x}$ for any $\overline{x} > \frac{1}{r-g}$. Thus, there exists $\overline{x}_1 > \overline{x}_*$, such that $\widetilde{v}(\widetilde{x}_1, \overline{x}_1) = \overline{x}_1$. Consequently,

$$\overline{x}_1 = f_0(\widetilde{x}_1/\overline{x}_1) \leq \overline{x}_*,$$

where the equality uses Proposition 3, and the inequality uses $\overline{x}_* = \max_{z\in[0,1]} f_0(z)$. Then we arrive at a contradiction as $\overline{x}_1 > \overline{x}_*$. Therefore,

$$\widetilde{v}(\widetilde{x}, \overline{x}_*) \leq \overline{x}_* = \widetilde{v}(\widetilde{x}_*, \overline{x}_*) \tag{IB.60}$$

for any $\widetilde{x} \in \mathbb{U}(\overline{x}_*)$. Then using (IB.9) and (IB.60), we conclude that $\overline{x}_* \in \mathbb{S}^*$ given in (IB.44). Since we have shown that $\overline{x}^* = \sup\{\overline{x} : \overline{x} \in \mathbb{S}^*\}$ in the proof of Theorem 4, we have $\overline{x}_* \leq \overline{x}^*$. Combining with (IB.59), we have $\overline{x}_* = \overline{x}^*$.

Therefore, $\overline{x}_* = \overline{x}^*$ is the default boundary in the MPE with the highest firm value. ■

Proof of Proposition 4-(i). For the sake of contradiction, assume there exist $1 \geq z_2 > z_1 \geq 0$, such that $\overline{x}_* = f_0(z_1) = f_0(z_2)$. Denote $\widetilde{x}_1 = z_1\overline{x}_*$, and $\widetilde{x}_2 = z_2\overline{x}_*$. Then by applying (IB.38) and Proposition 3, we obtain

$$\mathcal{R}(\widetilde{x}_i, \overline{x}_*) = \mathcal{H}(\widetilde{x}_i, \overline{x}_*), \tag{IB.61}$$

where $i = 1, 2$, $\mathcal{R}(\widetilde{x}, \widetilde{v})$ denotes the left side of (IB.8), and $\mathcal{H}(\widetilde{x}, \overline{x})$ denotes the right side of (IB.8). We prove that

$$\mathcal{R}(\widetilde{x}_0, \overline{x}_*) < \mathcal{H}(\widetilde{x}_0, \overline{x}_*) \tag{IB.62}$$

for some $\widetilde{x}_0 \in (\widetilde{x}_1, \widetilde{x}_2)$.

Using (IB.33), (IB.35), we have

$$\begin{aligned}&\mathcal{H}_{\widetilde{x}}(\widetilde{x}, \overline{x}) - \mathcal{R}_{\widetilde{x}}(\widetilde{x}, \overline{x}) \\ =&r + \lambda F\Big(\frac{\widetilde{x}}{\overline{x}}\Big) - \big[\mathbf{1}_{c(\widetilde{x};\overline{x})\leq 1}(1-\tau) + \mathbf{1}_{c(\widetilde{x};\overline{x})>1}\big]\Big[r + \lambda\frac{\widetilde{x}}{\overline{x}}\Big(1 - \frac{\ell}{\overline{x}}\Big)F'\Big(\frac{\widetilde{x}}{\overline{x}}\Big) + \lambda F\Big(\frac{\widetilde{x}}{\overline{x}}\Big)\Big].\end{aligned} \tag{IB.63}$$

If $c(\widetilde{x}_2; \overline{x}_*) > 1$, then there exists a sufficiently small $\varepsilon > 0$, such that $\mathcal{H}_{\widetilde{x}}(\widetilde{x}, \overline{x}_*) - \mathcal{R}_{\widetilde{x}}(\widetilde{x}, \overline{x}_*) = -\lambda\frac{\widetilde{x}}{\overline{x}_*}\Big(1 - \frac{\ell}{\overline{x}_*}\Big)F'\Big(\frac{\widetilde{x}}{\overline{x}_*}\Big) < 0$ for $\widetilde{x} \in (\widetilde{x}_2 - \varepsilon, \widetilde{x}_2)$. It follows that $\mathcal{H}(\widetilde{x}_0, \overline{x}_*) - \mathcal{R}(\widetilde{x}_0, \overline{x}_*) > \mathcal{H}(\widetilde{x}_2, \overline{x}_*) - \mathcal{R}(\widetilde{x}_2, \overline{x}_*) = 0$ for $\widetilde{x}_0 \in (\widetilde{x}_2 - \varepsilon, \widetilde{x}_2)$.

Thus, we only need to consider the case $c(\widetilde{x}_2; \overline{x}_*) \leq 1$. Then we have $c(\widetilde{x}; \overline{x}_*) < 1$ for any $\widetilde{x} \in [\widetilde{x}_1, \widetilde{x}_2)$, and then

$$\mathcal{H}_{\widetilde{x}}(\widetilde{x}, \overline{x}_*) - \mathcal{R}_{\widetilde{x}}(\widetilde{x}, \overline{x}_*) = \tau r + \lambda\Big[\tau F\Big(\frac{\widetilde{x}}{\overline{x}_*}\Big) - (1-\tau)\frac{\widetilde{x}}{\overline{x}_*}\Big(1 - \frac{\ell}{\overline{x}_*}\Big)F'\Big(\frac{\widetilde{x}}{\overline{x}_*}\Big)\Big]. \tag{IB.64}$$

If $\mathcal{H}_{\widetilde{x}}(\widetilde{x}_1, \overline{x}_*) - \mathcal{R}_{\widetilde{x}}(\widetilde{x}_1, \overline{x}_*) > 0$ or $\mathcal{H}_{\widetilde{x}}(\widetilde{x}_2-, \overline{x}_*) - \mathcal{R}_{\widetilde{x}}(\widetilde{x}_2-, \overline{x}_*) < 0$, then using (IB.61), we conclude that there exists $\widetilde{x}_0 \in (\widetilde{x}_1, \widetilde{x}_2)$, such that $\mathcal{R}(\widetilde{x}_0, \overline{x}_*) < \mathcal{H}(\widetilde{x}_0, \overline{x}_*)$. Thus, we only need to consider the case $\mathcal{H}_{\widetilde{x}}(\widetilde{x}_1, \overline{x}_*) - \mathcal{R}_{\widetilde{x}}(\widetilde{x}_1, \overline{x}_*) \leq 0$ and $\mathcal{H}_{\widetilde{x}}(\widetilde{x}_2-, \overline{x}_*) - \mathcal{R}_{\widetilde{x}}(\widetilde{x}_2-, \overline{x}_*) \geq 0$. It follows that

$$\tau r + \lambda\Big[\tau F\Big(\frac{\widetilde{x}_1}{\overline{x}_*}\Big) - (1-\tau)\frac{\widetilde{x}_1}{\overline{x}_*}\Big(1 - \frac{\ell}{\overline{x}_*}\Big)F'\Big(\frac{\widetilde{x}_1}{\overline{x}_*}\Big)\Big] \leq 0, \tag{IB.65}$$

$$\tau r + \lambda\Big[\tau F\Big(\frac{\widetilde{x}_2}{\overline{x}_*}\Big) - (1-\tau)\frac{\widetilde{x}_2}{\overline{x}_*}\Big(1 - \frac{\ell}{\overline{x}_*}\Big)F'\Big(\frac{\widetilde{x}_2}{\overline{x}_*}\Big)\Big] \geq 0. \tag{IB.66}$$

Using (IB.65), we have

$$\tau F\Big(\frac{\widetilde{x}_1}{\overline{x}_*}\Big) - (1-\tau)\frac{\widetilde{x}_1}{\overline{x}_*}\Big(1 - \frac{\ell}{\overline{x}_*}\Big)F'\Big(\frac{\widetilde{x}_1}{\overline{x}_*}\Big) \leq -\frac{\tau r}{\lambda} < 0. \tag{IB.67}$$

So that,

$$\begin{aligned}&\tau F\Big(\frac{\widetilde{x}_2}{\overline{x}_*}\Big) - (1-\tau)\frac{\widetilde{x}_2}{\overline{x}_*}\Big(1 - \frac{\ell}{\overline{x}_*}\Big)F'\Big(\frac{\widetilde{x}_2}{\overline{x}_*}\Big) \\ =&F\Big(\frac{\widetilde{x}_2}{\overline{x}_*}\Big)\left[\tau - (1-\tau)\Big(1 - \frac{\ell}{\overline{x}_*}\Big)\frac{\frac{\widetilde{x}_2}{\overline{x}_*}F'\big(\frac{\widetilde{x}_2}{\overline{x}_*}\big)}{F\big(\frac{\widetilde{x}_2}{\overline{x}_*}\big)}\right] \\ \leq&F\Big(\frac{\widetilde{x}_2}{\overline{x}_*}\Big)\left[\tau - (1-\tau)\Big(1 - \frac{\ell}{\overline{x}_*}\Big)\frac{\frac{\widetilde{x}_1}{\overline{x}_*}F'\big(\frac{\widetilde{x}_1}{\overline{x}_*}\big)}{F\big(\frac{\widetilde{x}_1}{\overline{x}_*}\big)}\right] \\ =&\frac{F\big(\frac{\widetilde{x}_2}{\overline{x}_*}\big)}{F\big(\frac{\widetilde{x}_1}{\overline{x}_*}\big)}\Big[\tau F\Big(\frac{\widetilde{x}_1}{\overline{x}_*}\Big) - (1-\tau)\frac{\widetilde{x}_1}{\overline{x}_*}\Big(1 - \frac{\ell}{\overline{x}_*}\Big)F'\Big(\frac{\widetilde{x}_1}{\overline{x}_*}\Big)\Big] < -\frac{\tau r}{\lambda},\end{aligned} \tag{IB.68}$$

where the first inequality uses Assumption 3, $\overline{x}_* \geq \widetilde{x}_2 > \widetilde{x}_1 \geq 0$, and $\overline{x}_* > \ell \geq 0$, and the second inequality uses (IB.67), $F'(z) > 0$ and $\widetilde{x}_2 > \widetilde{x}_1$. Comparing (IB.66) and (IB.68), we arrive at a contradiction.

Using (IB.62), we have

$$\mathcal{R}(\widetilde{x}_0, \widetilde{v}(\widetilde{x}_0, \overline{x}_*)) = \mathcal{H}(\widetilde{x}_0, \overline{x}_*) > \mathcal{R}(\widetilde{x}_0, \overline{x}_*) \tag{IB.69}$$

where the equality uses (IB.38). Since $\mathcal{R}_v(x, v) > 0$, we have $\widetilde{v}(\widetilde{x}_0, \overline{x}_*) > \overline{x}_*$. Using (IB.40), we can see that $\widetilde{v}(\widetilde{x}_0, \overline{x}) \leq \overline{x}$ for any $\overline{x} > \frac{1}{r-g}$. Thus, there exists $\overline{x}_1 > \overline{x}_*$, such that $\widetilde{v}(\widetilde{x}_0, \overline{x}_1) = \overline{x}_1$, which implies that

$$\overline{x}_1 = f_0(\widetilde{x}_0/\overline{x}_1) \leq \overline{x}_*,$$

where the equality uses Proposition 3, and the inequality uses the definition of $\overline{x}_*$ given in (IB.14). Then we arrive again at a contradiction. ■

Proof of Proposition 4-(ii). According to (IB.63), when $c(\widetilde{x};\overline{x}) \le 1$, we have[49]

$$\mathcal{H}_{\widetilde{x}}(\widetilde{x},\overline{x}) - \mathcal{R}_{\widetilde{x}}(\widetilde{x},\overline{x}) = \tau r + \lambda\Big[\tau F\Big(\frac{\widetilde{x}}{\overline{x}}\Big) - (1-\tau)\frac{\widetilde{x}}{\overline{x}}\Big(1 - \frac{\ell}{\overline{x}}\Big)F'\Big(\frac{\widetilde{x}}{\overline{x}}\Big)\Big]. \tag{IB.70}$$

Recall that (IB.12) holds for any $\widetilde{x} \in \widetilde{\mathbb{U}}$ and $\overline{x} \ge \widetilde{x}$ satisfying (IB.3). Let

$$\xi(z) = \frac{\tau r}{\lambda} + \tau F(z) - (1-\tau)z\Big(1 - \frac{\ell}{f_0(z)}\Big)F'(z). \tag{IB.71}$$

Then we have $\xi(z) = [\mathcal{H}_{\widetilde{x}}(zf_0(z), f_0(z)) - \mathcal{R}_{\widetilde{x}}(zf_0(z), f_0(z))]/\lambda$ when $c(zf_0(z); f_0(z)) \le 1$.

Step 1: We prove that either of the following cases hold:

(i) $\xi(z) > 0$ *for any* $z \in [0,1]$.

(ii) $\xi(z) > 0$ *for* $z \in [0,1)$, *and* $\xi(1) = 0$.

(iii) There exists $z_1 \in (0,1)$, *such that* $\xi(z_1) = 0$, $\xi(z) > 0$ *for* $z \in [0, z_1)$, *and* $\xi(z) < 0$ *for* $z \in (z_1, 1]$.

We derive from (IB.71) that

$$\begin{aligned}\frac{\ell(1-\tau)}{f_0(z)} =& \frac{(1-\tau)zF'(z) - \tau F(z) - (\tau r/\lambda - \xi(z))}{zF'(z)}\\ =& 1 - \tau - \frac{\tau/\lambda[r + \lambda F(z)] - \xi(z)}{zF'(z)}.\end{aligned} \tag{IB.72}$$

When $c(\widetilde{x};\overline{x}) \le 1$, using $c(\widetilde{x};\overline{x}) = r\widetilde{x} + \lambda\int_0^{\widetilde{x}/\overline{x}}(\widetilde{x} - \ell Z)\,dF(Z) = \widetilde{x}[r + \lambda F(\widetilde{x}/\overline{x})] - \ell\lambda\int_0^{\widetilde{x}/\overline{x}} Z\,dF(Z)$, we have $\theta(c(\widetilde{x};\overline{x})) = \tau - \tau c(\widetilde{x};\overline{x}) = \tau - \tau\Big[\widetilde{x}[r + \lambda F(\widetilde{x}/\overline{x})] - \ell\lambda\int_0^{\widetilde{x}/\overline{x}} Z\,dF(Z)\Big]$. Substituting it into (IB.3), we obtain that

$$\begin{aligned}(r-g)\overline{x} =& 1 - \tau + \tau\Big[\widetilde{x}[r + \lambda F(\widetilde{x}/\overline{x})] - \ell\lambda\int_0^{\widetilde{x}/\overline{x}} Z\,dF(Z)\Big] - \lambda(\overline{x} - \ell)\Big(\int_0^{\widetilde{x}/\overline{x}} ZdF(Z)\Big)\\ =& (1-\tau)\Big[1 + \lambda\ell\Big(\int_0^{\widetilde{x}/\overline{x}} ZdF(Z)\Big)\Big] + \tau\widetilde{x}[r + \lambda F(\widetilde{x}/\overline{x})] - \lambda\overline{x}\Big(\int_0^{\widetilde{x}/\overline{x}} ZdF(Z)\Big).\end{aligned}$$

Multiplying both sides of the above equation by $\ell/\overline{x}$ and using $z = \widetilde{x}/\overline{x}$ and $\overline{x} = f_0(z)$, we obtain

$$(r-g)\ell = \frac{\ell(1-\tau)}{f_0(z)}\Big[1 + \lambda\ell\Big(\int_0^z ZdF(Z)\Big)\Big] + \ell\tau z[r + \lambda F(z)] - \lambda\ell\int_0^z ZdF(Z). \tag{IB.73}$$

Substituting (IB.72) into (IB.73), we obtain

$$\begin{aligned}&(r-g)\ell\\ =& \Big[1 - \tau - \frac{\tau/\lambda[r + \lambda F(z)] - \xi(z)}{zF'(z)}\Big]\Big[1 + \lambda\ell\int_0^z ZdF(Z)\Big] + \ell\tau z[r + \lambda F(z)] - \lambda\ell\int_0^z ZdF(Z)\\ =& 1 - \tau - \tau\lambda\ell\int_0^z ZdF(Z) - \frac{\tau/\lambda[r + \lambda F(z)] - \xi(z)}{zF'(z)}\Big[1 + \lambda\ell\int_0^z ZdF(Z)\Big] + \ell\tau z[r + \lambda F(z)]\\ =& 1 - \tau - \tau\lambda\ell\int_0^z ZdF(Z) + \tau[r + \lambda F(z)]\Big[\ell z - \frac{1 + \lambda\ell\int_0^z ZdF(Z)}{\lambda zF'(z)}\Big] + \xi(z)\frac{1 + \lambda\ell\int_0^z ZdF(Z)}{zF'(z)}.\end{aligned} \tag{IB.74}$$

[49] When $c(\widetilde{x};\overline{x}) = 1$, we have $c(x;\overline{x}) < 1$ for $x < \widetilde{x}$, and $\mathcal{H}_{\widetilde{x}}(\widetilde{x},\overline{x}) - \mathcal{R}_{\widetilde{x}}(\widetilde{x},\overline{x})$ refers to the left partial derivative $\mathcal{H}_{\widetilde{x}}(\widetilde{x}-,\overline{x}) - \mathcal{R}_{\widetilde{x}}(\widetilde{x}-,\overline{x})$.

Let $\mathcal{V}(z) = -\lambda\ell\int_0^z ZdF(Z) + [r+\lambda F(z)]\Big[\ell z - \frac{1+\lambda\ell\int_0^z ZdF(Z)}{\lambda zF'(z)}\Big]$. Then (IB.74) is equivalent to

$$(r-g)\ell - (1-\tau) = \tau\mathcal{V}(z) + \xi(z)\frac{1+\lambda\ell\int_0^z ZdF(Z)}{zF'(z)}, \tag{IB.75}$$

where $(r-g)\ell-(1-\tau) < 0$ since $\frac{1-\tau}{r-g} > \ell$ and $r > g$. According to Assumption 3, we have $(zF'(z)/F(z))' \geq 0$, for $z \in (0,1)$. It follows that

$$F(z)(zF'(z))' - z(F'(z))^2 \geq 0, \quad z \in (0,1). \tag{IB.76}$$

For $z \in (0,1]$, we have

$$\begin{aligned}
&\mathcal{V}'(z)\\
=& -\lambda\ell zF'(z) + \lambda F'(z)\Big[\ell z - \frac{1+\lambda\ell\int_0^z ZdF(Z)}{\lambda zF'(z)}\Big] + [r+\lambda F(z)]\frac{1+\lambda\ell\int_0^z ZdF(Z)}{\lambda(zF'(z))^2}(zF'(z))'\\
=&\frac{1+\lambda\ell\int_0^z ZdF(Z)}{\lambda(zF'(z))^2}\Big[[r+\lambda F(z)](zF'(z))' - \lambda z(F'(z))^2\Big]\\
=&\frac{1+\lambda\ell\int_0^z ZdF(Z)}{\lambda(zF'(z))^2}\left[r(zF'(z))' + \lambda\Big(F(z)(zF'(z))' - z(F'(z))^2\Big)\right] > 0,
\end{aligned} \tag{IB.77}$$

where the inequality uses (IB.76) and $(zF'(z))' \geq z(F'(z))^2/F(z) > 0$ for $z > 0$. Since $\lim_{z\to 0} zF'(z) = 0$, we have $\lim_{z\to 0}\mathcal{V}(z) = -\infty$. When $\tau\mathcal{V}(1) < (r-g)\ell-(1-\tau)$, we have $\tau\mathcal{V}(z) < (r-g)\ell-(1-\tau)$ for any $z \in [0,1]$. Substituting it into (IB.75), we have $\xi(z) > 0$ for any $z \in [0,1]$. So that the result in case (i) holds.

Next, we consider the case $\tau\mathcal{V}(1) \geq (r-g)\ell-(1-\tau)$. Then the following equation admits a unique solution in the region $z \in [0,1]$, denoted by z_1.

$$(r-g)\ell - (1-\tau) = \tau\mathcal{V}(z). \tag{IB.78}$$

We have $z_1 \in (0,1]$ as $\lim_{z\to 0}\mathcal{V}(z) = -\infty$. As $\mathcal{V}(z)$ is increasing in z and $(r-g)\ell-(1-\tau) = \tau\mathcal{V}(z_1)$, we have $(r-g)\ell-(1-\tau) > \tau\mathcal{V}(z)$ for $z \in [0,z_1)$, and $(r-g)\ell-(1-\tau) < \tau\mathcal{V}(z)$ for $z \in (z_1,1]$.

When $z_1 = 1$, we derive from (IB.75) that $\xi(z) > 0$ for $z \in [0,1)$ and $\xi(1) = 0$. Then the result in case (ii) holds.

When $z_1 \in (0,1)$, we derive from (IB.75) that $\xi(z) > 0$ for $z \in [0,z_1)$, and $\xi(z) < 0$ for $z \in (z_1,1]$. Then the result in case (iii) holds.

Step 2: We prove that for the pair $(\widetilde{x},\overline{x})$ *satisfying* $\overline{x} \geq \widetilde{x} \geq 0$, $\overline{x} = \mathcal{A}(\overline{x}) = \widetilde{v}(\widetilde{x},\overline{x})$, *we have* $\widetilde{x} > 0$,

$$c(\widetilde{x};\overline{x}) \leq 1, \quad \textit{and} \quad \mathcal{H}_{\widetilde{x}}(\widetilde{x},\overline{x}) - \mathcal{R}_{\widetilde{x}}(\widetilde{x},\overline{x}) \geq 0, \tag{IB.79}$$

where $\mathcal{A}(\cdot)$ *is given in* (IB.9).

We have $\overline{x} > \ell \geq 0$ using (IB.47). We first prove that $\widetilde{x} > 0$. For the sake of contradiction, assume $\widetilde{x} = 0$. According to (IB.63), we have $\mathcal{H}_{\widetilde{x}}(0,\overline{x}) - \mathcal{R}_{\widetilde{x}}(0,\overline{x}) = \tau r > 0$. Then there exists $\varepsilon > 0$, such that

$$\mathcal{H}(\varepsilon,\overline{x}) - \mathcal{R}(\varepsilon,\overline{x}) > \mathcal{H}(0,\overline{x}) - \mathcal{R}(0,\overline{x}) = 0, \tag{IB.80}$$

where the equality uses $\overline{x} = \widetilde{v}(0,\overline{x})$. Since $\mathcal{R}(\varepsilon,\widetilde{v}(\varepsilon,\overline{x})) = \mathcal{H}(\varepsilon,\overline{x}) > \mathcal{R}(\varepsilon,\overline{x})$ and $\mathcal{R}_v > 0$, we have $\widetilde{v}(\varepsilon,\overline{x}) > \overline{x}$. Then we arrive at a contradiction. Thus, $\widetilde{x} > 0$.

Next, we prove that $\mathcal{H}_{\widetilde{x}}(\widetilde{x},\overline{x}) - \mathcal{R}_{\widetilde{x}}(\widetilde{x},\overline{x}) \geq 0$. For the sake of contradiction, assume $\mathcal{H}_{\widetilde{x}}(\widetilde{x},\overline{x}) - \mathcal{R}_{\widetilde{x}}(\widetilde{x},\overline{x}) < 0$. Then there exists $x_1 \in (0,\widetilde{x})$, such that

$$\mathcal{H}(x_1,\overline{x}) - \mathcal{R}(x_1,\overline{x}) > \mathcal{H}(\widetilde{x},\overline{x}) - \mathcal{R}(\widetilde{x},\overline{x}) = 0, \tag{IB.81}$$

where the equality uses $\overline{x} = \widetilde{v}(\widetilde{x}, \overline{x})$. Since $\mathcal{R}(x_1, \widetilde{v}(x_1, \overline{x})) = \mathcal{H}(x_1, \overline{x}) > \mathcal{R}(x_1, \overline{x})$ and $\mathcal{R}_v > 0$, we have $\widetilde{v}(x_1, \overline{x}) > \overline{x}$. Then we arrive at a contradiction. Thus, $\mathcal{H}_{\widetilde{x}}(\widetilde{x}, \overline{x}) - \mathcal{R}_{\widetilde{x}}(\widetilde{x}, \overline{x}) \geq 0$.

According to (IB.63), we have

$$\mathcal{H}_{\widetilde{x}}(\widetilde{x}, \overline{x}) - \mathcal{R}_{\widetilde{x}}(\widetilde{x}, \overline{x}) = -\lambda \frac{\widetilde{x}}{\overline{x}} \Big(1 - \frac{\ell}{\overline{x}}\Big) F'\Big(\frac{\widetilde{x}}{\overline{x}}\Big) < 0, \text{ if } \quad c(\widetilde{x}; \overline{x}) > 1. \tag{IB.82}$$

Since we have shown that $\mathcal{H}_{\widetilde{x}}(\widetilde{x}, \overline{x}) - \mathcal{R}_{\widetilde{x}}(\widetilde{x}, \overline{x}) \geq 0$, we can derive from (IB.82) that $c(\widetilde{x}, \overline{x}) \leq 1$, which proves (IB.79).

Step 3: We prove that for any pair $(\widetilde{x}, \overline{x})$ *satisfying* $\overline{x} > \widetilde{x} \geq 0$, $\overline{x} = \mathcal{A}(\overline{x}) = \widetilde{v}(\widetilde{x}, \overline{x})$, *and* $c(\widetilde{x}; \overline{x}) < 1$, *we have*

$$\mathcal{H}_{\widetilde{x}}(\widetilde{x}, \overline{x}) - \mathcal{R}_{\widetilde{x}}(\widetilde{x}, \overline{x}) = 0, \tag{IB.83}$$

where $\mathcal{A}(\cdot)$ *is given in* (IB.9).

For the sake of contradiction, assume that $\mathcal{H}_{\widetilde{x}}(\widetilde{x}, \overline{x}) - \mathcal{R}_{\widetilde{x}}(\widetilde{x}, \overline{x}) \neq 0$. Using $c(\widetilde{x}; \overline{x}) < 1$ and (IB.63), we can see that there exists a small $\varepsilon > 0$, such that $\widetilde{x} + \varepsilon < \overline{x}$, and $\mathcal{H}_{\widetilde{x}}(x, \overline{x}) - \mathcal{R}_{\widetilde{x}}(x, \overline{x})$ is continuous in $x \in (\widetilde{x} - \varepsilon, \widetilde{x} + \varepsilon)$. It follows that there exists $x \in (\widetilde{x} - \varepsilon, \widetilde{x} + \varepsilon)$, such that

$$\mathcal{H}(x, \overline{x}) - \mathcal{R}(x, \overline{x}) > \mathcal{H}(\widetilde{x}, \overline{x}) - \mathcal{R}(\widetilde{x}, \overline{x}) = 0, \tag{IB.84}$$

where the equality uses $\overline{x} = \widetilde{v}(\widetilde{x}, \overline{x})$. Since $\mathcal{R}_v > 0$, we have $\mathcal{H}(x, \overline{x}) > \mathcal{R}(x, \overline{x}) \geq \mathcal{R}(x, x)$, so that $x \in \mathbb{U}(\overline{x})$ using (IB.37). Since $\mathcal{R}(x, \widetilde{v}(x, \overline{x})) = \mathcal{H}(x, \overline{x}) > \mathcal{R}(x, \overline{x})$ and $\mathcal{R}_v > 0$, we have $\widetilde{v}(x, \overline{x}) > \overline{x} = \mathcal{A}(\overline{x})$. Recalling that $x \in \mathbb{U}(\overline{x})$ and (IB.9), we arrive at a contradiction. This proves (IB.83).

Step 4: We prove that there exists a unique $\overline{x}$ *such that* $\mathcal{A}(\overline{x}) = \overline{x}$.

For the sake of contradiction, assume there exist $\overline{x}_1 \neq \overline{x}_2$, such that $\mathcal{A}(\overline{x}_i) = \overline{x}_i$, $i = 1, 2$. Then, there exists $\widetilde{x}_i \in \widetilde{\mathbb{U}}$, such that $\overline{x}_i = \widetilde{v}(\widetilde{x}_i, \overline{x}_i) \geq \widetilde{v}(\widetilde{x}, \overline{x}_i)$ for any $\widetilde{x} \in \mathbb{U}(\overline{x}_i)$. According to (IB.12) and the result proved in step 2, there exist $0 < z_1 < z_2 \leq 1$, such that $\widetilde{x}_1 = z_1 f_0(z_1)$, $\overline{x}_1 = f_0(z_1)$, $\widetilde{x}_2 = z_2 f_0(z_2)$, $\overline{x}_2 = f_0(z_2)$. According to the result proved in step 2, we have $c(\widetilde{x}_i; \overline{x}_i) \leq 1$, $i = 1, 2$. Combining it with the result proved in step 2 of the proof of Proposition 3, we can see that

$$c(z f_0(z); f_0(z)) < 1, \quad z \in [0, z_2). \tag{IB.85}$$

In particular, $c(z_1 f_0(z_1); f_0(z_1)) < 1$, which together with $z_1 < 1$ and the result proved in step 3 implies that $\mathcal{H}_{\widetilde{x}}(z_1 f_0(z_1); f_0(z_1)) - \mathcal{R}_{\widetilde{x}}(z_1 f_0(z_1); f_0(z_1)) = 0$. Combining it with (IB.70) and (IB.71), we have $\xi(z_1) = 0$. Then we derive from the result proved in step 1 that $\xi(z) < 0$ for any $z \in (z_1, 1]$. Using $\widetilde{x}_2 = z_2 f_0(z_2)$, $\overline{x}_2 = f_0(z_2)$ and the result proved in step 2, we conclude that $\mathcal{H}_{\widetilde{x}}(z_2 f_0(z_2); f_0(z_2)) - \mathcal{R}_{\widetilde{x}}(z_2 f_0(z_2); f_0(z_2)) \geq 0$, and thus $\xi(z_2) \geq 0$. So that we arrive at a contradiction.

According to the proof in Theorem 4, we know that the set of default boundaries in all equilibria $\in \mathcal{S}_x$ can be represented by (IB.44). In step 4 above, we have proved the uniqueness of $\overline{x}$, and thus $\overline{x} = \overline{x}_*$, which is defined in (IB.14). In Proposition 4-(i), we have proved the uniqueness of the target debt-EBIT ratio achieving the highest equilibrium value $\overline{x}_*$. This proves the uniqueness of equilibrium. ■

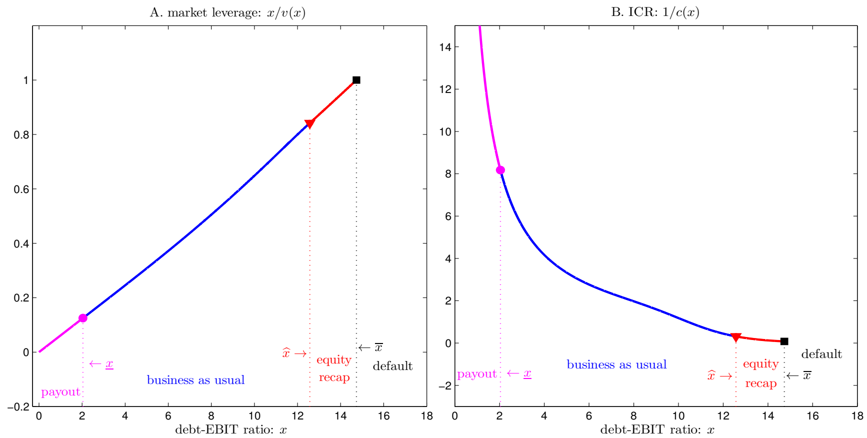

Figure 18: Market leverage ML, and interest coverage ratio ICR.

IC Market Leverage and Interest Coverage Ratio

Accounting covenants. A common liquidity metric used by practitioners to measure a firm's ability to service its debt is the *interest coverage ratio*, which we denote by ICR_t:

$$ICR_t = \frac{Y_t}{C_t}\,. \tag{IC.1}$$

The ICR_t measures the firm's ability to service its debt interest payments out of current EBIT. We can also evaluate the firm's ability to meet the sum of interest and tax payments with its contemporaneous EBIT with the following *liquidity coverage ratio*:

$$LCR_t = \frac{Y_t}{C_t + \Theta_t}\,. \tag{IC.2}$$

When $LCR_t > 1$, the firm's debt balance X_t automatically decreases.[50] ML_t is known as the loan-to-value (LTV) ratio in real estate finance. These ratios are widely used in debt covenants.[51]

Panels A and B of Figure 18 plot market leverage ML and the interest coverage ratio ICR as functions of x, respectively, for our baseline model. As expected, market leverage ML increases with the debt-EBIT ratio x and ICR increases significantly as x decreases.

[50] Strebulaev (2007) defines financial distress as situations where the firm's ICR_t falls below one and requires the firm to take corrective actions (e.g., inefficient asset sales) to alleviate debt burden under financial distress. We share his view that liquidity measures such as ICR and LCR are critically important for capital structure decisions. However, in Strebulaev (2007) the firm continuously issues equity or makes dividend payments to shareholders and there is no earnings retention. Leverage dynamics in his model are fundamentally different from ours in which earnings retention, debt rollover, equity issuance, and payout jointly determine leverage dynamics.

[51] It is straightforward to introduce other covenants such as cash-flow-based covenants that limit the firm's borrowing. Lian and Ma (2021) document that 80% of debt (for US nonfinancial firms) is backed predominantly by cash flows from operations (cash flow/earnings based lending). Our model is well suited to capture these empirically important cash flow based lending activities.

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