Strategic Investment under Uncertainty with First- and Second-mover Advantages*

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Abstract

We develop a real-option duopoly model with second-mover advantage. When later entrants benefit from observing early movers, firms enter using mixed strategies, generating two disconnected probabilistic-entry regions alternating with waiting regions. These predictions imply that innovation increases monotonically with demand when second-mover advantage is strong but becomes non-monotonic when it is weak, consistent with evidence linking patent protection and innovation. The mixed-strategy equilibrium is robust, surviving reputation refinements of Kreps et al. (1982) and Abreu and Gul (2000), and our dynamic extension of Harsanyi's purification argument. Finally, we show that anticipating information spillovers from early entrants also generates second-mover advantage.

Keywords: real-option game, mixed strategies, information spillover, wars of attrition, incomplete-information game, reputation, purification

JEL codes: E22, G13, G31, L13

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1 Introduction

In strategically competitive industries, the first mover is often perceived to have an advantage over followers.[1] However, in some circumstances it is advantageous to be a second mover. For example, by observing and imitating the first mover's successful strategies while avoiding its mistakes, a firm can dynamically adapt and improve its own strategy.

To capture the benefits of learning from peers' successes and failures, we assume that later entrants incur lower costs than early entrants, and we develop, to our knowledge, the first tractable real-option duopoly model in which equilibrium entry occurs in mixed strategies. Our analysis in Sections 2 and 3 features a mixed-strategy equilibrium that naturally arises from the firm's incentives to enter as Follower. In real-option games, such as those studied by Grenadier (1996) and Weeds (2002), firms employ pure strategies, entering the market as soon as demand exceeds an endogenous threshold. By contrast, in our mixed-strategy equilibrium, a firm enters probabilistically. Our analysis extends and adapts the classic war-of-attrition analysis to a dynamic duopoly entry game with stochastic market demand. In Section 4, we derive empirical predictions of our mixed-strategy equilibrium and show that they are consistent with evidence on corporate innovations; see, e.g., Fabrizio and Tsolmon (2014).

Our model features two *ex ante* identical firms, a and b, that discount cash flows at the rate $r > 0$ and compete to enter a new product market where demand, X_t, evolves stochastically over time t. Each firm can pay a fixed cost to enter the market at any time. The first entrant (Leader) captures the entire demand until the other (Follower) enters. When both firms are in the market, they equally share the total market demand. Total market demand may change as the industry transitions from a monopoly to a duopoly structure. We model second-mover advantage by assuming that Follower's entry cost (K_F) is lower than Leader's entry cost (K_L), so that the entry-cost ratio $K_L/K_F > 1$. There are three Markov subgame-perfect equilibria (MPE): one symmetric equilibrium and two asymmetric equilibria, all of which we characterize in closed form. The symmetric equilibrium is unique and features mixed strategies. As we

[1]For example, being the first mover may enable a firm to establish strong brand recognition and product loyalty. Also, by entering first, a firm makes it less profitable for its competitors to enter. As a result, competitors voluntarily wait until market demand reaches a higher level before entering the market. In equilibrium, the first mover earns monopoly profits until other firms enter.

show below, a firm's entry strategy is nonlinear and varies non-monotonically with market demand. To explain our model's mechanism, we divide the range of market demand into four regions: high, intermediate high, intermediate low, and low.

First, consider the region where market demand is high. In this region, Follower enters immediately because its entry option is 'deep in the money' and should be exercised as soon as Leader enters. This raises the question: does a firm have an incentive to enter as Leader in this high demand region when being Follower is strictly better off? The answer is yes, but only probabilistically. The intuition is as follows.

Recall that a firm's goal is not to outperform its competitor, but rather to maximize its own value. A firm weighs the benefit of waiting, which is to become Follower and capture the second-mover advantage, against the cost of waiting, which is to forgo a large profit. In equilibrium, while competing in a non-cooperative game, both firms strike a balance between the two pure strategies—waiting and entering—by entering *probabilistically* as Leader. This suggests the existence of an endogenous threshold for market demand X_t, denoted by $\overline{x}$, above which firms enter using mixed strategies. That is, in the $x \geq \overline{x}$ region, firms enter probabilistically, and once Leader is determined, the other firm immediately enters, leaving only duopoly rents for Leader.

Second, consider the other end of market demand when X_t is low. In this region, firms wait for the standard option-value-of-waiting reason as in McDonald and Siegel (1986) and Dixit and Pindyck (1994). This implies the existence of another endogenous threshold, $\widetilde{x}$, below which firms choose to wait. In this $x < \widetilde{x}$ region, firms wait with probability one.

Third, consider the two intermediate regions that lie between the high- and low market-demand regions discussed above. In the region to the right of the low market-demand region, market demand is large enough for a single firm to operate as a monopolist, but not large enough for two firms to profitably operate in the market. Therefore, once one firm has entered, the other firm rationally waits to preserve its option value. Anticipating this equilibrium response, both firms trade off (a) the benefits of entering sooner to capture monopoly rents and (b) the cost of giving up second-mover advantage, making them indifferent between entering immediately as Leader and waiting for another period. This gives rise to a second mixed-strategy region: $x \in [\widetilde{x}, \underline{x}]$, with an endogenous upper boundary $\underline{x}$. Notably, Leader makes profits as a monopolist in this region, unlike in the first mixed-strategy region where firms split the entire industry profits. Economically, this region captures the intuition that a firm's early entry serves as a natural deterrent to its competitor.

Importantly, in this intermediate low region, a firm is averse to uncertainty in market demand, causing its value to be concave in market demand. This is the opposite of the well-known convexity result in standard single-firm and game-theoretic real-option models, e.g., Grenadier (1996). The intuition is as follows. First, since Follower's entry is a call option whose value is convex in market demand, Leader must have a short position in this call, making its value concave in market demand. Second, prior to Leader's entry, since a firm is indifferent between becoming Leader and waiting to enter for another period, its value must equal Leader's. In summary, it is the indifference condition for firm entry that underpins the mixed strategy and renders the firm endogenously risk averse in this region.

Next, we turn to the fourth region, $x \in (\underline{x}, \overline{x})$, where market demand is moderately high. This region is sandwiched between the two mixed-strategy regions. On the one hand, compared to the intermediate low market-demand region to its left, the expected monopoly rents are lower because higher demand makes Follower more willing to enter, thereby reducing the expected duration of Leader's monopoly rents. On the other hand, relative to the high market-demand region to its right, demand is not yet high enough for a firm to be willing to share profits with its competitor. As a result, firms are discouraged from entering as Leader. Taking both considerations into account, firms choose to wait with probability one in this intermediate high market-demand region. Intuitively, as probabilistic entry is optimal in both neighboring regions, firms optimally wait until market demand either rises above $\overline{x}$ or falls below $\underline{x}$. Finally, as firms are indifferent between entering probabilistically or waiting at the two boundaries, $\underline{x}$ and $\overline{x}$, we can determine them using the smooth-pasting conditions.

To fully characterize the solution, we need to pin down the equilibrium entry probability in the two mixed-strategy regions. We do so using two conditions. First, when using a mixed strategy, a firm must be indifferent between the strategies it chooses to mix, ensuring that its value is independent of its own entry rate; this indifference allows us to solve for firm value, since the payoff of entering as Leader is known. Second, the rate at which a firm enters probabilistically is chosen to make its competitor indifferent between entering as

Leader and waiting for another period.[2] In equilibrium, firms enter at the same rate, $\lambda^*(X_t)$, as Leader, which is in general not monotonic in X_t. This is because Follower's incentive to enter increases with X_t, which in turn reduces the expected value of Leader's monopoly rents, thus undermining a firm's *ex ante* incentives to enter as Leader.

In summary, there are four regions in equilibrium: two waiting regions alternating with two probabilistic-entry regions. In the low market-demand region, firms always wait, preserving the standard option value of waiting. Therefore, firm value is convex in market demand in this region. In the intermediate low region, firms are endogenously risk averse, making their value concave in market demand, as they are indifferent between entering as Leader and waiting for another period. In the (third) intermediate high market-demand region, firm value prior to Leader's entry is again convex, as firms want to preserve both the standard option value of waiting and the second-mover advantage. Finally, in the high market-demand region, firm value is linear, since the total market demand is so high that monopoly profits are unattainable; here firms are indifferent between entering as Leader and waiting, so their value equals Leader's net payoff, which is linear in demand.

Having characterized the key properties of the mixed-strategy equilibrium, we turn to empirical predictions of this equilibrium in Section 4. Fabrizio and Tsolmon (2014) find that in industries with weak patent protection, innovation increases with market demand, whereas in industries with strong patent protection, innovation is not monotonic in demand. Thus, the responsiveness of innovations to market demand critically depends on the degree of patent protection. To interpret these empirical findings through the lens of our model, we use the strength of patent protection as a proxy for an industry's second-mover advantage: the stronger the patent protection, the weaker the second-mover advantage.

Weak patent protection makes it easier for Follower to imitate Leader's successful strategies. As a result, firms only enter the market when demand is sufficiently high to support both firms profitably. In contrast, strong patent protection restricts Follower's imitation, thus allowing Leader to earn monopoly rents. While Leader's profits as an incumbent increase with market demand, the duration of monopoly rents decreases as demand rises. These two

[2] See Tadelis (2013) for a textbook treatment on the mixed-strategy equilibrium and discussions for these two equilibrium conditions.

opposing forces—higher current profits and shorter monopoly duration—cause a firm's innovation or entry propensity to respond non-monotonically to changes in market demand when patent protection is strong, i.e., when second-mover advantage is weak. In summary, our model generates predictions consistent with the findings of Fabrizio and Tsolmon (2014).

Next, we provide two equilibrium refinements by extending our complete-information model in Sections 2-3 to two distinct incomplete-information settings. In Section 5, we develop a reputation model following the seminal 'gang of four' paper by Kreps, Milgrom, Roberts, and Wilson (1982), as well as Abreu and Gul (2000). Specifically, we assume that a firm may be crazy (with some probability, however small), and a crazy firm always seeks to 'win' by entering as Follower with probability one. Importantly, as the probability that a firm is crazy approaches zero, the unique equilibrium in our reputation model converges to the symmetric mixed-strategy equilibrium of the complete-information model. This demonstrates the robustness of the mixed-strategy equilibrium discussed in Section 3.

How should we interpret the mixed-strategy equilibrium in our complete-information model? One interpretation is literal: firms 'roll the dice' when deciding whether to enter the market. Alternatively, it can be seen as a metaphor—each privately informed firm selects its pure strategy as if its competitor were playing a mixed strategy, while the competitor, also privately informed, likewise employs a pure strategy. This insight motivates our second equilibrium refinement. In Section 6, we develop a dynamic incomplete-information game in which each firm knows its own entry cost as Leader but does not know its competitor's entry cost. By extending the classic purification analysis of Harsanyi (1973) to our dynamic setting, we show that as type uncertainty vanishes—eliminating entry-cost uncertainty—the pure-strategy equilibrium of this game converges to the mixed-strategy equilibrium of the complete-information game analyzed in Section 3. This dynamic purification-based extension provides further Bayesian-Nash-equilibrium-based support for the mixed-strategy in our baseline complete-information game.

In the model of Sections 2-3, we focused on how learning from observing peers' experiences lowers a firm's entry cost and generates equilibrium second-mover advantage under complete and symmetric information. However, in practice, a firm's entry can reveal its private signals about market demand, which affect its peer's profitability and incentives to enter. Using a horizontal infill oil and gas wells setting, Décaire and Wittry (2025) find that firms actively learn about *"(a) peers' beliefs regarding well profitability (i.e., partially revealing private information)"* and *"(b) information regarding the wells' realized outcome (i.e., production)"* (see page 9 in Décaire and Wittry (2025).)

To capture (*a*) and (*b*) above, in Section 7 we model endogenous information spillover in a setting with two key features. First, Follower forms a more accurate estimate of its own profitability based on Leader's private signals, as inferred from Leader's entry. Second, upon Leader's entry, Follower learns the true (unobserved, common) value that indicates whether the project is likely to be viable—for example, whether the well is a dry hole, as in Décaire and Wittry (2025)'s empirical study. Anticipating these information spillovers, a firm strategically delays entry to learn from early entrants. Finally, in Appendix F, we develop an extended model with *ex ante* firm heterogeneity and show that its predictions are consistent with the empirical findings in Décaire and Wittry (2025).

Related literature. Our paper is closely related to the literature on real-option games. Grenadier (1996) develops a real-option duopoly model to examine real-estate development cascades and overbuilding. Weeds (2002) studies irreversible investment in competing research projects with uncertain returns under a winner-takes-all patent system. Bustamante (2015) analyzes how strategic interaction among *ex ante heterogeneous* firms in product markets affects industry investment dynamics and expected returns. These models focus on first-mover advantage.

Grenadier (2002) and Back and Paulsen (2009) analyze continuous-time games in which firms accumulate capital in an oligopolistic industry. They show that competition accelerate firms' investment rates, with corporate investment characterized by a running maximum of capital stock. Whether an equilibrium is subgame perfect is a key issue in the real-option games literature. Back and Paulsen (2009) demonstrate that the perfectly competitive outcome, produced by closed-loop strategies that are firms' mutually best responses, is Markov subgame perfect in their model. These studies also focus on first-mover advantage, but model strategic interactions using singular-control rather than optimal-stopping games.[3]

[3]Kogan (2001, 2004) develop production-based general-equilibrium asset-pricing models, using singular controls to model investment irreversibility.

There is also a growing body of literature that integrates industrial organization considerations into asset pricing models. Dou, Ji, and Wu (2022) extend the standard Lucas-tree asset pricing model to incorporate endogenous strategic competition. Chen, Dou, Guo, and Ji (2024) investigate how strategic competition and financial distress interact dynamically.

Our work is further related to real-option models with imperfect information. In Grenadier (1999), the qualities of firm signals are known in advance, making the sequence of private information revelation certain in equilibrium, with the most informed firm revealing its information first. In contrast, in our model presented in Section 7, firm signals are unknown and drawn from the same distribution, resulting in a stochastic revelation sequence. A firm enters the market only when it infers that its signal is superior to its competitor's. Lambrecht and Perraudin (2003) develop an incomplete-information duopoly model in which the first mover takes all, characterizing a firm's optimal real-option exercising strategy in the absence of a second-mover advantage. Finally, our model is also related to real-option signaling models, such as Grenadier and Malenko (2011), Morellec and Schürhoff (2011), and Bustamante (2012). Unlike these models, our incomplete-information models examine imperfect competition between firms with private information and feature an equilibrium second-mover advantage.

2 Complete-information Model

In this section, we set up an entry game in which two *ex ante* identical firms decide when to enter a new market. Let τ_L and τ_F denote the stochastic times at which Leader and Follower enter the market, respectively. By definition, $\tau_F \geq \tau_L$. The determination of which firm becomes Leader is endogenous and random.

At the core of our model is the second-mover advantage. By observing Leader's successes and failures, Follower can benefit from a learning curve, which lowers its own production costs or allows it to operate a more efficient production technology. We capture this second-mover advantage by letting Follower's entry cost be lower than Leader's. Let $K_L > 0$ and $K_F > 0$ denote the lump-sum entry costs that Leader and Follower must pay at their respective entry times τ_L and τ_F. Let R denote the entry-cost ratio:

$$R = K_L/K_F \,. \tag{1}$$

To ensure a second-mover advantage, it is necessary to require $R > 1$.

2.1 Industry Structure and Market Demand

The industry structure consists of three phases. First, before either firm enters ($t < \tau_L$), neither firm receives any cash flow. Second, after Leader enters at τ_L and before Follower enters at τ_F, the industry operates as a monopoly. During this monopoly stage, Leader captures the total industry profits, $\{X_t; t \in [\tau_L, \tau_F)\}$, where $\{X_t\}$ follows a geometric Brownian motion (GBM) as in McDonald and Siegel (1986) and Dixit and Pindyck (1994):

$$dX_t = \mu X_t dt + \sigma X_t d\mathcal{Z}_t \,, \quad X_0 = x_0 > 0 \,. \tag{2}$$

The parameters μ and $\sigma > 0$ are the drift and volatility of the growth rate of X, $\{\mathcal{Z}_t; t \geq 0\}$ is a standard Brownian motion.

Third, after Follower enters at τ_F, the industry becomes a duopoly and the total profits change from X_t to $2DX_t$, where $D \in (0, 1)$, and $(2D - 1)$ measures the change in the total industry profits as the industry transitions from a monopoly to a duopoly. We divide the total industry profits $2DX_t$ into $D_L X_t < X_t$ for Leader and $D_F X_t$ for Follower, so that $D_L + D_F = 2D$. While the industry profits may expand (if $D \geq 1/2$) or shrink (if $D < 1/2$), Leader's profits will be reduced by Follower's entry. To ease exposition, we further assume that the two firms equally split industry profits ($D_L = D_F$), so each receives profits at the rate of $\{DX_t; t \geq \tau_F\}$, as in Dixit and Pindyck (1994)'s duopoly model (see Chapter 9).

Let τ_a and τ_b respectively denote firm a's and b's stochastic entry time before Leader is determined. Leader's entry time is then given by

$$\tau_L = \min\{\tau_a, \tau_b\} = \tau_a \wedge \tau_b \,. \tag{3}$$

Both firms are risk-neutral and discount profits at the constant rate r. As in the standard real-option models, we require $r > \mu$ and $r > 0$. Let $\Pi(x)$ denote the present value of X:

$$\Pi(x) = \mathbb{E}_t^x \left[\int_t^\infty e^{-r(s-t)} X_s ds \right] = \frac{x}{r - \mu}. \tag{4}$$

Below we summarize these assumptions, which apply throughout our analysis:[4]

$$\textit{Assumptions}: \quad r > \mu, \quad r > 0, \quad \sigma > 0, \quad K_L > 0, \quad K_F > 0, \quad D \in (0, 1) \,. \tag{5}$$

[4]As we show below, all key theoretical results and empirical predictions in our analysis hold regardless of the sign of the drift μ.

2.2 Duopoly Model Solution Procedure

We solve our duopoly model using backward induction. After both firms have entered ($t > \tau_F$), they equally split profits and thus are valued at $D\Pi(x)$. Next, we calculate Follower's and Leader's values after Leader's entry but before Follower's entry, i.e., for the (τ_L, τ_F) period.

Follower's pre-entry and Leader's post-entry values: $F(x)$ and $L(x)$. Follower's value in the (τ_L, τ_F) period is given by:

$$F(x) = \max_{\tau_F \geq t} \mathbb{E}_t^x \left[\int_{\tau_F}^{\infty} e^{-r(s-t)} D X_s ds - e^{-r(\tau_F - t)} K_F \right] , \tag{6}$$

where $X_t = x > 0$. Let τ_F^* denote Follower's optimal entry time for (6). Taking τ_F^* as given, we define Leader's post-entry value function, $L(x)$, for $t \in (\tau_L, \tau_F^*)$ as:

$$L(x) = \mathbb{E}_t^x \left[\int_t^{\infty} e^{-r(s-t)} X_s ds - \int_{\tau_F^*}^{\infty} e^{-r(s-t)} (1-D) X_s ds \right] , \tag{7}$$

where the first term in (7) represents the time-t value if Leader were a perpetual monopoly and the second term represents the time-t value taken away by Follower upon entry at τ_F^*.

2.2.1 Step 1: Solving $F(x)$, $L(x)$, and Follower's Optimal Entry Time τ_F^*.

Using backward induction, we first jointly solve Follower's optimal entry time τ_F^* and its closed-form value function $F(x)$, and then calculate Leader's post-entry value $L(x)$.

Follower's optimal entry threshold τ_F^* and pre-entry value $F(x)$. Follower's entry decision boils down to the classic single firm's real-option problem (McDonald and Siegel, 1986; Dixit and Pindyck, 1994), in which Follower incurs the cost K_F at its chosen entry time τ_F^* and subsequently receives a stochastic flow payoff of DX_t for all $t \geq \tau_F$. Follower's value $F(x)$ is thus given by:

$$F(x) = (D\Pi(x_F) - K_F) \left(\frac{x}{x_F} \right)^{\beta} , \quad x < x_F, \tag{8}$$

$$F(x) = D\Pi(x) - K_F , \quad x \geq x_F, \tag{9}$$

where the optimal entry threshold, x_F, is given by

$$x_F = \frac{1}{D} \frac{\beta}{\beta - 1} (r - \mu) K_F \tag{10}$$

and $\beta > 1$ measures optionality and is given by

$$\beta = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2} . \tag{11}$$

As in standard real-option models, Follower's pre-entry value $F(x)$ is increasing and convex. The higher the volatility σ, the higher the value $F(x)$.

Leader's post-entry value $L(x)$. Solving $L(x)$ defined in (7), we obtain

$$L(x) = \Pi(x) - (1 - D)\,\Pi(x_F) \left(\frac{x}{x_F}\right)^{\beta} , \quad x < x_F , \tag{12}$$

$$L(x) = D\Pi(x) , \quad x \geq x_F . \tag{13}$$

In the $x \geq x_F$ region, both Leader and Follower are in the market, equally splitting the market profits and valued at $D\Pi(x)$. In the $x < x_F$ region, Leader's time-t value $L(x)$ equals the difference between $\Pi(x)$ and $(1 - D)\,\Pi(x_F) \left(\frac{x}{x_F}\right)^{\beta}$, the latter of which equals the value of Leader's lost profits due to Follower's entry. Note that solving $L(x)$ is purely a valuation problem, as no decision by Leader is involved. Leader's value $L(x)$ is concave in the $x < x_F$ region but linear in the $x \geq x_F$ region. Therefore, $L(x)$ is not globally concave. This property has important equilibrium implications as we will show later.

Next, we turn to Step 2, the final and key step of our analysis for the $[0, \tau_L)$ period. In this period, firms formulate their optimal entry strategies into a market with no incumbents.

2.2.2 Step 2: Determining Leader and Its Entry Time τ_L^*

For a pair of entry strategies (τ_a, τ_b), firm i's continuation value $J_i(x)$ at time t is given by

$$\mathbb{E}_t^x \left[e^{-r(\tau_L - t)} \left(\mathbf{1}_{\tau_i < \tau_{-i}} (L(X_{\tau_i}) - K_L) + \mathbf{1}_{\tau_i > \tau_{-i}} F(X_{\tau_{-i}}) + \mathbf{1}_{\tau_i = \tau_{-i}} \frac{L(X_{\tau_i}) - K_L + F(X_{\tau_i})}{2} \right) \right] , \tag{14}$$

where $\tau_L = \tau_i \wedge \tau_{-i}$, $X_t = x > 0$, and $\mathbf{1}_A$ is an indicator function that equals one if event A occurs and zero otherwise. The first term in (14) describes the event where firm i becomes Leader, the second term describes the event where firm i becomes Follower, and the last term accounts for the scenario where the two firms enter at the same time. Firm i chooses its optimal entry time τ_i to maximize its value given in (14), taking into account the best response of its competitor, firm $-i$. Next, we define second-mover advantage.[5]

[5]Our definition of second-mover advantage builds on the definition of second-mover advantage in a two-period setting with pre-assigned Leader and Follower in Gal-Or (1985, 1987).

Definition 1 A second-mover advantage exists at time t if $S(X_t) > 0$, where

$$S(x) = F(x) - (L(x) - K_L). \tag{15}$$

We can generalize our baseline model by assuming that Follower's entry cost decreases with time elapsed since Leader's entry.[6] This captures the idea that the longer Follower learns from Leader, the greater the benefit, resulting in a lower entry cost (see Appendix G for our analysis of this generalized model). The main economic predictions of second-mover advantage, such as the mixed-strategy equilibrium, continue to hold in this alternative formulation of our baseline complete-information model.

3 Complete-information Model Solution

In this section, we first provide a general characterization of the solution and then focus on the symmetric mixed-strategy equilibrium solution.

3.1 Characterizing Model Solution via Three Cases

Depending on the value of R, the solution falls into one of the three cases, Case A, Case B, and Case C. Next, we describe these three cases of our model solution.

Proposition 1 *Let* $R_{AB} = D^{-1}\big[\beta\big(1-D\big)+D\big]^{-\frac{1}{\beta-1}} > 1$. *The solution falls into one of the following three cases.*

A. *If* $R > R_{AB}$, *then* $S(x) > 0$ *for all* $x > 0$.

B. *If* $1 < R < R_{AB}$, *then* $S(x) < 0$ *for* $\widehat{x}_L < x < \widehat{x}_F$, *and* $S(x) > 0$ *for* $x < \widehat{x}_L$ *or* $x > \widehat{x}_F$, *where* $\widehat{x}_F > \widehat{x}_L$ *are the two roots for* $S(x) = 0$.

C. *If* $R < 1$, *then* $S(x) < 0$ *for* $x > \widehat{x}_L$ *and* $S(x) > 0$ *for* $x < \widehat{x}_L$, *where* $\widehat{x}_L \in (0, x_F)$ *is the unique root of* $S(x) = 0$.

[6] Patent protection is often of fixed duration, which implies that the second-mover advantage starts small but increases over time as the patent gets closer to its expiration date. A well-known case is the ulcer-relief drug Zantac (Berndt, Pindyck, and Azoulay, 2003). In Appendix G, we extend our baseline model here to allow for a time-varying second-mover advantage.

To highlight the effect of second-mover advantage on firm entry and value, we focus on Case A where $S(x) > 0$ for all $x > 0$ in the main body of this paper. In Appendix IA.B, we summarize and discuss the solution for Case B where first- and second-mover advantages arise in equilibrium for different levels of market demand. We relegate the solution and analysis for Case C to Internet Appendix IA.D.

3.2 MPE: Definition and Closed-Form Solution

In Case A, $S(x) > 0$ holds for all $x > 0$. Between waiting and entering as Leader, a firm strictly prefers waiting, as it yields a higher payoff. But there is no Follower without Leader. Does this mean there is no equilibrium? No, the above reasoning only rules out symmetric pure-strategy equilibrium. We show that there exist two asymmetric pure-strategy equilibria and a symmetric *mixed-strategy* equilibrium. Below we focus on the symmetric equilibrium.

How does a firm enter using a mixed strategy in a continuous-time model? It does so probabilistically with a rate over a small time increment dt. Our mixed-strategy equilibrium is related to the mixed-strategy equilibrium in classic war-of-attrition games (Levin, 2004; Polak, 2008), which feature constant equilibrium exit rates.[7] As market demand X_t is stochastic in our model, the entry rate for our mixed-strategy equilibrium depends on X_t. Let $\lambda_i(X_t)$ denote the rate at which firm i becomes Leader over a small time interval $[t, t + dt]$ where $t < \tau_L$. Firm i's entry time τ_i is a doubly stochastic process as its associated rate $\{\lambda_i(X_t)\}_{t\geq 0}$ is also stochastic (Duffie, 2001). Next, we define feasible Markov mixed strategies and the Markov perfect mixed-strategy equilibrium.

Definition 2 An entry rate $\lambda_i(x)$ is a measurable function from $\mathbb{R}_+$ to $\mathbb{R}_+$. A pair of Markov strategy $(\lambda_a(\,\cdot\,), \lambda_b(\,\cdot\,))$ is feasible if and only if for any $t > 0$, $\int_0^t \lambda_i(X_s)ds < \infty$ almost surely. Let Φ denote the set of all feasible Markov mixed strategies.

Definition 3 Let $J_i(x; \lambda_a, \lambda_b)$ denote firm i's continuation value at time t defined in (14) for a given $X_t = x > 0$ and a feasible Markov mixed strategy pair (λ_a, λ_b). A feasible strategy pair $(\lambda_a^*, \lambda_b^*)$ is a *Markov subgame perfect mixed-strategy equilibrium* if for any $x > 0$, the following

[7] For war-of-attrition models, see Ghemawat and Nalebuff (1985, 1990), Fudenberg and Tirole (1986), and Hendricks, Weiss, and Wilson (1988) with deterministic payoffs and Steg (2015) with stochastic payoffs.

conditions hold:

$$J_a(x;\lambda_a^*,\lambda_b^*) \geq J_a(x;\lambda_a,\lambda_b^*), \quad \forall\,(\lambda_a,\lambda_b^*)\in\Phi, \tag{16}$$

$$J_b(x;\lambda_a^*,\lambda_b^*) \geq J_b(x;\lambda_a^*,\lambda_b), \quad \forall\,(\lambda_a^*,\lambda_b)\in\Phi. \tag{17}$$

Let $V_i(x)$ denote firm i's equilibrium value function: $V_i(x) = J_i(x;\lambda_a^*,\lambda_b^*)$.

Next, we summarize the key results in Theorem 1.

Theorem 1 *There exist two asymmetric pure-strategy Markov perfect equilibria and a unique symmetric mixed-strategy Markov perfect equilibrium.*[8] *In the symmetric equilibrium,* $V_a(x) = V_b(x) = V_*(x)$*, where* $V_*(x)$ *is the unique solution for the following variational-inequality problem in the* $x \geq 0$ *domain:*

$$\max\left\{\underbrace{\frac{\sigma^2x^2}{2}V_*''(x)+\mu x V_*'(x)-rV_*(x)}_{\text{equals 0 in waiting regions}},\quad \underbrace{(L(x)-K_L)-V_*(x)}_{\text{equals 0 in mixed-entry regions}}\right\}=0\,. \tag{18}$$

The equilibrium strategy is given by $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$*, where* $\lambda^*(x) = 0$ *in the* $V_*(x) > L(x) - K_L$ *region, and firms enter probabilistically at a strictly positive rate of* $\lambda^*(x)$*, as given below, in the* $V_*(x) = L(x) - K_L$ *region:*

$$\lambda^*(x) = \frac{1}{S(x)}\left[\underbrace{\left(\mathbf{1}_{x<x_F}+D\mathbf{1}_{x\geq x_F}\right)x}_{\text{Leader's operating profit}} - rK_L\right]. \tag{19}$$

First, we explain the valuation equation (18) for $V_*(x)$. When choosing to wait, a firm must be better off not entering, so $V_*(x) > L(x)-K_L$. As waiting implies zero profit, with risk neutrality, the standard valuation method suggests that the annuity value of $V_*(x)$, $rV_*(x)$, equals the expected change in firm value: $\frac{\sigma^2x^2}{2}V_*''(x)+\mu x V_*'(x)$ (Duffie, 2001). This explains the first term in (18). When playing mixed strategies, a firm must be indifferent between entering and not entering, so $V_*(x) = L(x) - K_L$. This explains the second term in (18).

Next, we provide an economically intuitive interpretation of the equilibrium entry rate $\lambda^*(x)$ given in (19). Consider firm i's decision over an infinitesimal time interval $(t, t+dt)$. Since the firm is using a mixed strategy, it must be indifferent between the following two strategies: (1) entering immediately at t and (2) waiting in the period $(t, t+dt)$ and then entering at $t+dt$. Under strategy (1), the firm becomes Leader at t, thus receiving a net payoff of $L(X_t) - K_L$. Under strategy (2), the firm receives two possible net payoffs at

[8] See Section 3.4 and Appendix A.3 for closed-form characterizations of the two pure-strategy equilibria.

$t + dt$: either $F(X_{t+dt})$ as Follower if its competitor enters during $(t, t + dt)$, which occurs with probability $\lambda^*_{-i}(X_t)dt$, or $L(X_{t+dt}) - K_L$ as Leader, which occurs with the remaining probability $1 - \lambda^*_{-i}(X_t)dt$. Discounting these payoffs to time t using the discount factor e^{-rdt}, firm i's expected payoff at time t is

$$\mathbb{E}_t[e^{-rdt}F(X_{t+dt})]\lambda^*_{-i}(X_t)dt + \mathbb{E}_t[e^{-rdt}(L(X_{t+dt}) - K_L)](1 - \lambda^*_{-i}(X_t)dt). \tag{20}$$

Equating the payoffs from the two strategies, ignoring all second-order terms, and simplifying the algebra, we obtain[9]

$$\underbrace{S(X_t)}_{\text{Reward of being Follower}} \cdot \lambda^*_{-i}(X_t)dt = \Big(\underbrace{\big(\mathbf{1}_{X_t<x_F} + D\mathbf{1}_{X_t\geq x_F}\big)X_t}_{\text{Leader's operating profit}} - rK_L\Big)dt. \tag{21}$$

Equation (21) balances two forms of flow costs over $(t, t + dt)$: the expected forgone second-mover advantage if entering as Leader at t (given by the left side of (21)) and the foregone flow profits of not entering as Leader at t (given by the right side of (21)). As in the mixed-strategy equilibrium, entry strategies are symmetric, we thus have: $\lambda^*_a(x) = \lambda^*_b(x) = \lambda^*(x)$ where $\lambda^*(x)$ is given by (19).

Conceptually, in our model firms play a war-of-attrition game (see Levin (2004) for a PhD teaching note on wars of attrition) and Leader is the loser of the game. There are two key differences between our model and the standard war-of-attrition games. First, ours is an *entry* game with stochastic and *endogenous* payoffs. Second, the equilibrium value function is nonconvex, the mixed-entry regions can be disconnected, and the entry rate is non-monotonic in market demand.

Next, we discuss in detail the solution summarized in Theorem 1. There are four mutually exclusive equilibrium regions, two waiting regions alternating with two mixed-entry regions.

[9]First, $\mathbb{E}_t[e^{-rdt}F(X_{t+dt})]\lambda^*_{-i}(X_t)dt = (1 - rdt)F(X_t)\lambda^*_{-i}(X_t)dt + O((dt)^2) = F(X_t)\lambda^*_{-i}(X_t)dt + O((dt)^2)$. Second, using Itô's formula, we have:

$$\begin{aligned}
&\mathbb{E}_t[e^{-rdt}(L(X_{t+dt}) - K_L)](1 - \lambda^*_{-i}(X_t)dt) \\
=&(1 - rdt)\Big(L(X_t) + \big(\mu X_t L'(X_t) + \sigma^2 X_t^2 L''(X_t)/2\big)dt - K_L\Big)(1 - \lambda^*_{-i}(X_t)dt) + O((dt)^2) \\
=&\Big(L(X_t) - K_L\Big)(1 - \lambda^*_{-i}(X_t)dt) - \Big(\big(\mathbf{1}_{X_t<x_F} + D\mathbf{1}_{X_t\geq x_F}\big)X_t - rK_L\Big)dt + O((dt)^2).
\end{aligned}$$

Substituting the above into (20), and ignoring $O((dt)^2)$, we can obtain the equation (21).

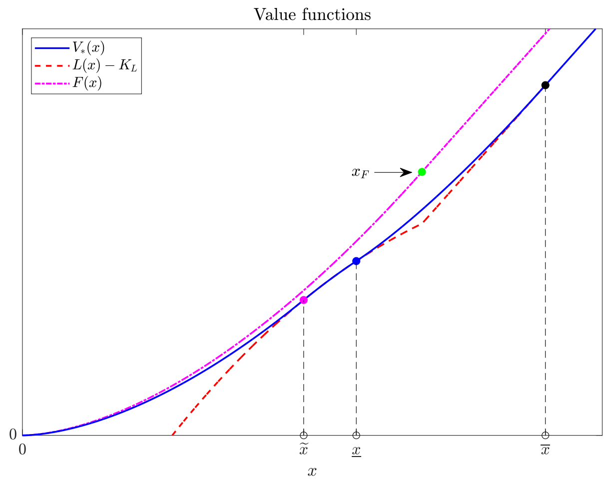

Figure 1: DETERMINING THE VALUE FUNCTION $V_*(x)$ IN THE SYMMETRIC EQUILIBRIUM. The equilibrium value function is $V_*(x)$, the solid blue line. It smoothly pastes onto the dashed red line, which gives the net payoff of becoming Leader, $L(x) - K_L$, at three endogenous thresholds: $\widetilde{x}$, $\underline{x}$, and $\overline{x}$. Follower's value is $F(x)$, the convex dash dotted magenta line, where x_F is the Follower's optimal entry threshold.

3.3 Four-Region Solution

Below we use an intuitive smooth-pasting approach to pin down the firm value $V_*(x)$ in Figure 1 and obtain our four-region solution implied by Theorem 1.[10] First, we plot $L(x)-K_L$, which is concave in the $x < x_F$ region and linear in the $x \geq x_F$ region (the dashed red line). Second, we plot Follower's value $F(x)$, which is convex in the $x < x_F$ region and linear in the $x \geq x_F$ region (the magenta dash-dotted line). Because the magenta dash-dotted line is above the dashed red line, indicating that $S(x) > 0$ holds for all $x > 0$, neither firm wants to be Leader with probability one. However, without Leader, there is no Follower. In a *symmetric* equilibrium, both firms must enter probabilistically. This is the third step, which involves drawing a line from the origin and smoothly pasting it onto the dashed red line $L(x) - K_L$. This gives a solid blue line for the pre-τ_L^* firm value $V_a(x) = V_b(x) = V_*(x)$ with three endogenous thresholds: $\widetilde{x}$, $\underline{x}$, and $\overline{x}$, which in turn define the four equilibrium regions.

As x increases from zero to ∞, a firm finds itself in one of the four mutually exclusive

[10] In Figures 1 and 2, we set $D = 0.55$, $K_L = 1.28$ and $K_F = 1$, and choose commonly used values: $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$ per annum, following standard practice in the real-options and contingent-claim literature, e.g., Grenadier (1996) and Leland (1994). As the main purpose of these two figures is to highlight the shapes of firm value and the equilibrium entry rates, we focus on various regions, the endogenous thresholds, concavity/convexity of firm value, and thus omit numbers on both axes.

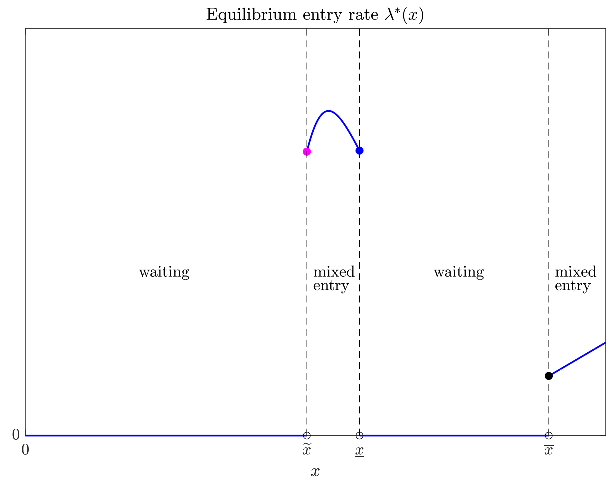

Figure 2: EQUILIBRIUM ENTRY RATE $\lambda^*(x)$ IN THE SYMMETRIC EQUILIBRIUM.

regions: 1.) a standard option-value-of-waiting region; 2.) a mixed-entry region with a stochastic duration of monopoly profits for Leader; 3.) a second waiting region; and 4.) a mixed-entry region with no monopoly profits for Leader.

3.3.1 Option-value-of-waiting Region

The first region from the left of x is $x \in [0, \widetilde{x})$, where firm value $V_*(x)$ is given by

$$V_*(x) = \left(\frac{x}{\widetilde{x}}\right)^{\beta} (L(\widetilde{x}) - K_L) \ , \quad x \in [0, \widetilde{x}) \, , \tag{22}$$

and the threshold $\widetilde{x}$ is given by:

$$\widetilde{x} = \frac{\beta}{\beta - 1}(r - \mu)K_L. \tag{23}$$

Note that $\widetilde{x}$ is the optimal investment threshold of a monopolist with entry cost K_L, as described in the classical real-option problem by McDonald and Siegel (1986) and Dixit and Pindyck (1994). Firm value equals the product of (a.) the present value of a dollar paid at $\tau := \inf\{s : X_s = \widetilde{x}\}$ and (b.) the net payoff if the firm enters at τ as Leader.[11] In this region, firm value is convex in x and the firm preserves the option value by waiting.

[11] Using the standard value-matching and smooth-pasting conditions at $x = \widetilde{x}$, i.e., $V_*(\widetilde{x}) = L(\widetilde{x}) - K_L$ and $V_*'(\widetilde{x}) = L'(\widetilde{x})$, we obtain firm value $V_*(x)$ given in (22) and the threshold $\widetilde{x}$ is given in (23).

3.3.2 Two Disconnected Mixed-entry Regions

Mixed-entry regions with no monopoly profits for Leader. When market demand is high (i.e., when $x \geq \overline{x}$), both firms play mixed strategies even though neither wants to be Leader. Why? This is because waiting is even more costly than entering probabilistically as the forgone operating profit is simply too high. Moreover, as soon as one firm enters, the other (lucky) firm immediately enters.

In sum, for $t > \tau_L^*$, Leader earns no monopoly rents, and $F(x) = D\Pi(x) - K_F$ and $L(x) = D\Pi(x)$ when $x \geq \overline{x}$. For $t \leq \tau_L^*$, as firms enter probabilistically, we have

$$V_*(x) \quad = \quad D\Pi(x) - K_L\,, \quad x \geq \overline{x}\,. \tag{24}$$

Substituting $S(x) = K_L - K_F$ into (19), we obtain the following equilibrium entry rate:

$$\lambda^*(x) = \frac{Dx - rK_L}{K_L - K_F} > 0\,, \quad x \geq \overline{x}. \tag{25}$$

The intuition is as follows. Because $x \geq \overline{x} > x_F$, Leader's net income is given by the numerator in (25) and the net value loss when becoming Leader is $S(x) = K_L - K_F$, which is the denominator in (25). In equilibrium, the linear $\lambda^*(x)$ entry rate makes the firm indifferent between entering and waiting. The higher the value of x, the greater the costs of forgoing one-period profit and thus the more likely firms are to enter to end the game sooner.

Mixed-entry regions with a stochastic duration of monopoly profits for Leader. What if X_t is intermediate, i.e., in the $x \in [\widetilde{x}, \underline{x}]$ region? In this case, the market demand is large enough for firms to prefer entering probabilistically rather than waiting. While the market demand is sufficient for one firm, it is not large enough for both firms to enter. This follows from the equilibrium property that $x_F > \underline{x}$. Once Leader is determined at τ_L^*, the other firm optimally waits, allowing Leader to collect monopoly rents until Follower optimally enters at $\tau_F^* = \inf\{s : X_s \geq x_F\}$.

In sum, firms enter probabilistically in this region, and a firm's pre-entry value is given by $V_*(x) = L(x) - K_L$, which is concave. The equilibrium entry rate function is:

$$\lambda^*(x) = \frac{x - rK_L}{S(x)}, \quad x \in [\widetilde{x}, \underline{x}], \tag{26}$$

where $S(x) = F(x) - (L(x) - K_L)$. The numerator in (26) is the forgone monopoly profit $(x - rK_L)$ due to waiting. The denominator in (26) is the second-mover's advantage $S(x)$, which is nonlinear and increasing in $x \in [\widetilde{x}, \underline{x}]$. As a result, the equilibrium entry rate $\lambda^*(x)$ that makes firms willing to enter probabilistically as Leader is non-monotonic in x. This non-monotonic entry likelihood prediction differs from the increasing entry likelihood prediction for the other mixed-entry ($x \geq \overline{x}$) region. Figure 2 demonstrates the different entry rate functions $\lambda^*(x)$ in the two mixed-entry regions.

Finally, we turn to the other waiting region that lies between the two mixed-entry regions.

3.3.3 Waiting Before Entering into either Mixed-entry Region

When x is intermediate high, i.e., in the $x \in (\underline{x}, \overline{x})$ region, firms prefer waiting over entering probabilistically. This is because X_t is close to Follower's optimal (pure-strategy) entry threshold x_F and therefore a firm's benefit of waiting dominates the cost of forgoing net income that it could earn as Leader. As there is no profit for either firm in this waiting region, the following ordinary differential equation (ODE) characterizes $V_*(x)$:

$$\frac{\sigma^2 x^2}{2} V_*''(x) + \mu x V_*'(x) - r V_*(x) = 0 \ , \quad x \in (\underline{x}, \overline{x}), \tag{27}$$

subject to the standard value-matching and smooth-pasting conditions at the two endogenous thresholds: $\underline{x}$ and $\overline{x}$ (see Lemma 1 of Appendix A.4). We obtain the following closed-form expression:

$$V_*(x) = \Theta(x; \underline{x}, \overline{x}), \quad x \in (\underline{x}, \overline{x}) \,, \tag{28}$$

where $\Theta(x; a, b)$ for any $x \in [a, b]$ is given by

$$\Theta(x; a, b) = \theta_1(a, b) x^\beta + \theta_2(a, b) x^\gamma \,, \tag{29}$$

$\theta_1(a, b)$ and $\theta_2(a, b)$ are two functions given in (A.12), $\beta > 1$ is given in (11), and

$$\gamma = \frac{-(\mu - \frac{1}{2}\sigma^2) - \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2} \,. \tag{30}$$

Note that firm value $V_*(x)$ is convex in this region. Intuitively, in this region, higher volatility increases firm value as it raises the probability that the firm enters either mixed-entry region, allowing firms to end the inefficient waiting game sooner. In Lemma 1 of Appendix A.4, we prove $\widetilde{x} < \underline{x} < x_F < \overline{x}$, characterize the pair $(\underline{x}, \overline{x})$ defining the waiting region, and prove that the value function is convex ($V_*''(x) > 0$) in this region.

In equilibrium, $V_*(x)$ changes the sign of its curvature *twice* by smoothly pasting onto

Leader's net payoff line $L(x) - K_L$ *three* times at $\widetilde{x}$, $\underline{x}$, and $\overline{x}$. That is, the shape of $V_*(x)$ changes from convex in the $x < \widetilde{x}$ region to concave in the $x \in (\widetilde{x}, \underline{x})$ region and then back to convex in the $x \in (\underline{x}, \overline{x})$ region before becoming linear in the $x \geq \overline{x}$ region.

Solution summary. In sum, there are four regions in general for Case A where the second-mover advantage dominates: 1.) In the $x < \widetilde{x}$ region, firms wait; 2.) In the $x \in [\widetilde{x}, \underline{x}]$ region, firms enter probabilistically with Leader earning equilibrium monopoly rents; 3.) In the $x \in (\underline{x}, \overline{x})$ region, firms wait; and 4.) In the $x \geq \overline{x}$ region, firms enter probabilistically with Follower entering immediately after Leader and leaving no monopoly rents to Leader.

Two-region solution as a special case. When entry-cost ratio R is sufficiently high, the equilibrium second-mover advantage is significant, and the four-region solution simplifies to a two-region solution. In this case, firms either enter probabilistically when market demand is high or wait otherwise. Because R is high, once a firm enters, the other firm follows immediately, eliminating Leader's monopoly rents. Therefore, there are only two regions: a waiting region and a mixed-entry region (with no monopoly rents for Leader). We refer readers to Appendix A.2 for more details.

3.4 Asymmetric Pure-strategy Equilibria

At the end of this section, we characterize the asymmetric pure-strategy equilibria.

Consider the asymmetric pure-strategy equilibrium where firm a is Leader and firm b is Follower.[12] In this equilibrium, $\tau_a = \tau_L$ and let $J_L(x)$ denote Leader's value. Firm a's problem, as given in (14), simplifies to

$$J_L(x) = \max_{\tau \geq t} \mathbb{E}_t^x \left[e^{-r(\tau - t)} (L(X_\tau) - K_L) \right] . \tag{31}$$

Using Lemma 5 in Internet Appendix IA.J, Leader's value $J_L(x)$ in this equilibrium equals firm value $V_i(x) = V_*(x)$ in the symmetric mixed-strategy equilibrium: $J_L(x) = V_*(x)$ where $V_*(x)$ is the unique solution for the variational inequality (18). Leader's optimal entry time is $\tau_L^* = \inf\{s \geq t : X_s \in \mathcal{R}^E\}$, where $\mathcal{R}^E = \{x > 0 : V_*(x) = L(x) - K_L\}$.

[12] This equilibrium is supported by beliefs that firm a is Leader and firm b is Follower with probability one. Making firm a Follower and firm b Leader, we obtain the other pure-strategy equilibrium.

Now we turn to Follower's problem. Taking Leader's strategy τ_L^* as given, let $J_F(x)$ denote Follower's optimal value for the following problem:

$$J_F(x) = \max_{\tau_F \geq \tau_L^*} \mathbb{E}_t^x \left[e^{-r(\tau_F - t)} \Big(D\Pi(X_{\tau_F}) - K_F \Big) \right] . \tag{32}$$

Solving (32), we obtain Follower's optimal entry time: $\tau_F^* = \inf\{s \geq \tau_L^* : X_s \geq x_F\}$, where x_F is given in (10). Next, we summarize the pure-strategy equilibria for the four-region case.

Proposition 2 *There are two asymmetric pure-strategy equilibria. In an asymmetric pure-strategy equilibrium, Leader enters at* $\tau_L^* = \inf\{s \geq t : X_s \in [\widetilde{x}, \underline{x}] \cup [\overline{x}, \infty)\}$*, where* $\widetilde{x}$ *is given by* (23)*, and* $\underline{x}$ *and* $\overline{x}$ *are given in Lemma 1. Leader's value is given by* $J_L(x) = V_*(x)$*, where* $V_*(x) = L(x) - K_L$ *for* $x \in [\widetilde{x}, \underline{x}] \cup [\overline{x}, \infty)$*, and* $V_*(x)$ *is given by* (22) *for* $x < \widetilde{x}$ *and by* (28) *for* $x \in (\underline{x}, \overline{x})$*. Follower enters at* $\tau_F^* = \inf\{s \geq \tau_L^* : X_s \geq x_F\}$ *and Follower's value* $J_F(x)$ *is given by*

$$J_F(x) = F(x), \quad x \leq \underline{x}, \tag{33}$$

$$J_F(x) = \frac{F(\underline{x})\overline{x}^\gamma - F(\overline{x})\underline{x}^\gamma}{\underline{x}^\beta \overline{x}^\gamma - \underline{x}^\gamma \overline{x}^\beta} x^\beta + \frac{F(\overline{x})\underline{x}^\beta - F(\underline{x})\overline{x}^\beta}{\underline{x}^\beta \overline{x}^\gamma - \underline{x}^\gamma \overline{x}^\beta} x^\gamma, \quad x \in (\underline{x}, \overline{x}), \tag{34}$$

$$J_F(x) = F(x) = D\Pi(x) - K_F, \quad x \geq \overline{x}\,. \tag{35}$$

See Appendix A.3 for the two-region case.

4 Empirical Predictions: Innovation and Patent Protection

Fabrizio and Tsolmon (2014) find that in industries with weak patent protection, innovation increases with market demand whereas in industries with strong patent protection, innovation is hardly responsive to market demand. That is, how innovations respond to market demand critically depends on the degree of patent protection. Below we show that our model's predictions are consistent with these findings.

We use the measure of patent protection in Fabrizio and Tsolmon (2014) as a proxy for an industry's second-mover advantage, where stronger patent protection corresponds to a weaker second-mover advantage. Intuitively, strong patent protection makes it harder for Follower to imitate, effectively increasing Follower's entry cost, which in turn allows Leader to earn

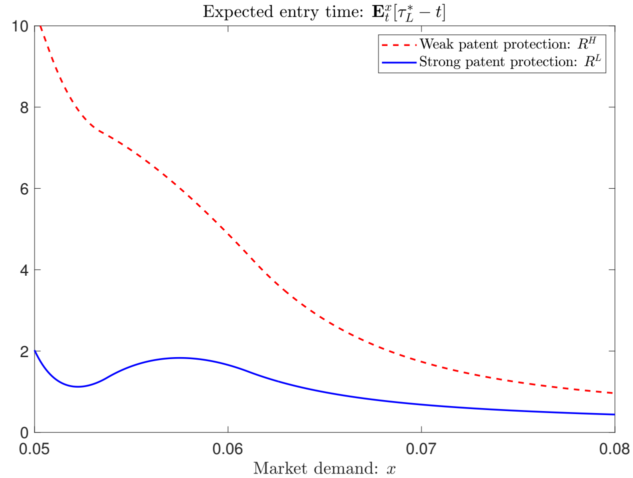

Figure 3: Expected Entry Time. Parameter values are $D = 0.85$, $K_L = RK_F$, $K_F = 1$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$. The solid blue line and the dashed red line correspond to the cases: $R = R^L = 1.024$ and $R = R^H = 1.072$, respectively.

monopoly rents.

Figure 3 plots Leader's expected entry time as a function of market demand. When patent protection is weak (the dashed red line), a firm's entry propensity increases with market demand. The intuition is that, when second-mover advantage R is sufficiently high, such that Leader earns no monopoly rents in equilibrium, a firm tends to innovate when market demand is high. In contrast, when patent protection is strong (the solid blue line), innovation propensity becomes *non-monotonic* in market demand, as shown in Section 3.

The intuition is as follows. Strong patent protection allows monopoly rents, which incentivizes earlier entry. By entering sooner when demand is lower, a firm can establish itself as a monopoly and deter competition. Interestingly, while reducing firm profits, lower market demand extends the duration of Leader's monopoly rents, as the competitor delays entry in response to weaker demand. The interplay of these two opposing forces renders a firm's entry (innovation) propensity non-monotonic in market demand.[13] Indeed, this mechanism

[13] Firm entry in Grenadier (1996) is also non-monotonic in market demand. However, the mechanisms driving the non-monotonicity result in our model and in Grenadier (1996) are quite different. The non-monotonicity in his model arises from the firms' incentives to continue collecting profits from existing assets. In contrast, the non-monotonicity in our model is driven by a relatively small second-mover advantage, corresponding to strong patent protection in Fabrizio and Tsolmon (2014). The intuition in our model is that even when market demand is low, a firm may still want to innovate under strong patent protection, as it expects to collect monopoly rents for an extended period. See Internet Appendix IA.E for a detailed comparison between Grenadier (1996) and our model.

underpins the four-region equilibrium solution for the complete-information game in Section 3, demonstrating that firm entry does not follow a simple increasing pattern with market demand when patent protection is strong.[14]

In summary, we have sharpened our complete-information model's prediction and strengthened its connection to empirical evidence on corporate innovations, as reported in Fabrizio and Tsolmon (2014).

5 Reputation and Unique Bayesian Nash Equilibrium

In this section, we develop a reputation model,[15] in which a rational firm has incentives to behave as if it were 'crazy,' with two goals in mind. First, we use this reputation model to uniquely select the mixed-strategy equilibrium out of the three MPEs in our complete-information model of Sections 2-3. Second, we use this model to generate new and economically intuitive results that are attainable only in a mixed-strategy equilibrium.

As Follower's payoff is always larger than Leader's, i.e., $S(x) > 0$ for all $x > 0$, the 'winner' of the entry game is always Follower. Building on Abreu and Gul (2000), we call a firm 'crazy' if it always 'wins' the war-of-attrition game by entering as Follower. Other than this behavioral assumption, we assume that this firm makes an optimal entry decision conditional on being Follower. That is, the crazy type only enters (weakly) after its competitor enters at τ_L, no matter how costly waiting is. Mathematically, the firm solves its stopping-time problem defined in (6) only for time $t \geq \tau_L$. There are two scenarios. When $X_{\tau_L} < x_F$, crazy type waits first and enters until the market demand shock X_t reaches its endogenous entry threshold x_F so that it enters at $\tau_F = \inf\{t \geq \tau_L : X_t \geq x_F\}$. When $X_{\tau_L} \geq x_F$, crazy type enters as soon as Leader does: $\tau_F = \tau_L+$. In the former scenario the option value of

[14]This non-monotonicity does not arise in a pure-strategy equilibrium, where Leader's entry is governed by a cutoff (threshold) strategy. When initial market demand is low, Leader enters at time τ_L^* with probability one if and only if X_t reaches the endogenous entry threshold. In other words, market demand at τ_L^* must equal Leader's entry threshold—a singleton. This implication is stark and counterfactual.

[15]The seminal 'gang of four' paper by Kreps, Milgrom, Roberts, and Wilson (1982) initiated the reputation literature. In that paper, the authors show that if the prisoner's dilemma game is repeated for long enough, then the players will cooperate in almost all periods, as long as a firm has a reputation for being crazy (with a strictly positive probability, however small it is). A crazy firm plays cooperatively until the other player deviates. Reputation models have been widely analyzed in bargaining and other related settings; see, e.g., Kambe (1999), Abreu and Gul (2000), and Abreu and Pearce (2007).

waiting is strictly positive and zero in the latter. Next we set up the model.

Model setup. Each firm takes into account that its competitor could be either crazy or rational. This leads to a dynamic incomplete-information game, which we introduce below. Let τ_i denote firm i's entry time as Leader. We focus on the symmetric common prior case (see Internet Appendix IA.H for the asymmetric prior case). That is, at $t = 0$, each firm is crazy with probability $\pi_0 \in (0, 1)$ and rational with probability $1 - \pi_0$.

Model solution. Let $\mathbb{P}_t(\,\cdot\,)$ denote the time-t probability that only depends on $\{X_s\}_{s\in[0,t]}$. The posterior that firm i is crazy, π_t^i, is[16]

$$\pi_t^i := \mathbb{P}_t(\text{firm } i \text{ is crazy} \mid \tau_L > t) = \frac{\mathbb{P}_t(\text{firm } i \text{ is crazy}, \tau_L > t)}{\mathbb{P}_t(\tau_L > t)} = \frac{\mathbb{P}_t(\text{firm } i \text{ is crazy})}{\mathbb{P}_t(\tau_i > t)}\,. \tag{36}$$

A key object in our reputation model is the (first) time that firm $-i$ learns that firm i is crazy with probability one, which we denote by $T_i = \inf\{t \geq 0 : \pi_t^i = 1\}$. As $\pi_{T_i}^i = 1$, conditional on having no Leader by $t = T_i$, a rational firm $-i$ must enter immediately. This implies that firm $-i$'s type must be revealed before T_i, meaning $T_{-i} \leq T_i$. By symmetry, we have $T_{-i} \geq T_i$. Therefore, $T_a^* = T_b^*$ in equilibrium. Let $T^* := T_a^* = T_b^*$.

Because a crazy firm never enters as Leader, we only need to characterize the rational types' strategies. From the perspective of firm $-i$, there are two scenarios: (1) firm i, if crazy, never enters as Leader; (2) firm i, if rational, enters probabilistically at the rate, denoted by $\Lambda_i(X_t, \pi_t^a, \pi_t^b)$, similar to that in our complete-information game of Section 3. Averaging these two scenarios, firm $-i$ expects that its competitor, firm i, enters at a rate, $\lambda_i(X_t, \pi_t^a, \pi_t^b) = (1 - \pi_t^i)\Lambda_i(X_t, \pi_t^a, \pi_t^b) + \pi_t^i \times 0$. Conditional on $t < \tau_L \wedge T^*$, the belief that firm i is crazy, π_t^i, thus evolves as follows:[17]

$$d\pi_t^i = \Lambda_i(X_t, \pi_t^a, \pi_t^b)\pi_t^i(1 - \pi_t^i)dt. \tag{37}$$

Next, we summarize the perfect Bayesian Nash equilibrium solution (for brevity, we relegate the formal definition of perfect Bayesian Nash equilibrium to Appendix C).

[16] First, we have $\mathbb{P}_t(\tau_L > t) = \mathbb{P}_t(\tau_a > t, \tau_b > t) = \mathbb{P}_t(\tau_a > t)\mathbb{P}_t(\tau_b > t)$. Second, $\mathbb{P}_t(\text{firm } i \text{ is crazy}, \tau_L > t) = \mathbb{P}_t(\text{firm } i \text{ is crazy}, \tau_{-i} > t) = \mathbb{P}_t(\text{firm } i \text{ is crazy})\mathbb{P}_t(\tau_{-i} > t)$.

[17] Because belief is a martingale, $\pi_t^i = \big(1 - \lambda_i(X_t, \pi_t^a, \pi_t^b)dt\big)\pi_{t+dt}^i + \lambda_i(X_t, \pi_t^a, \pi_t^b)dt \times 0$ holds over a small interval dt. Substituting $\big(1-\lambda_i(X_t, \pi_t^a, \pi_t^b)dt\big)^{-1} = 1+\lambda_i(X_t, \pi_t^a, \pi_t^b)dt$ and $\lambda_i(X_t, \pi_t^a, \pi_t^b) = (1-\pi_t^i)\Lambda_i(X_t, \pi_t^a, \pi_t^b)$ into the preceding equation and simplifying the expression, we obtain (37). The associated cumulative distribution function (CDF) for τ_i is $G_i(t) = 1 - e^{-\int_0^t \lambda_i(X_s, \pi_s^a, \pi_s^b)ds}$.

Theorem 2 *There exists a unique perfect Bayesian Nash equilibrium in which a rational firm i's equilibrium value is $V_*(x)$, where $V_*(x)$ is the unique solution for the variational-inequality problem* (18)*. The rational firm i' entry rate function is*

$$\Lambda_i^*(x, \pi^a, \pi^b) = \frac{1}{S(x)(1-\pi^i)}\Big[\big(\mathbf{1}_{x<x_F} + D\mathbf{1}_{x\geq x_F}\big)x - rK_L\Big] \tag{38}$$

in the $V_*(x) = L(x) - K_L$ region and $\Lambda_i^*(x, \pi^a, \pi^b) = 0$ in the $V_*(x) > L(x) - K_L$ region. Firm $-i$ thus anticipates that firm i, which can be rational or crazy, enters at an equilibrium entry rate of $(1-\pi^i)\Lambda_i^*(x, \pi^a, \pi^b) = \lambda^*(x)$, where $\lambda^*(x)$ is given in Theorem 1. In equilibrium, the posterior beliefs are given by

$$\pi_t^a = \pi_t^b = \pi_t = \pi_0 e^{\int_0^t \lambda^*(X_u)du}, \quad t < \tau_L^* \wedge T^*, \tag{39}$$

where $T^* := \min\{T_a^*, T_b^*\}$ is finite almost surely and given by

$$T^* = \inf\left\{t \geq 0 : \int_0^t \lambda^*(X_u)du = -\ln \pi_0\right\}. \tag{40}$$

Equilibrium properties. Below we highlight three key equilibrium properties. First, there is no pure-strategy equilibrium. We prove this by contradiction. Suppose a rational firm a enters using a pure strategy. Without loss of generality, let the initial value of X_0 be in the entry region so that firm a enters at time 0, meaning $T_a^* = 0$. Because $T_a^* = T_b^*$ in any equilibrium, $T_b^* = 0$ has to hold. However, simultaneous entry is clearly not an equilibrium, as a rational firm can deviate by just waiting for an infinitesimal period to become Follower.

Second, the equilibrium value of a rational firm $V_*(X_t)$ is the same as that in the mixed-strategy equilibrium of Theorem 1. Also, a rational firm expects that its competitor, which can be rational or crazy, enters at an equilibrium rate, which takes the same form as the equilibrium entry rate in the mixed-strategy equilibrium for the complete-information game.[18] The intuition is as follows. Increasing the probability that its competitor is crazy, π_t lowers a firm's probability of becoming Follower, but also shortens the expected time of ending the attrition game. These two opposing effects exactly offset each other, leaving firm value $V_*(X_t)$ and the expected equilibrium entry rate $\lambda^*(X_t)$ independent of π_t.

Third, the probability that a firm is crazy increases exponentially over time at a rate of

[18] As a special case, as the probabilities that firms are crazy approach zero: $\pi_t^a = \pi_t^b \to 0$, a rational firm i's entry rate $\Lambda_i^*(X_t, \pi_t^a, \pi_t^b)$ in our reputation model converges to the equilibrium entry rate for the complete-information game, $\lambda^*(X_t)$, as given in Theorem 1.

$\lambda^*(X_t)$. This is intuitive, as (37) shows that the equilibrium belief that a firm is crazy increases proportionally at the rate of $\lambda^*(X_t)$, which equals $\Lambda_i^*(X_t, \pi_t, \pi_t)$ multiplied by the probability that it is rational $(1-\pi_t)$. As beliefs are symmetric: $\pi_t^a = \pi_t^b$, both firms' types are revealed at T^*, when the cumulative integral of $\lambda^*(X_t)$, given in Theorem 1, starting from time 0 reaches $-\ln \pi_0$ for the first time, as shown in (40). Finally, we point out that both types are revealed in finite time in that T^* is finite almost surely.[19]

We now turn to the new, intuitive results that this reputation model generates and further highlight the economic appeal of mixed-equilibrium strategies. In our model, no matter how unlikely the crazy type is, a rational type always adopts the strategy described in Theorem 2, making its competitor's inference of its true type impossible.[20] In contrast, the alternative of adopting a pure strategy will fully reveal its type in equilibrium, which is suboptimal for the firm.[21] This is because its competitor can then fully capture second-mover advantage by credibly adopting Follower's optimal strategy in a pure-strategy equilibrium. Therefore, even though its competitor is not necessarily convinced that the firm is crazy, it is still optimal for this firm to build its reputation (of being possibly crazy) by adopting the strategy described in Theorem 2. Finally, since there is a real possibility that a firm is crazy and thus will never enter as Leader, a rational competitor thus cannot afford to wait indefinitely. This is why a rational firm enters probabilistically when the market demand is high.

Next, we highlight how our analysis of this reputation model helps us better understand the differences between mixed- and pure-strategy equilibria in the complete-information game of Section 3. Specifically, we have shown that compared with the pure-strategy equilibria, the mixed-strategy equilibrium in Section 3 is the more natural and robust equilibrium. This is because the symmetric mixed-strategy equilibrium characterized in Theorem 1 for the baseline complete-information model corresponds to the limit of the unique equilibrium in this newly

[19] To show this result, we proceed in three steps. First, using the equilibrium result given in (25), we have $\lambda^*(x) = \frac{Dx - rK_L}{K_L - K_F} \geq \lambda^*(\overline{x}) > 0$ for any $x \geq \overline{x}$, which implies $\int_0^t \lambda^*(X_s)ds \geq \lambda^*(\overline{x}) \int_0^t \mathbf{1}_{X_s \geq \overline{x}} ds$. Second, note the limiting result: $\int_0^t \mathbf{1}_{X_s \geq \overline{x}} ds \to \infty$ as $t \to \infty$, which follows from a property for a GBM process $\{X_t\}$: for any constant $x > 0$, the occupation time in the $X_t \geq x$ region is infinite: $\int_0^\infty \mathbf{1}_{X_s \geq x} ds = \infty$; see, e.g., Revuz and Yor (2013). Third, we prove by contradiction: if T^* were infinite, then $\int_0^{T^*} \lambda^*(X_u)du$ would equal ∞, contradicting with (40). In conclusion, T^* given in (40) must be finite almost surely.

[20] Of course, this statement only holds prior to T^* when firm types are revealed in equilibrium as discussed earlier in this section.

[21] A pure strategy fully reveals the firm's type because a rational type has to enter when the market demand reaches the prescribed entry threshold.

developed reputation model (as the probability of the crazy type decreases towards zero). And moreover, there is no asymmetric pure-strategy equilibrium in our reputation model provided that there is a strictly positive probability that a firm is crazy. Finally, we point out that this new reputation model also generates predictions consistent with empirical findings in Fabrizio and Tsolmon (2014), as the expected equilibrium entry rate $\lambda^*(X_t)$ summarized above in Theorem 2 is the same as the mixed-strategy equilibrium entry rate given in Section 3.

6 Interpreting Mixed Strategies in Section 3 by Extending Harsanyi's Purification Insight

Can we provide an intuitive interpretation of the mixed-strategy equilibrium in our complete-information game of Section 3 by relating it to a pure-strategy equilibrium in a more general game setting? The answer is yes. To demonstrate this, we extend the classic purification analysis of Harsanyi (1973) to our dynamic framework by introducing firm-type uncertainty into the complete-information entry game studied in Sections 2–3.[22] Using the solution to this enriched game and letting entry-cost uncertainty shrink to zero, we characterize its limiting outcome and show that it coincides with the mixed-strategy equilibrium of our original complete-information model.

Now we develop a dynamic Bayesian game in which each firm knows its own entry cost (as Leader) but does not know its competitor's entry cost. We set Follower's entry cost K_F to a known constant and make the standard 'common prior' assumption (Fudenberg and Tirole, 1991) for a firm's entry cost as Leader: K_L^i for $i = a, b$. That is, K_L^i is firm i's private information, and K_L^a and K_L^b are independently and identically distributed. Let $\Psi(k)$ denote the CDF for firm type, defined on the support of $[\underline{k}, \overline{k}]$, satisfying $\Psi(\underline{k}) = 0$, $\Psi(\overline{k}) = 1$, and $\Psi'(\,\cdot\,) > 0$. It is helpful to define the conditional density (mass) of firms whose entry cost lies within the $(k, k + dk)$ interval:

$$m(k) = \frac{\Psi'(k)}{1 - \Psi(k)}\,. \tag{41}$$

A key insight is that in a pure-strategy equilibrium the firm with a lower entry cost enters

[22]For textbook treatments on purification-based interpretations of mixed-strategy equilibria in static complete-information games, see Fudenberg and Tirole (1991) (pages 233-234) and Tadelis (2013) (pages 264-266).

first. Under incomplete information, even though a firm uses pure strategies, its entry is probabilistic from its competitor's perspective. Below we summarize equilibrium belief updating and entry strategy (for brevity, we relegate perfect Bayesian Nash equilibrium definition and the equilibrium value to Appendix D).

Theorem 3 *There exists a Bayesian Nash equilibrium in which firm i whose type is K_L^i enters (using a pure strategy) at $\tau_i^* = \inf\{t \geq 0 : K_t^* \geq K_L^i\}$, where K_t^* evolves as follows:*

$$m(K_t^*)dK_t^* = \lambda^*(X_t; K_t^*)dt\,, \quad K_0^* = \underline{k}\,. \tag{42}$$

In (42), $\lambda^*(x;k)$ is the mixed-strategy entry rate given in Theorem 1 for the complete-information game in which Leader's entry cost is k. From firm $-i$'s perspective, firm i's entry time has the following CDF:

$$\mathbb{P}_t(\tau_i^* \leq t) = 1 - e^{-\int_0^t \lambda^*(X_s; K_s^*)ds}. \tag{43}$$

Theorem 3 establishes the existence of a process K_t^* that characterizes the evolution of the entry cost for the firm entering at time t in equilibrium. We refer to this firm as the marginal firm at time t. A firm's optimal strategy is to enter if its entry cost at time t is weakly lower than this marginal firm's entry cost K_t^*, but to wait otherwise. Although a firm uses this pure strategy, its entry is observed to be random by its competitor. This is because the competitor does not observe the firm's type and must dynamically update its belief, causing the perceived entry likelihood to be stochastic.

To intuitively interpret our main results in Theorem 3, below we extend Harsanyi's original purification argument for static games to our dynamic setting. Fix an infinitesimal time period $(t, t+dt)$ prior to Leader's entry at τ_L and consider firm i's entry decision this period. Facing a continuum of firm $-i$'s types, K_L^{-i}, drawn from $\frac{\Psi(k)}{1-\Psi(K_t^*)}$, $k \in (K_t^*, \overline{k}]$, firm i anticipates that only those types of firm $-i$ with low entry costs choose to enter. We thus conjecture and then verify that there exists an endogenous threshold, K_{t+dt}^*, such that only those types whose entry costs are lower than K_{t+dt}^* choose to enter: $K_L^{-i} < K_{t+dt}^*$. Therefore, firm i assigns the following probability that its competitor enters this period conditional on having not entered prior to t:

$$\mathbb{P}_t(\tau^*_{-i} \in (t, t+dt)) = \mathbb{P}_t(K_L^{-i} < K^*_{t+dt} \mid K_L^{-i} > K^*_t) = \frac{\Psi(K^*_{t+dt}) - \Psi(K^*_t)}{1 - \Psi(K^*_t)} = m(K^*_t) dK^*_t, \tag{44}$$

where $m(\,\cdot\,)$ is the conditional hazard rate for $\Psi(\,\cdot\,)$ given in (41).

For the above conjecture to be an equilibrium, firm i whose type lies within (K^*_t, K^*_{t+dt}) has to be indifferent between entering this period or not. This equilibrium restriction implies $\mathbb{P}_t(\tau^*_{-i} \in (t, t+dt)) = \lambda^*(X_t; K^*_t)dt$, where $\lambda^*(X_t; K^*_t)$ is given in Theorem 1, which allows us to pin down the equilibrium dynamics of K^*_t for the marginal firm type.[23]

Finally, we show that the mixed-strategy equilibrium in the complete-information game of Section 3 is the limit of the perfect Bayesian Nash equilibrium characterized in Theorem 3, as we eliminate entry cost uncertainty. To see this result, consider a prior distribution $\Psi(\,\cdot\,)$ with a support of $(K_L - \varepsilon, K_L + \varepsilon)$ where $\varepsilon > 0$. By shrinking ε, we can see from (44) that $\lim_{\varepsilon \to 0} \mathbb{P}_t(\tau^*_i \in (t, t+dt)) = \lambda^*(X_t; K_L)dt$. That is, for a given $\{X_t\}$ path, as $\varepsilon \to 0$, a firm's entry rate in the incomplete-information game is the same as that in the complete-information game of Section 3.

In summary, by extending Harsanyi (1973)'s purification insight for static models to our dynamic setting, we can interpret the mixed-strategy equilibrium for our complete-information model as the limit of a pure-strategy equilibrium in an incomplete-information game, as we eliminate type uncertainty. Our purification-based analysis provides another rationale for us to work with the mixed-strategy equilibrium in our complete information model. Moreover, as the distribution of entry time given in this incomplete-information game is observationally similar to that in our complete-information model in Section 2, we thus conclude that the incomplete-information game developed in this section generates predictions that are consistent with

[23] Fix an infinitesimal time period $(t, t+dt)$. If entering as Leader, the firm receives a net payoff value of $L(X_t) - K^i_L$. If it waits in the period $(t, t+dt)$ and enters at $t+dt$ provided no firm has entered yet, it receives two possible net payoffs at $t+dt$: either $F(X_{t+dt})$ as Follower if its competitor enters in the period $(t, t+dt)$, which occurs with probability $\mathbb{P}_t(\tau^*_{-i} \in (t, t+dt))$, or $L(X_{t+dt}) - K^i_L$ as Leader, which occurs with the remaining probability $\mathbb{P}_t(\tau^*_{-i} \geq t+dt) = 1 - \mathbb{P}_t(\tau^*_{-i} \in (t, t+dt))$. Therefore, discounting the payoffs above to time t using the discount factor e^{-rdt}, we obtain firm i's time-t expected payoff

$$\mathbb{E}_t[e^{-rdt} F(X_{t+dt})]\mathbb{P}_t(\tau^*_{-i} \in (t, t+dt)) + \mathbb{E}_t[e^{-rdt}(L(X_{t+dt}) - K^i_L)]\mathbb{P}_t(\tau^*_{-i} \geq t+dt) = L(X_t) - K^i_L,$$

where the equality follows from the firm's indifference between entering or not as Leader. Ignoring all second-order terms in the equation above and simplifying the algebra, we obtain the result.

empirical findings in Fabrizio and Tsolmon (2014).

7 Information Spillover and Second-mover Advantage

In this section, we develop an entry model in which second-mover advantage naturally arises from firms' incentives to learn about information that is only released by Leader's entry. Our model is motivated by Décaire and Wittry (2025), who use a horizontal infill oil and gas wells setting[24] to empirically show that firms actively learn about *"(a) peers' beliefs regarding well profitability (i.e., partially revealing private information)"* and *"(b) information regarding the wells' realized outcome (i.e., production)."*

7.1 Model Setting

To capture (a) described above, we assume that the true profitability, Q, is constant but unknown to the firms, each of which has private information about well profitability. Let Q_i denote firm i's signal about Q, drawn from a cumulative distribution function $\Psi(z)$, satisfying the usual conditions.[25] To ease exposition, we assume that both firms place equal weights on their own signal and their peer's when estimating Q:

$$\mathbb{E}[Q \mid Q_a, Q_b] = \frac{Q_a + Q_b}{2}. \tag{45}$$

Regarding (b) described above, Décaire and Wittry (2025) document that *"once the first phase of drilling is completed (typically within a few weeks), firms learn whether the project is likely to be a dry hole, $\cdots$, or whether the project is likely to be viable."* To capture this information spillover, we introduce a binary random variable, θ, indicating whether a well is a dry hole, i.e., $\theta = 0$ with probability $1 - p \in (0, 1)$, or not $(\theta = 1)$, and its value is revealed to the public only at τ_L.

In summary, to capture both (a) and (b) types of learning described above, we work with three random variables, θ, Q_a and Q_b, which are independent of each other under the common prior assumption and incorporated into the baseline model of Section 2.[26] Each firm can, at

[24] See Kellogg (2014) and Décaire et al. (2020) for related empirical work using this setting.

[25] Specifically, $\Psi(\underline{q}) = 0$, $\Psi(\overline{q}) = 1$, and $\Psi'(q) > 0$ for any $q \in [\underline{q}, \overline{q}]$, where $\overline{q} > \underline{q} \geq 0$.

[26] To focus on the impact of pure information spillover and to align with the empirical analysis in Décaire and Wittry (2025), we assume that the two firms' entry costs are equal: $K_L = K_F = K > 0$, and moreover, Follower's entry does not influence Leader's profits.

any time, pay a cost of K to obtain a perpetual stochastic profit at the rate of $\theta Q X_t$, where the $\{X_t\}$ process given in (2) is publicly observed and independent of (θ, Q_a, Q_b). Conditional on the well not being a dry hole ($\theta = 1$) as revealed at τ_L, the amount of oil in the well, Q, remains unknown to firms after τ_L. Note that the setting here is one featuring a "common value."

Firm i's value at τ_L conditional on being Leader: $L_{\tau_L}(X_{\tau_L}, Q_i)$. At the moment of entry ($t = \tau_L$) as Leader, the value of firm i is

$$L_{\tau_L}(X_{\tau_L}, Q_i) = \mathbb{E}_{\tau_L}\left[\int_{\tau_L}^{\infty} e^{-r(s-t)}\theta Q X_s ds \mid Q_i, \tau_i < \tau_{-i}\right] = p\mathbb{Q}_i^L \frac{X_{\tau_L}}{r-\mu}, \tag{46}$$

where p is the probability that the well is not a dry hole and

$$\mathbb{Q}_i^L = \mathbb{E}_{\tau_L}[Q \mid Q_i, \tau_i < \tau_{-i}] \tag{47}$$

is firm i's belief about Q at τ_L conditional on not only its own signal Q_i but also on $\tau_i < \tau_{-i}$ since it enters as Leader.

Firm i's value at τ_L conditional on being Follower: $F_{\tau_L}(X_{\tau_L}, Q_i)$. At $t = \tau_L$, conditional on being Follower ($\tau_i > \tau_{-i}$), firm i's belief about Q at τ_L equals

$$\mathbb{Q}_i^F = \mathbb{E}_{\tau_L}[Q \mid Q_i, \tau_i > \tau_{-i}], \tag{48}$$

which remains constant for all $t \geq \tau_L$. This implies that firm i's problem at any $t \geq \tau_L$ boils down to choosing its entry time τ_F to solve a standard real-option problem (Dixit and Pindyck, 1994) with a cash-flow payoff process $\{\mathbb{Q}_i^F X_t\}$ and a one-time investment cost K, provided that the oil well is revealed at τ_L not to be a dry hole: $\theta = 1$. Therefore, we can write firm i's value at τ_L as $F_{\tau_L}(X_{\tau_L}, Q_i)$ given as follows:

$$F_{\tau_L}(X_{\tau_L}, Q_i) = p\left(\frac{\mathbb{Q}_i^F x_i^F}{r-\mu} - K\right)\left(\frac{X_{\tau_L}}{x_i^F}\right)^{\beta}, \quad X_{\tau_L} < x_i^F, \tag{49}$$

$$F_{\tau_L}(X_{\tau_L}, Q_i) = p\left(\frac{\mathbb{Q}_i^F X_{\tau_L}}{r-\mu} - K\right), \quad X_{\tau_L} \geq x_i^F, \tag{50}$$

where $x_i^F = \frac{\beta}{\beta-1}(r-\mu)\frac{K}{\mathbb{Q}_i^F}$.

Next, we analyze the pure-strategy equilibrium.

7.2 Equilibrium

Let $\mathcal{I}(Q_i)$ denote firm i's equilibrium belief about Q as a function of its signal Q_i conditional on being Leader at τ_L^*. Intuitively, the firm with the higher private signal Q_i enters as Leader as it benefits more from entry. Under this conjecture, which we verify later, firm i's belief about Q at τ_L^* defined in (47) is then given by

$$\mathbb{Q}_i^{L_*} := \mathcal{I}(Q_i) = \int_{\underline{q}}^{Q_i} \frac{Q_i + z}{2} \frac{d\Psi(z)}{\Psi(Q_i)}, \tag{51}$$

where the integrand follows (45) and the upper limit for the integral uses our conjecture that Follower's signal is lower than Leader's, which is Q_i in this case.

Anticipating that in equilibrium Leader must be the one with the stronger signal, both firms wait longer than they would had they purely relied on their own signals. In effect, equilibrium entry delay in our setting resembles 'shading the bids' to address "winner's curse" in the auction literature (see, e.g., Krishna (2009), for a textbook treatment.)

The following theorem summarizes the perfect Bayesian Nash equilibrium solution (see Appendix E for a proof.)

Theorem 4 *There exists a perfect Bayesian Nash equilibrium in which firm i's entry time is* $\tau_i^* = \inf\{t \geq 0 : \mathbf{Q}_t^* \leq Q_i\}$, *where the* $\{\mathbf{Q}_t^*\}$ *process is given by*

$$d\mathbf{Q}_t^* = \frac{-\Psi(\mathbf{Q}_t^*)}{\Psi'(\mathbf{Q}_t^*)} \Lambda^*(X_t, \mathbf{Q}_t^*)dt, \quad \mathbf{Q}_0^* = \overline{q}. \tag{52}$$

Here, $\Lambda^*(x, q) = 0$ *for* $x < \overline{x}(q) = \frac{\beta}{\beta - 1}(r - \mu)\frac{K}{p\mathcal{I}(q)}$ *and* $\Lambda^*(x, q) = \frac{p\mathcal{I}(q)x - rK}{(1-p)K}$ *for* $x > \overline{x}(q)$.

The $\{\mathbf{Q}_t^*\}$ process characterized above in Theorem 4 describes the equilibrium evolution of the signal of the (marginal) firm that is indifferent between entering as Leader immediately and waiting with the hope that its competitor enters the next instant.

Next, we discuss how a firm's waiting for its peer's information to be revealed generates equilibrium second-mover advantage.

7.3 Equilibrium Second-mover Advantage

At $t = \tau_L^*$, conditional being Follower, firm i infers $Q_{-i} = \mathbf{Q}^*_{\tau_L^*}$, and its equilibrium belief, $\mathbb{Q}_i^{F*}$, is given by

$$\mathbb{Q}_i^{F*} = \frac{Q_i + \mathbf{Q}^*_{\tau_L^*}}{2} \geq \mathbb{Q}_i^{L*}, \tag{53}$$

where the inequality follows from the equilibrium property that the firm with a stronger signal enters first: $Q_i \leq Q_{-i} = \mathbf{Q}^*_{\tau_L^*}$. Comparing firm i's value as Leader, (46), and its value as Follower, (49)-(50), we can show that second-mover advantage arises in equilibrium. Mathematically,

$$F_{\tau_L^*}(X_{\tau_L^*}, Q_i) \geq p\left(\frac{\mathbb{Q}_i^{F*} X_{\tau_L^*}}{r-\mu} - K\right) \geq p\left(\frac{\mathbb{Q}_i^{L*} X_{\tau_L^*}}{r-\mu} - K\right) > L_{\tau_L^*}(X_{\tau_L^*}, Q_i) - K, \tag{54}$$

as Leader's value at entry time τ_L^* is given by $L_{\tau_L^*}(X_{\tau_L^*}, Q_i) = p\mathbb{Q}_i^{L*}\frac{X_{\tau_L^*}}{r-\mu}$ (see (46)). The first inequality in (54) follows from the property that the firm's option value is always weakly positive, the second inequality follows from $\mathbb{Q}_i^{F*} \geq \mathbb{Q}_i^{L*}$ as shown in (53), and the last inequality is due to the possibility of a dry well ($p < 1$).

In summary, a firm is better off being Follower than Leader at τ_L^*, which is formally shown in (54): $F_{\tau_L^*}(X_{\tau_L^*}, Q_i) > L_{\tau_L^*}(X_{\tau_L^*}, Q_i) - K$. Second-mover advantage arises from two sources of information spillover. First, by waiting its peer to enter, a firm can learn about whether the well is a dry hole (the value of θ), thus effectively managing catastrophic (dry hole, e.g., $\theta = 0$) risk. Second, by waiting its peer to enter, a firm can fully infer its peer's signal, thus obtaining a more precise estimate for the well's quality by combining its own signal and its peer's. In contrast, if entering as Leader, a firm can only infer that its peer's signal is lower than its own, not the value of its peer's signal. That is, Follower's entry is a more informative decision than Leader's entry.

Finally, in Appendix F, we extend the setting analyzed here in which the two firms are symmetric to an *ex ante* asymmetric setting in which one firm is known to have a higher valuation for information revealed by Leader's entry. Using the solution of this extended model, we further discuss its predictions which are consistent with the following empirical findings in Décaire and Wittry (2025): (1) "*first-movers tend to receive the strongest signal among their regional peers. Moreover, they are the most likely to benefit from the newly* *revealed information, as they own the largest number of options in the region."* and (2) *"an increase in the quantity of information expected to be released by peers increases firms' incentive to delay investment decisions."*

8 Conclusion

Second-mover advantage fundamentally shapes firm entry and competition. When firms benefit from learning from early entrants, a mixed-strategy equilibrium naturally emerges. We develop a tractable duopoly entry model featuring two disconnected probabilistic-entry regions separated by two waiting regions, with firm value changing from convex to concave, then back to convex and finally linear as market demand increases.

Our model's predictions are consistent with Fabrizio and Tsolmon (2014)'s finding that corporate innovation increases with market demand in industries with weaker patent protection but is less responsive in industries with stronger patent protection.

To demonstrate robustness, we develop two dynamic Bayesian games: (1) a reputation model in which firms may be "crazy" and enter only as Follower, building on Kreps et al. (1982), and Abreu and Gul (2000), and (2) an incomplete-information game with dynamic purification, extending Harsanyi (1973)'s insight to our duopoly setting. The mixed-strategy equilibrium survives both refinements, strengthening the plausibility of our analysis.

In addition to learning from peers' experiences, firms may also benefit from early entrants' entry decisions, which reveal valuable private information about market demand. We develop an equilibrium model with endogenous information spillovers and show that second-mover advantage naturally arises in this setting.

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Appendices

A Equilibria for Case A: $R > R_{AB}$

A.1 Equilibrium Definition for the General Case

Definition 4 A Markov entry strategy for firm $i \in \{a, b\}$ is a pair: $\varphi_i = (\mathcal{E}_i, \lambda_i(x))$, where $\mathcal{E}_i \subseteq \mathbb{R}_+$ is a closed set and the entry rate $\lambda_i(x)$ is a measurable function from $\mathbb{R}_+ \backslash \mathcal{E}_i$ to $\mathbb{R}_+$. Firm i enters the market for sure when $X_t \in \mathcal{E}_i$ and randomly at an intensity rate of $\lambda_i(X_t)$ when $X_t \notin \mathcal{E}_i$. A Markov strategy pair $(\varphi_a, \varphi_b) = \{(\mathcal{E}_a, \lambda_a), (\mathcal{E}_b, \lambda_b)\}$ is feasible if and only if $\int_0^t \lambda_i(X_s)ds < \infty$ almost surely for any $t < \inf\{s \geq 0 : X_s \in \mathcal{E}_a \cup \mathcal{E}_b\}$. Let Φ denote the set of all feasible entry strategies.

Given $X_0 = x_0 > 0$ and a feasible Markov strategy pair $(\varphi_a, \varphi_b) = \{(\mathcal{E}_a, \lambda_a), (\mathcal{E}_b, \lambda_b)\}$, the entry time pair (τ_a, τ_b) is determined by the joint distribution:

$$\mathbb{P}^{x_0}(\tau_a \leq t_a, \tau_b \leq t_b) = \mathbb{E}^{x_0}\left[(G_a(t_a) - G_a(0))(G_b(t_b) - G_b(0))\right], \quad t_a \geq 0, t_b \geq 0, \tag{A.1}$$

where $G_i(t)$ is the conditional distribution of firm i's entry time τ_i given $\{X_s; s \geq 0\}$:

$$G_i(t) = 1 - \left(1 - \mathbf{1}_{\{X_s \in \mathcal{E}_i \text{ for some } s \leq t\}}\right) e^{-\int_0^t \lambda_i(X_u)du}. \tag{A.2}$$

Definition 5 Let $J_i(x; \varphi_a, \varphi_b)$ denote firm i's value at time t defined in (14) for a given $X_t = x > 0$ and a feasible Markov strategy pair $(\varphi_a, \varphi_b) = \{(\mathcal{E}_a, \lambda_a), (\mathcal{E}_b, \lambda_b)\}$. A feasible entry strategy pair $\{\varphi_a^*, \varphi_b^*\}$ forms a *Markov perfect equilibrium* if for any $x > 0$, we have

$$J_a(x; \varphi_a^*, \varphi_b^*) \geq J_a(x; \varphi_a, \varphi_b^*), \quad \forall\, \varphi_a = (\mathcal{E}_a, \lambda_a)\ s.t.\ \{\varphi_a, \varphi_b^*\} \in \Phi, \tag{A.3}$$

$$J_b(x; \varphi_a^*, \varphi_b^*) \geq J_b(x; \varphi_a^*, \varphi_b), \quad \forall\, \varphi_b = (\mathcal{E}_b, \lambda_b)\ s.t.\ \{\varphi_a^*, \varphi_b\} \in \Phi. \tag{A.4}$$

A.2 Two-Region Mixed-strategy Equilibrium: Case A

In this appendix, we report the solution for a degenerate subcase of Case A. We can show that there are only two regions if the second-mover advantage is sufficiently large in that $R > R_{A_1A_2}$, where $R_{A_1A_2}$ is given by

$$R_{A_1A_2} = \left(\frac{D^{1-\beta} - D}{(1-D)\beta}\right)^{\frac{1}{\beta-1}}. \tag{A.5}$$

When $R > R_{A_1A_2}$, there exists a threshold $\overline{x}$ dividing the $x > 0$ real line into two regions: 1.) the waiting region where $x < \overline{x}$ and $V_*(x) > L(x) - K_L$ and 2.) the probabilistic entry region where $x \geq \overline{x}$ and $V_*(x) = L(x) - K_L$. The variational inequality (18) simplifies to the ODE (27) in the waiting ($x < \overline{x}$) region, subject to the value-matching and smooth-pasting conditions in footnote 11 at the threshold $\overline{x}$.

In the probabilistic entry region $x \geq \overline{x}$, Leader's value is given by

$$V_*(x) \quad = \quad D\Pi(x) - K_L\,, \quad x \geq \overline{x}\,. \tag{A.6}$$

Using Leader's linear net payoff function at entry in the $x \geq \overline{x}$ region, (A.6), and solving the ODE (27) in the $x < \overline{x}$ region subject to the value-matching and smooth-pasting conditions in footnote 11, we obtain the closed-form expressions for $V_*(x)$:

$$V_*(x) \quad = \quad \left(\frac{x}{\overline{x}}\right)^{\beta} \left(D\Pi(\overline{x}) - K_L\right)\,, \quad x < \overline{x}\,, \tag{A.7}$$

where the threshold $\overline{x}$ is given by

$$\overline{x} \quad = \quad \frac{1}{D}\frac{\beta}{\beta - 1}(r - \mu)K_L\,. \tag{A.8}$$

Now we verify the equilibrium result that as soon as one firm enters probabilistically, the other also immediately enters. This is because $\overline{x} > x_F$, which follows from a comparison of (A.8) for $\overline{x}$ and (10) for x_F under the condition: $R > R_{A_1A_2} > 1$. Finally, substituting (9) and (13) into (19) gives the equilibrium entry rate (25) in the region $x \geq \overline{x}$. Next we summarize the solution for the case where $R > R_{A_1A_2}$.

Proposition 3 *For $R > R_{A_1A_2}$, there exists a symmetric Markov perfect mixed-strategy equilibrium where the threshold $\overline{x}$ given in (A.8) divides the $x > 0$ real line into two solution regions. In the $x < \overline{x}$ region, firms wait ($\lambda^*(x) = 0$) and $V_a(x) = V_b(x) = V_*(x)$, where $V_*(x)$ is given in* (A.7)*. In the $x \geq \overline{x}$, region, firms enter probabilistically at the same rate, $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$, where $\lambda^*(x)$ is given in* (25)*, and $V_a(x) = V_b(x) = V_*(x)$, where $V_*(x)$ is given in* (A.6)*. As soon as one firm enters, the other enters immediately: $\tau_F^* = \tau_L^* +$.*

Graphical Illustration. In Panel A of Figure 4, we plot $L(x) - K_L$ using the dashed red line and $F(x)$ using the magenta dash-dotted line. We pin down firm value $V_a(x) = V_b(x) = V_*(x)$ by smoothly pasting a convex curve (from the origin) onto the $L(x) - K_L$ payoff line. Doing so determines the endogenous threshold $\overline{x}$ (the solid black dot): To the left of $\overline{x}$ is the

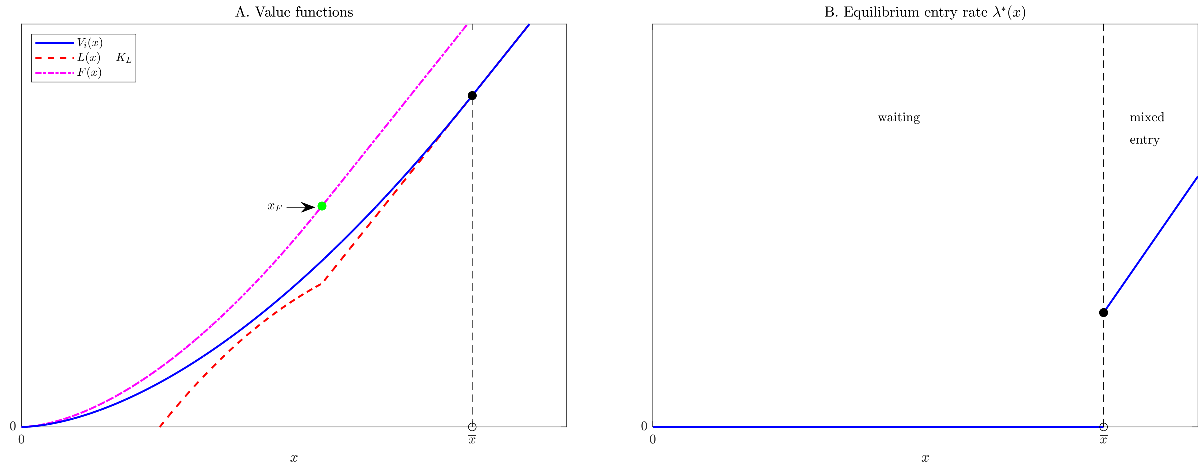

Figure 4: Value functions $V_a(x) = V_b(x) = V_*(x)$ and entry rates $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$ in the symmetric Markov perfect mixed-strategy equilibrium in degenerate subcase of Case A. Parameter values are $D = 0.55$, $K_L = 1.5$, $K_F = 1$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$.

increasing convex $V_i(x)$ (the black solid line) and to the right of $\overline{x}$ is the straight net payoff line $L(x) - K_L = D\Pi(x) - K_L$ (the blue solid straight line).

Panel B of Figure 4 plots the equilibrium entry rate $\lambda^*(x)$ that supports $V_i(x) = V_*(x)$ obtained in Panel A. The vertical dashed line in Panel B divides the solution into two regions. To the left of $\overline{x}$ is the waiting region where $\lambda^*(x) = 0$. To the right of $\overline{x}$ is the probabilistic entry region where $\lambda^*(x) = \frac{Dx - rK_L}{K_L - K_F}$.

A.3 Two-Region Pure-strategy Equilibria in Case A

In this appendix, we provide closed-form solutions of asymmetric pure-strategy equilibria for the case $R > R_{A_1A_2}$, where $R_{A_1A_2}$ is given in (A.5).

When $R > R_{A_1A_2}$, Leader's pure-strategy entry region is $[\overline{x}, \infty)$, where $\overline{x}$ is given in (A.8). Because Leader's optimal entry threshold is larger than Follower's: $\overline{x} > x_F$, Follower enters immediately after Leader does: $\tau_F^* = \tau_L^* +$. Therefore, Follower's value, $J_F(x)$, is given by the following closed-form expressions:

$$J_F(x) = F(x) = D\Pi(x) - K_F, \quad x \geq \overline{x}, \tag{A.9}$$

$$J_F(x) = \left(\frac{x}{\overline{x}}\right)^{\beta} F(\overline{x}) = \left(\frac{x}{\overline{x}}\right)^{\beta} (D\Pi(\overline{x}) - K_F), \quad x < \overline{x}\,. \tag{A.10}$$

We summarize the key results for the pure-strategy equilibria when $R > R_{A_1A_2}$.

Proposition 4 *There are two asymmetric pure-strategy equilibria for $R > R_{A_1A_2}$. Leader enters at $\tau_L^* = \inf\{s \geq t : X_s \geq \overline{x}\}$ and Leader's value is $J_L(x) = V_*(x)$, where $\overline{x}$ is given in* (A.8) *and $V_*(x)$ is given in* (A.6)-(A.7). *Because Follower's entry threshold x_F is lower than Leader's threshold: $\overline{x} > x_F$, Follower enters immediately after Leader ($\tau_F^* = \tau_L^*+$) and Follower's value $J_F(x)$ is given by* (A.9)-(A.10).

A.4 Additional Technical Results for Thresholds $\underline{x}$ and $\overline{x}$

In this appendix, we provide additional technical details for the thresholds $\underline{x}$ and $\overline{x}$ in the mixed-strategy equilibrium.

Formulas for $\theta_1(a,b)$ and $\theta_2(a,b)$ in (29). Let $\tau_{ab} = \inf\{s \geq t : X_s \leq a \text{ or } X_s \geq b\}$ for a given pair (a,b) satisfying $0 < a < b$ and let $\Theta(x;a,b)$ denote the following present value:

$$\Theta(x;a,b) = \mathbb{E}_t^x[e^{-r(\tau_{ab}-t)}(L(X_{\tau_{ab}}) - K_L)], \quad x \in [a,b]. \tag{A.11}$$

We can show that $\Theta(x;a,b)$ is given by (29), where $\theta_1(a,b)$ and $\theta_2(a,b)$ solve:

$$\theta_1 a^\beta + \theta_2 a^\gamma = L(a) - K_L \quad \text{and} \quad \theta_1 b^\beta + \theta_2 b^\gamma = L(b) - K_L.$$

Solving the above equations yields

$$\theta_1(a,b) = \frac{(L(a)-K_L)b^\gamma - (L(b)-K_L)a^\gamma}{a^\beta b^\gamma - a^\gamma b^\beta} \text{ and } \theta_2(a,b) = \frac{(L(b)-K_L)a^\beta - (L(a)-K_L)b^\beta}{a^\beta b^\gamma - a^\gamma b^\beta}. \tag{A.12}$$

Next, we present a lemma for the mixed-strategy equilibrium.

Lemma 1 *For any $R \in [1, R_{A_1A_2})$, there exists a unique pair of thresholds $(\underline{x}, \overline{x})$ in the domain $(\widetilde{x}, x_F) \times (\frac{1}{D}\widetilde{x}, \infty)$ satisfying*[27]

$$\Theta(\underline{x};\underline{x},\overline{x}) = L(\underline{x}) - K_L, \qquad \Theta_x(\underline{x};\underline{x},\overline{x}) = L'(\underline{x}), \tag{A.13}$$

$$\Theta(\overline{x};\underline{x},\overline{x}) = D\Pi(\overline{x}) - K_L, \qquad \Theta_x(\overline{x};\underline{x},\overline{x}) = D\Pi'(\overline{x}), \tag{A.14}$$

where $\Theta(x;a,b)$, $\widetilde{x}$, and x_F are given by (29), (23), *and* (10), *respectively. Moreover, $\underline{x}$ and $\overline{x}$ are continuously differentiable as functions of R in $R \in [1, R_{A_1A_2})$, and they are given by*

$$\underline{x} = (1+u)\frac{\beta}{\beta-1}(r-\mu)K_L \quad \textit{and} \quad \overline{x} = (1+U)\frac{1}{D}\frac{\beta}{\beta-1}(r-\mu)K_L, \tag{A.15}$$

[27] Recall that in this case, we have $x_F \leq \frac{1}{D}\widetilde{x}$, which implies $\overline{x} > x_F$.

where (u, U) *is the unique solution pair to the following system of equations in the domain* $(0, \frac{1}{DR} - 1) \times (0, \infty)$*:*

$$U(1+U)^{-\gamma} = \left(\frac{1}{D}\right)^{\gamma} u(1+u)^{-\gamma}, \tag{A.16}$$

$$H(U) = \left(\frac{1}{D}\right)^{\beta} H(u) - \left(\frac{1}{D} - 1\right)\frac{\beta}{\beta - 1} R^{\beta-1}. \tag{A.17}$$

The function $H(\cdot)$ *in (A.17) is defined as follows:*

$$H(z) = \frac{(1-\gamma)\frac{\beta}{\beta-1}(1+z) + \gamma}{(\beta-\gamma)(1+z)^{\beta}}. \tag{A.18}$$

We have $\Theta_{xx}(x; \underline{x}, \overline{x}) > 0$ *for any* $x \in [\underline{x}, \overline{x}]$*. When* $R = R_{A_1A_2}$*, we have* $\underline{x} = \widetilde{x}$*,* $\overline{x} = \frac{1}{D}\widetilde{x}$*, and for any* $x \in [\underline{x}, \overline{x}]$*,* $\Theta(x; \underline{x}, \overline{x}) = V_*(x)$*, where* $V_*(x)$ *is given by* (A.6)-(A.7)*.*

B Policy Implications

In this appendix, we analyze our model's policy implications in two steps. First, we develop a cooperative benchmark where a planner maximizes the total social surplus.[28] By construction, the cooperative benchmark outcome serves as the upper bound for the total surplus. While desirable, it is unlikely to be achieved in practice. This is because this outcome requires either the firms to merge or the government to dictate corporate decisions.

The second step of our policy analysis is to consider a setting in which the government chooses a subsidy policy and firms subsequently make their entry decisions. This benchmark introduces a Stackelberg game between the government and the firms into the entry-timing game of Section 3. When deciding on its entry, a firm takes into account both the government policies and its competitor's entry strategy.

Cooperative benchmark. For a given Leader's entry time τ_L and Follower's entry time $\tau_F \geq \tau_L$, the total market capitalization of the industry is given by

$$W(x; K_L, K_F) = \mathbb{E}_t^x \left[\int_{\tau_L}^{\tau_F} e^{-r(s-t)} X_s ds + \int_{\tau_F}^{\infty} e^{-r(s-t)} 2D X_s ds - K_L e^{-r(\tau_L - t)} - K_F e^{-r(\tau_F - t)} \right]. \tag{B.1}$$

[28]We interpret first-best (efficient) investment at the firm level as welfare-maximizing. As the focus of this paper is on the production side, we set aside the consumers' endogenous response to changes in the industry structure. Our cooperative benchmark-based policy analysis is related to Weeds (2002), who studies policy implications for R&D using a similar benchmark under perfect coordination.

The government chooses τ_L and τ_F to maximize (B.1). Let $(\tau_L^{\text{co}}, \tau_F^{\text{co}})$ denote the optimal strategies and let $W^{\text{co}}(x)$ denote the associated total surplus value. Closed-form solutions for $(\tau_L^{\text{co}}, \tau_F^{\text{co}})$ and $W^{\text{co}}(x)$ are provided in Proposition IA.3 of Appendix IA.G.

Due to the second-mover advantage, firms have incentives to wait too long in the hope that their competitor enters first. To mitigate this investment inefficiency, the planner subsidizes Leader to provide incentives for firms to enter sooner. In practice, governments sometimes subsidize investments to stimulate innovations; see, e.g., Bloom et al. (2002), González et al. (2005), and Howell (2017), for evidence. Below we develop a model with entry subsidy.

A Model with entry subsidy. Let δ_L denote the subsidy: a lump-sum transfer at τ_L from the government to Leader. The government chooses δ_L to maximize:

$$\max_{\delta_L} \; W(x_0; K_L - \delta_L, K_F) - \mathbb{E}[\delta_L e^{-r\tau_L}], \tag{B.2}$$

where $W(x; K_L, K_F)$ is given in (B.1).

Taking the subsidy policy δ_L as given, firms maximize their own market values by choosing their entry strategies. The solution for the symmetric mixed-strategy equilibrium is characterized in Theorem 1 but with Leader's entry cost being $K_L - \delta_L$ rather than K_L. We focus on the case where $x_0 > 0$ is small so the entry game starts in the option-value-of-waiting region.

Below, we first assess the industry value loss relative to the cooperative benchmark and then analyze the effectiveness of the subsidy policy in mitigating inefficiency.

Industry value loss. As in Weeds (2002), we use the cooperative benchmark to measure the loss of the total industry value caused by duopoly competition. By dividing the total market capitalization of the duopoly industry, $V_a(x) + V_b(x)$, by the total market capitalization of the industry in the cooperative benchmark, $W^{\text{co}}(x)$, and subtracting this ratio from one, we obtain the following industry-value-loss measure:

$$\Delta(x) = 1 - \frac{V_a(x) + V_b(x)}{W^{\text{co}}(x)} = 1 - \frac{2V_*(x)}{W^{\text{co}}(x)}, \tag{B.3}$$

where $V_*(x)$ is given in Theorem 1.

In panel A of Figure 5, we plot $\Delta(x)$ for x in the option-value-of-waiting region (the first region from the left of x, characterized in Section 3.3). We see that absent policy intervention, competition causes a significant industry value loss. As we increase the entry-cost ratio R from

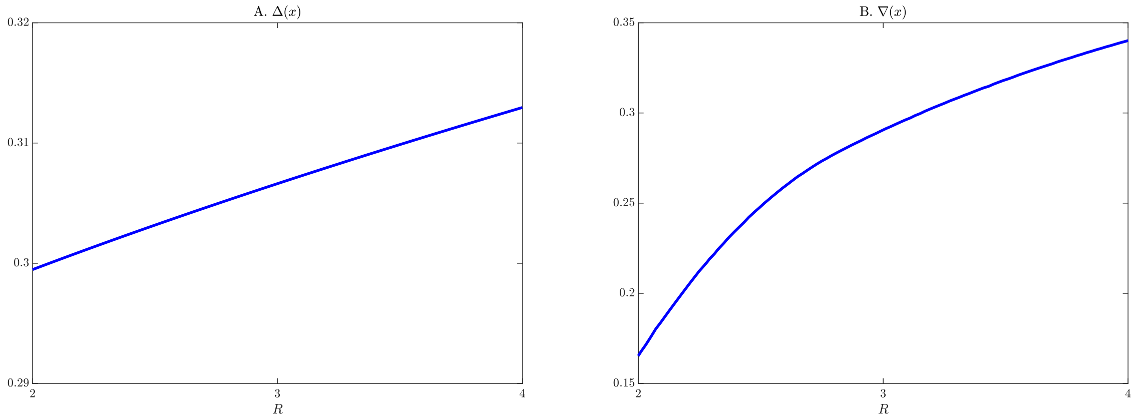

Figure 5: Industry Value Loss: $\Delta(x)$ defined in (B.3) and effectiveness of the subsidy policy: $\nabla(x)$ defined in (B.4). Parameter values are $D = 0.55$, $K_L = RK_F$, $K_F = 1$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$.

$R = 2$ to $R = 4$, the percentage value loss for the industry, $\Delta(x)$, increases from 29.9% to 31.3%. The larger the entry-cost ratio R, the greater the value erosion $\Delta(x)$.

The total industry value is eroded by a firm's incentives to delay its entry. Indeed, Leader's entry time in the symmetric mixed-strategy equilibrium, τ_L^*, is larger than Leader's entry time in the cooperative benchmark τ_L^{CO} with probability one (see Appendix IA.G for details). Note that firms use pure entry strategies in the cooperative benchmark, whereas they use mixed-entry strategies under duopoly competition. Even with the same entry thresholds, mixed-entry strategies take longer because entries are probabilistic. Given the significant value loss from market competition, how effective is the subsidy policy at mitigating the inefficient delay?

Effectiveness of the subsidy policy. Let $W^{\text{Sub}}(x) = W(x; K_L - \delta_L^{\text{Sub}}, K_F) - \mathbb{E}^x[\delta_L^{\text{Sub}} e^{-r\tau_L}]$ denote the total market capitalization under the optimal subsidy policy δ_L^{Sub}. By dividing $W^{\text{Sub}}(x)$ by the total market capitalization of the industry absent policy intervention, $V_a(x) + V_b(x)$, and subtracting this ratio from one, we obtain the following policy effectiveness measure:

$$\nabla(x) = \frac{W^{\text{Sub}}(x)}{V_a(x) + V_b(x)} - 1 = \frac{W^{\text{Sub}}(x)}{2V_*(x)} - 1, \tag{B.4}$$

where $V_*(x)$ is given in Theorem 1. In panel B of Figure 5, we plot $\nabla(x)$ in the option-value-of-waiting region. We see that the subsidy policy significantly increases the total market capitalization of the industry. As we increase the entry-cost ratio R from $R = 2$ to $R = 4$, the policy increases the market capitalization of the industry, $\nabla(x)$, from 16.3% to 33.9%.

In summary, we have analyzed the policy implications of our complete-information model by using both a standard cooperative benchmark and our proposed subsidy policy. We show that an effective and feasible government intervention can substantially mitigate inefficient entry delay and create value.

C Technical Details for Section 5

In this appendix, we provide technical details for the reputation model in Section 5. We follow Definition 4 to define a feasible Markov strategy pair $(\varphi_a, \varphi_b) = \{(\mathcal{E}_a, \lambda_a), (\mathcal{E}_b, \lambda_b)\}$ and an entry time pair (τ_a, τ_b) via the CDF $(G_a(t), G_b(t))$ in (A.1)-(A.2), where τ_i is firm i's entry time assessed from firm $-i$'s perspective and is *unconditional* on the type of firm i. Given firm i's entry time τ_i, we let $\widehat{\tau}_i$ denote the corresponding rational firm i's entry time defined via its CDF $\widehat{G}_i(t) = \mathbb{P}_t(\widehat{\tau}_i \leq t) = \frac{G_i(t)}{1-\pi_0^i}$, where the second equality follows from

$$G_i(t) = \mathbb{P}_t(\tau_i \leq t) = \mathbb{P}_t(\tau_i \leq t \mid \text{firm } i \text{ is rational})\mathbb{P}_t(\text{firm } i \text{ is rational}) = \widehat{G}_i(t)(1 - \pi_0^i). \tag{C.1}$$

If no firm enters the market before $T_i = \inf\{t \geq 0 : \pi_t^i = 1\}$, then firm $-i$ will believe firm i is crazy for sure. As a result, at time $t \geq T_i$, the rational firm $-i$ is willing to enter as Leader by solving the problem (31). Thus, we only need to check the off-equilibrium deviation of the rational firm before the revelation time $T = \min\{T_a, T_b\}$.[29]

A rational firm i's continuation value function at time $t < T = \min\{T_a, T_b\}$ is given by

$$\mathbb{E}_t^x\Bigg[e^{-r(\tau_L - t)}\left(\mathbf{1}_{\widehat{\tau}_i < \tau_{-i}}(L(X_{\widehat{\tau}_i}) - K_L) + \mathbf{1}_{\widehat{\tau}_i > \tau_{-i}}F(X_{\tau_{-i}}) + \mathbf{1}_{\widehat{\tau}_i = \tau_{-i}}\frac{L(X_{\tau_L}) - K_L + F(X_{\tau_L})}{2}\right)\mathbf{1}_{\tau_L \leq T} + e^{-r(T-t)}J_L(X_T)\mathbf{1}_{\tau_L > T}\Bigg], \tag{C.2}$$

where $\tau_L = \widehat{\tau}_i \wedge \tau_{-i}$, $X_t = x > 0$, and $J_L(x)$ is given in (31). In the following, we use $T^{\varphi_a,\varphi_b} = \min\{T_a^{\varphi_a,\varphi_b}, T_b^{\varphi_a,\varphi_b}\}$ to emphasize the dependence of revelation time on strategy (φ_a, φ_b), where $T_i^{\varphi_a,\varphi_b}$ is the first time that firm $-i$ learns that firm i is crazy with probability one under strategy (φ_a, φ_b). When $T = T^{\varphi_a,\varphi_b}$ in (C.2), the last term in (C.2) is zero. This is because the rational firm will enter as Leader at time $t \leq T^{\varphi_a,\varphi_b}$ for sure. Indeed, the last

[29] Lemma 5 in Internet Appendix IA.K shows that $J_L(x) = V_*(x)$ and $\tau_L^* = \inf\{t : X(t) \in \mathcal{R}^E\}$ is the optimal stopping time to problem (31), where $V_*(x)$ is given in Theorem 1 and $\mathcal{R}^E = \{x > 0 : V_*(x) = L(x) - K_L\}$.

term in (C.2) is used only at off-equilibrium deviation.

Next, we define the Bayesian Nash equilibrium for the reputation model in Section 5.

Definition 6 Let $J_i(x; \varphi_a, \varphi_b, T)$ denote a rational firm i's value at time t, as defined in (C.2), for a given $X_t = x > 0$, revelation time T, and a feasible Markov strategy pair $(\varphi_a, \varphi_b) = \{(\mathcal{E}_a, \lambda_a), (\mathcal{E}_b, \lambda_b)\}$. A feasible entry strategy pair $\{\varphi_a^*, \varphi_b^*\}$ forms a *perfect Bayesian Nash equilibrium* if for any $x > 0$, we have

$$J_a(x; \varphi_a^*, \varphi_b^*, T^*) \geq J_a(x; \varphi_a, \varphi_b^*, T^*), \quad \forall\, (\varphi_a, \varphi_b^*) \in \Phi, \tag{C.3}$$

$$J_b(x; \varphi_a^*, \varphi_b^*, T^*) \geq J_b(x; \varphi_a^*, \varphi_b, T^*), \quad \forall\, (\varphi_a^*, \varphi_b) \in \Phi, \tag{C.4}$$

where $T^* = T^{\varphi_a^*, \varphi_b^*}$ is the revelation time under the strategy pair $\{\varphi_a^*, \varphi_b^*\}$. Let $V_i(x)$ denote the rational firm i's equilibrium value function: $V_i(x) = J_i(x; \varphi_a^*, \varphi_b^*, T^*)$.

Note that even when a rational firm i deviates from φ_i^* to another strategy φ_i, the revelation time $T^* := T^{\varphi_a^*, \varphi_b^*}$ remains unaltered because its competitor, firm $-i$, cannot detect whether a deviation has occurred before the entry of firm i.

D Technical Details for Section 6

In this appendix, we provide technical details for the purification analysis in Section 6. For illustration, we focus only on Case A where $\overline{k} > \underline{k} > K_F R_{AB}$.

D.1 Definition of Bayesian Nash Equilibrium

Definition 7 An entry strategy for firm $i \in \{a, b\}$ is a measurable mapping $\varphi_i(K_L^i, \omega)$: $[\underline{k}, \overline{k}] \times \Omega \to [0, \infty)$, such that for each $K_L^i \in [\underline{k}, \overline{k}]$, $\tau_i(K_L^i) = \varphi_i(K_L^i, \cdot)$ is a stopping time.

Given a strategy pair (φ_a, φ_b), firm i's entry time τ_i, as assessed from firm $-i$'s perspective, is determined by

$$G_i(t) := \mathbb{P}_t(\tau_i \leq t) = \int_{\underline{k}}^{\overline{k}} \mathbb{P}_t(\tau_i(k) \leq t \mid K_L^i = k) d\Psi(k). \tag{D.1}$$

Let $K_t^i = \Psi^{-1}(G_i(t))$, and let Φ denote the set of all feasible entry strategies, in which (X_t, K_t^a, K_t^b) is a Markov process.

For a given pair of entry strategy (τ_a, τ_b) and firm i's type K_L^i, firm i's continuation value function at time t with $X_t = x$, $K_t^a = k_a$, and $K_t^b = k_b$ is given by

$$\begin{aligned}
&\mathbb{E}_t^{x,k_a,k_b}\Bigg[e^{-r(\tau_L - t)}\Bigg(\mathbf{1}_{\tau_i(K_L^i)<\tau_{-i}(K_L^{-i})}\big(L(X_{\tau_L}) - K_L^i\big) + \mathbf{1}_{\tau_i(K_L^i)>\tau_{-i}(K_L^{-i})}F(X_{\tau_L}) \\
&\qquad + \mathbf{1}_{\tau_i(K_L^i)=\tau_{-i}(K_L^{-i})}\frac{L(X_{\tau_L}) - K_L^i + F(X_{\tau_L})}{2}\Bigg)\Bigg] \\
=&\mathbb{E}_t^{x,k_a,k_b}\Bigg[\int_{\underline{k}}^{\overline{k}} e^{-r(\tau_i(K_L^i)\wedge\tau_{-i}(k) - t)}\Bigg(\mathbf{1}_{\tau_i(K_L^i)<\tau_{-i}(k)}\big(L(X_{\tau_i(K_L^i)}) - K_L^i\big) + \mathbf{1}_{\tau_i(K_L^i)>\tau_{-i}(k)}F(X_{\tau_{-i}(k)}) \\
&\qquad + \mathbf{1}_{\tau_i(K_L^i)=\tau_{-i}(k)}\frac{L(X_{\tau_i(K_L^i)}) - K_L^i + F(X_{\tau_i(K_L^i)})}{2}\Bigg)d\Psi_t^{-i}(k)\Bigg],
\end{aligned} \tag{D.2}$$

where $\tau_L = \tau_i(K_L^i) \wedge \tau_{-i}(K_L^{-i})$ and $\Psi_t^{-i}(k) = \mathbb{P}_t(K_L^{-i} \le k \mid \tau_{-i}(K_L^{-i}) > t)$ is the posterior CDF of K_L^{-i} at time t assessed from firm i's perspective.

Definition 8 Let $J_i(x, k_a, k_b, K_L^i; \varphi_a, \varphi_b)$ denote firm i's value at time t, as defined in (D.2), for a given $X_t = x > 0$, $K_t^a = k_a$, $K_t^b = k_b$, and a feasible Markov strategy pair (φ_a, φ_b). A feasible strategy pair $\{\varphi_a^*, \varphi_b^*\}$ forms a *Bayesian Nash equilibrium* if for any $K_L^a, K_L^b \in (\underline{k}, \overline{k})$, $x > 0$, and possible pair (k_a, k_b), we have

$$J_a(x, k_a, k_b, K_L^a; \varphi_a^*, \varphi_b^*) \ge J_a(x, k_a, k_b, K_L^a; \varphi_a, \varphi_b^*), \quad \forall\, (\varphi_a, \varphi_b^*) \in \Phi, \tag{D.3}$$

$$J_b(x, k_a, k_b, K_L^b; \varphi_a^*, \varphi_b^*) \ge J_b(x, k_a, k_b, K_L^b; \varphi_a^*, \varphi_b), \quad \forall\, (\varphi_a^*, \varphi_b) \in \Phi. \tag{D.4}$$

Note that even if firm i deviates from φ_i^* to another strategy φ_i, the dynamics of K_t^a and K_t^b remain unchanged because its competitor, firm $-i$, can never detect the deviation before firm i enters. More precisely, if we use $K_t^{i,\varphi_a,\varphi_b}$ to emphasize the dependence on strategy (φ_a, φ_b), then $K_t^i = k_i$ refers to $K_t^{i,\varphi_a^*,\varphi_b^*} = k_i$ when we evaluate both sides of (D.3) and (D.4).

D.2 Belief Updating

Intuitively, firm i with a higher cost K_L^i enters later. This suggests a monotonically increasing relationship between firm i's entry cost K_L^i and its entry time $\tau_i(K_L^i)$. Recall the CDF defined in (D.1) and note that $1 - G_i(t)$, representing the probability that firm i remains in the game up to time t, should equal the proportion of the types of firm i that remain in the game at time t. Then firm $-i$ infers firm i's entry as follows:

$$\tau_i(K_L^i) = \inf\{t \geq 0 : K_t^i \geq K_L^i\}, \tag{D.5}$$

where $K_t^i = \Psi^{-1}(G_i(t))$. Since $K_t^i < K_L^i$ is equivalent to $\tau_i(K_L^i) > t$, K_t^i is the lowest possible value of K_L^i conditional on no Leader before time t.

A firm updates its belief about its competitor's (unknown but constant) entry cost, starting from a prior $\Psi(k)$ where $k \in [\underline{k}, \overline{k}]$. The posterior CDF of K_L^i at time t (conditional on no Leader) is given by

$$\Psi_t^i(k) = \frac{\mathbb{P}_t(K_L^i \leq k, \tau_i(K_L^i) > t)}{\mathbb{P}_t(\tau_i(K_L^i) > t)} = \frac{\mathbb{P}_t(K_L^i \leq k, K_L^i > K_t^i)}{\mathbb{P}_t(K_L^i > K_t^i)} = \frac{\Psi(k) - \Psi(K_t^i)}{1 - \Psi(K_t^i)}, \quad k > K_t^i. \tag{D.6}$$

Next, we derive the dynamic equation of K_t^i. As the waiting (to enter) game unfolds, the equilibrium posterior belief at $t > 0$ requires that only those types with $K_L^i > K_t^i$ are still in the game, so that the mass of firm i's type decreases from one at $t = 0$ to $1 - \Psi(K_t^i)$. Additionally, over a small increment $(t, t + dt)$, only those whose types lie inside the interval $(K_t^i, K_t^i + dK_t^i)$ enter. Therefore, given $\{X_s; s \leq t + dt\}$, firm $-i$ infers that firm i's conditional entry probability over the $(t, t + dt)$ interval, denoted by $\Lambda_t^i dt$, must be given by the ratio between $\Psi(K_t^i + dK_t^i) - \Psi(K_t^i)$ and $1 - \Psi(K_t^i)$:

$$\Lambda_t^i dt = \frac{\Psi(K_t^i + dK_t^i) - \Psi(K_t^i)}{1 - \Psi(K_t^i)} = \Psi_t^i(K_t^i + dK_t^i) - \Psi_t^i(K_t^i), \tag{D.7}$$

where $\Psi_t^i(\cdot)$ is the posterior CDF of K_L^i and is given in (D.6). As $\Psi(K_t^i) = G_i(t)$ and $\Psi(K_t^i + dK_t^i) = G_i(t + dt)$, rewriting (D.7) and taking $dt \to 0$, we obtain $\Lambda_t^i = \frac{G_i'(t)}{1 - G_i(t)}$. That is, Λ_t^i is the hazard rate implied by the CDF $G_i(t)$ for firm i's entry time τ_i conditional on $\{X_s; s \leq t\}$, so that $G_i(t) = 1 - e^{-\int_0^t \Lambda_s^i ds}$. Using $\Psi(K_t^i + dK_t^i) - \Psi(K_t^i) = \Psi'(K_t^i) dK_t^i$, we obtain the following dynamics for K_t^i from (D.7):

$$dK_t^i = \frac{1 - \Psi(K_t^i)}{\Psi'(K_t^i)} \Lambda_t^i dt. \tag{D.8}$$

D.3 Equilibrium Solution

Below we characterize a symmetric Bayesian Nash equilibrium in which $\Lambda_t^i = \Lambda^*(X_t, K_t^i)$ for $i = a, b$, where $K_t^a = K_t^b = K_t^*$. We demonstrate that $\Lambda^*(x, k) = \lambda^*(x; k)$, thus obtaining (42) and (43).

Denote $V_i(x,k)$ as firm i's equilibrium value function for $X_t = x$ and $K_t^* = k$. For $K_t^* = k \geq K_L^i$, firm i believes that its competitor has a higher entry cost with probability one. Hence, firm i is willing to enter as Leader by solving the following problem:

$$J_L(x; K_L^i) = \max_{\tau \geq t} \mathbb{E}_t^x \left[e^{-r(\tau - t)} (L(X_\tau) - K_L^i) \right]. \tag{D.9}$$

Lemma 5 in Internet Appendix IA.K shows that $J_L(x; K_L^i)$ is the same as $V_*(x; K_L^i)$, the equilibrium value in Theorem 1 but with K_L replaced by K_L^i. It follows that for $K_t^* = k \geq K_L^i$,

$$V_i(x,k) = J_L(x; K_L^i) = V_*(x; K_L^i). \tag{D.10}$$

When $K_t^* = k < K_L^i$, firm i chooses to wait. Using (D.8) with $\Lambda_t^i = \Lambda^*(X_t, K_t^i)$, we can derive the HJB equation for $V_i(x,k)$ as follows:[30]

$$\begin{aligned} rV_i(x,k) =& \frac{\sigma^2 x^2}{2} \frac{\partial^2 V_i(x,k)}{\partial x^2} + \mu x \frac{\partial V_i(x,k)}{\partial x} \\ &+ \Lambda^*(x,k)[F(x) - V_i(x,k)] + \frac{1 - \Psi(k)}{\Psi'(k)} \Lambda^*(x,k) \frac{\partial V_i(x,k)}{\partial k} \end{aligned} \tag{D.11}$$

for $k < K_L^i$. Because firm i is indifferent between entering and waiting at the entry threshold $k = K_L^i$, both (D.10) and (D.11) hold at $k = K_L^i$, i.e., $V_i(x,k) = V_*(x; K_L^i)$ satisfies equation (D.11). Then, we infer that $\Lambda^*(x, K_L^i)$ is the same as $\lambda^*(x; K_L^i)$, which is the equilibrium entry rate given in Theorem 1 with Leader's entry cost K_L^i. Since K_L^i takes any value within $(\underline{k}, \overline{k})$, we infer $\Lambda^*(x,k) = \lambda^*(x;k)$ for all $x > 0$ and $k \in (\underline{k}, \overline{k})$.

Finally, to deepen our understanding of the key model predictions, we plot the equilibrium belief updating and firm entry τ_L^* in Figure 6. We choose a uniform distribution with a CDF of $\Psi(k) = k - 1.5$ for $k \in (\underline{k}, \overline{k}) = (1.5, 2.5)$ and consider $K_L^{-i} > K_L^i = K_L = 1.55$ with $K_F = 1$. For the triplet (r, μ, σ) and D, we use the same (annualized) parameter values as those in Figures 1-2: $r = 4\%$, $\mu = 2\%$, $\sigma = 10\%$, and $D = 0.55$. This implies that $R_{A_1 A_2} = 1.403$. Hence, we have $\underline{k} > R_{A_1 A_2} K_F$, and for all $k \in (\underline{k}, \overline{k})$, $\lambda^*(x;k) = 0$ in the region where $x < \overline{x}(k) = \frac{1}{D} \frac{\beta}{\beta - 1}(r - \mu)k$ and $\lambda^*(x;k) = \frac{Dx - rk}{k - K_F}$ in the region where $x > \overline{x}(k)$. We simulate a $\{X_t\}_{t \in [0,4]}$ path in panel A. The evolution process of K_t^*, as given in (42), is illustrated in panel B of Figure 6. We see that $K_t^* = \underline{k} = 1.5$ for all $t \leq t_1 = 1.112$ because $\max_{t \in [0, t_1]} X_t < \overline{x}(\underline{k})$. We see that K_t^* increases for all $t \in (t_1, t_2) = (1.112, 1.401)$ as over this period $X_t > \overline{x}(K_t^*)$.

[30]The boundary conditions are $V_i(x,k) \to \frac{\Psi(K_L^i) - \Psi(k)}{1 - \Psi(k)} F(x) + \frac{1 - \Psi(K_L^i)}{1 - \Psi(k)} (L(x) - K_L^i)$ as $x \to \infty, k < K_L^i$, $V_i(0,k) = 0$, $k < K_L^i$, and $V_i(x, K_L^i) = V_*(x; K_L^i)$.

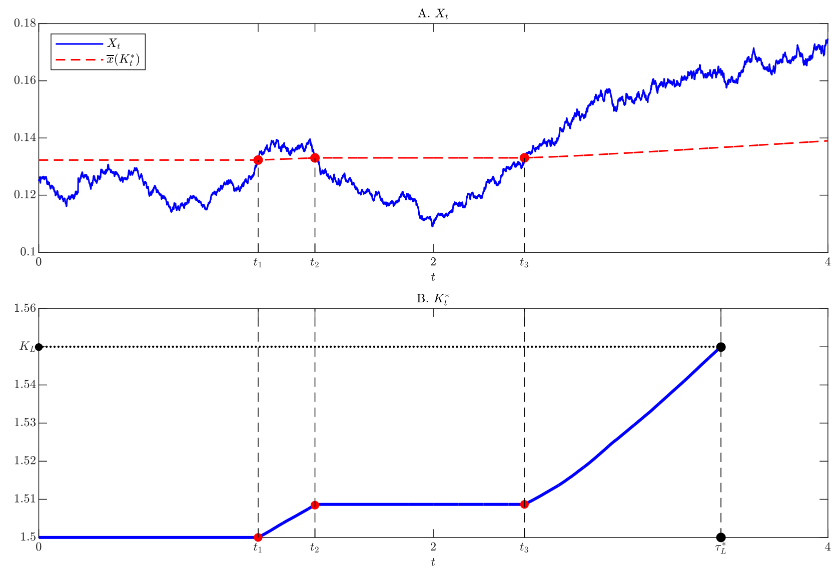

Figure 6: Simulation of X_t and the corresponding processes K_t^*. We consider a uniform distribution with a CDF of $\Psi(k) = k - 1.5$ for $k \in (\underline{k}, \overline{k}) = (1.5, 2.5)$. Parameter values are $D = 0.55$, $K_L^{-i} > K_L^i = K_L = 1.55$, $K_F = 1$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$.

E Technical Details for Section 7

In this appendix, we provide technical details for the analysis in Section 7.

E.1 Definition of Bayesian Nash Equilibrium

Definition 9 An entry strategy for firm $i \in \{a, b\}$ is a measurable mapping $\varphi_i(Q_i, \omega)$: $[\underline{q}, \overline{q}] \times \Omega \to [0, \infty)$, such that for each $Q_i \in [\underline{q}, \overline{q}]$, $\tau_i(Q_i) = \varphi_i(Q_i, \cdot)$ is a stopping time.

Given a strategy pair (φ_a, φ_b), firm i's entry time τ_i, as assessed from firm $-i$'s perspective, is determined by

$$G_i(t) := \mathbb{P}_t(\tau_i \leq t) = \int_{\underline{q}}^{\overline{q}} \mathbb{P}_t(\tau_i(q) \leq t \mid Q_i = q) d\Psi(q). \tag{E.1}$$

Let $\mathbf{Q}_t^i = \Psi^{-1}(1 - G_i(t))$, and let Φ denote the set of all feasible entry strategies, in which $(X_t, \mathbf{Q}_t^a, \mathbf{Q}_t^b)$ is a Markov process.

For a given pair of entry strategy (τ_a, τ_b) and firm i's private signal Q_i, firm i's continuation value function at time t with $X_t = x$, $\mathbf{Q}_t^a = q_a$, and $\mathbf{Q}_t^b = q_b$ is given by

$$\mathbb{E}_t^{x, q_a, q_b}\Bigg[e^{-r(\tau_L - t)}\Bigg(\mathbf{1}_{\tau_i(Q_i) < \tau_{-i}(Q_{-i})}\big(L_{\tau_L}(X_{\tau_L}, Q_i) - K\big) + \mathbf{1}_{\tau_i(Q_i) > \tau_{-i}(Q_{-i})} F_{\tau_L}(X_{\tau_L}, Q_i)$$

$$+ \mathbf{1}_{\tau_i(Q_i)=\tau_{-i}(Q_{-i})} \frac{L_{\tau_L}(X_{\tau_L}, Q_i) - K + F_{\tau_L}(X_{\tau_L}, Q_i)}{2} \Big) \Big]$$

$$= \mathbb{E}_t^{x,q_a,q_b} \Bigg[\int_{\underline{q}}^{\overline{q}} e^{-r(\tau_i(Q_i)\wedge\tau_{-i}(q)-t)} \Bigg(\mathbf{1}_{\tau_i(Q_i)<\tau_{-i}(q)} \big(L_{\tau_L}(X_{\tau_L}, Q_i) - K \big) + \mathbf{1}_{\tau_i(Q_i)>\tau_{-i}(q)} F_{\tau_L}(X_{\tau_L}, Q_i)$$

$$+ \mathbf{1}_{\tau_i(Q_i)=\tau_{-i}(q)} \frac{L_{\tau_L}(X_{\tau_L}, Q_i) - K + F_{\tau_L}(X_{\tau_L}, Q_i)}{2} \Bigg) d\Psi_t^{-i}(q) \Bigg], \tag{E.2}$$

where $L_{\tau_L}(X_{\tau_L}, Q_i)$ is given in (46), $F_{\tau_L}(X_{\tau_L}, Q_i)$ is given in (49)-(50), $\tau_L = \tau_i(Q_i) \wedge \tau_{-i}(Q_{-i})$ and $\Psi_t^{-i}(q) = \mathbb{P}_t(Q_{-i} \leq q \mid \tau_{-i}(Q_{-i}) > t)$ is the posterior CDF of Q_{-i} at time t assessed from firm i's perspective.

Definition 10 Let $J_i(x, q_a, q_b, Q_i; \varphi_a, \varphi_b)$ denote firm i's value at time t, as defined in (E.2), for a given $X_t = x > 0$, $\mathbf{Q}_t^a = q_a$, $\mathbf{Q}_t^b = q_b$, and a feasible Markov strategy pair (φ_a, φ_b). A feasible strategy pair $\{\varphi_a^*, \varphi_b^*\}$ forms a *Bayesian Nash equilibrium* if for any $Q_a, Q_b \in (\underline{q}, \overline{q})$, $x > 0$, and possible pair (q_a, q_b), we have

$$J_a(x, q_a, q_b, Q_a; \varphi_a^*, \varphi_b^*) \geq J_a(x, q_a, q_b, Q_a; \varphi_a, \varphi_b^*), \quad \forall\, (\varphi_a, \varphi_b^*) \in \Phi, \tag{E.3}$$

$$J_b(x, q_a, q_b, Q_b; \varphi_a^*, \varphi_b^*) \geq J_b(x, q_a, q_b, Q_b; \varphi_a^*, \varphi_b), \quad \forall\, (\varphi_a^*, \varphi_b) \in \Phi. \tag{E.4}$$

Note that even if firm i deviates from φ_i^* to another strategy φ_i, the dynamics of $\mathbf{Q}_t^a$ and $\mathbf{Q}_t^b$ remain unchanged because its competitor, firm $-i$, can never detect the deviation before firm i enters. More precisely, if we use $\mathbf{Q}_t^{i,\varphi_a,\varphi_b}$ to emphasize the dependence on strategy (φ_a, φ_b), then $\mathbf{Q}_t^i = q_i$ refers to $\mathbf{Q}_t^{i,\varphi_a^*,\varphi_b^*} = q_i$ when we evaluate both sides of (E.3) and (E.4).

E.2 Belief Updating

Intuitively, firm i with a lower signal Q_i enters later. This suggests a monotonically decreasing relationship between firm i's signal Q_i and its entry time $\tau_i(Q_i)$. Recall the CDF defined in (E.1) and note that $1 - G_i(t)$, representing the probability that firm i remains in the game up to time t, should equal the proportion of the types of firm i that remain in the game at time t. Then firm $-i$ infers firm i's entry as follows:

$$\tau_i(Q_i) = \inf\{t \geq 0 : \mathbf{Q}_t^i \leq Q_i\}, \tag{E.5}$$

where $\mathbf{Q}_t^i = \Psi^{-1}(1 - G_i(t))$. Since $\mathbf{Q}_t^i > Q_i$ is equivalent to $\tau_i(Q_i) > t$, $\mathbf{Q}_t^i$ is the highest possible value of Q_i conditional on no Leader before time t.

A firm updates its belief about its competitor's private signal, starting from a prior $\Psi(q)$ where $q \in [\underline{q}, \overline{q}]$. The posterior CDF of Q_i at time t (conditional on no Leader) is given by

$$\Psi_t^i(q) = \frac{\mathbb{P}_t(Q_i \leq q, \tau_i(Q_i) > t)}{\mathbb{P}_t(\tau_i(Q_i) > t)} = \frac{\mathbb{P}_t(Q_i \leq q, Q_i < \mathbf{Q}_t^i)}{\mathbb{P}_t(Q_i < \mathbf{Q}_t^i)} = \frac{\Psi(q)}{\Psi(\mathbf{Q}_t^i)}, \quad q < \mathbf{Q}_t^i. \tag{E.6}$$

Next, we derive the dynamic equation of $\mathbf{Q}_t^i$. As the waiting (to enter) game unfolds, the equilibrium posterior belief at $t > 0$ requires that only those types with $Q_i < \mathbf{Q}_t^i$ are still in the game, so that the mass of firm i's type decreases from one at $t = 0$ to $\Psi(\mathbf{Q}_t^i)$. Additionally, over a small increment $(t, t + dt)$, only those whose types lie inside the interval $(\mathbf{Q}_t^i + d\mathbf{Q}_t^i, \mathbf{Q}_t^i)$ enter. Therefore, given $\{X_s; s \leq t + dt\}$, firm $-i$ infers that firm i's conditional entry probability over the $(t, t + dt)$ interval, denoted by $\Lambda_t^i dt$, must be given by the ratio between $\Psi(\mathbf{Q}_t^i) - \Psi(\mathbf{Q}_t^i + d\mathbf{Q}_t^i)$ and $\Psi(\mathbf{Q}_t^i)$:

$$\Lambda_t^i dt = \frac{\Psi(\mathbf{Q}_t^i) - \Psi(\mathbf{Q}_t^i + d\mathbf{Q}_t^i)}{\Psi(\mathbf{Q}_t^i)} = \Psi_t^i(\mathbf{Q}_t^i) - \Psi_t^i(\mathbf{Q}_t^i + d\mathbf{Q}_t^i), \tag{E.7}$$

where $\Psi_t^i(q) = \Psi(q)/\Psi(\mathbf{Q}_t^i)$ is the posterior CDF of Q_i and is given in (E.6). As $\Psi(\mathbf{Q}_t^i) = 1 - G_i(t)$ and $\Psi(\mathbf{Q}_t^i + d\mathbf{Q}_t^i) = 1 - G_i(t + dt)$, rewriting (E.7) and taking $dt \to 0$, we obtain $\Lambda_t^i = \frac{G_i'(t)}{1 - G_i(t)}$. That is, Λ_t^i is the hazard rate implied by the CDF $G_i(t)$ for firm i's entry time τ_i conditional on $\{X_s; s \leq t\}$, so that $G_i(t) = 1 - e^{-\int_0^t \Lambda_s^i ds}$. Using $\Psi(\mathbf{Q}_t^i + d\mathbf{Q}_t^i) - \Psi(\mathbf{Q}_t^i) = \Psi'(\mathbf{Q}_t^i) d\mathbf{Q}_t^i$, we obtain the following dynamics for $\mathbf{Q}_t^i$ from (E.7):

$$d\mathbf{Q}_t^i = \frac{-\Psi(\mathbf{Q}_t^i)}{\Psi'(\mathbf{Q}_t^i)} \Lambda_t^i dt. \tag{E.8}$$

E.3 Equilibrium Solution

Below we characterize a symmetric Bayesian Nash equilibrium in which $\Lambda_t^i = \Lambda^*(X_t, \mathbf{Q}_t^i)$ for $i = a, b$, where $\mathbf{Q}_t^a = \mathbf{Q}_t^b = \mathbf{Q}_t^*$. At time $t = \tau_L$, conditional on being Leader, firm i's estimation of Q is given by

$$\begin{aligned} \mathbb{Q}_i^L &= \mathbb{E}_{\tau_L}[Q \mid Q_i, \tau_i < \tau_{-i}] = \mathbb{E}_{\tau_L}\left[\frac{Q_i + Q_{-i}}{2} \mid Q_i, \tau_i < \tau_{-i}\right] \\ &= \frac{Q_i}{2} + \mathbb{E}_t\left[\frac{Q_{-i}}{2} \mid Q_{-i} < \mathbf{Q}_{\tau_L}^*\right] = \frac{Q_i}{2} + \int_{\underline{q}}^{\mathbf{Q}_{\tau_L}^*} \frac{z}{2} \frac{d\Psi(z)}{\Psi(\mathbf{Q}_{\tau_L}^*)}. \end{aligned} \tag{E.9}$$

We denote

$$\mathcal{W}(Q_i, q) = \frac{Q_i}{2} + \int_{\underline{q}}^{q} \frac{z}{2} \frac{d\Psi(z)}{\Psi(q)}. \tag{E.10}$$

Denote $V_i(x, q)$ as firm i's equilibrium value function for $X_t = x$ and $\mathbf{Q}_t^* = q$. For $\mathbf{Q}_t^* =$

$q \leq Q_i$, firm i believes that its competitor has a lower signal with probability one. Then firm i is willing to enter as Leader by solving the following problem:

$$\begin{aligned} J_L(x; Q_i, \mathbf{Q}_t^*) &= \max_{\tau_L \geq t} \mathbb{E}_t^x \left[e^{-r(\tau_L - t)} (L_{\tau_L}(X_{\tau_L}, Q_i) - K) \right] \\ &= \max_{\tau_L \geq t} \mathbb{E}_t^x \left[e^{-r(\tau_L - t)} (p\mathcal{W}(Q_i, \mathbf{Q}_{\tau_L}^*) \Pi(X_{\tau_L}) - K) \right] \\ &\leq \max_{\tau_L \geq t} \mathbb{E}_t^x \left[e^{-r(\tau_L - t)} (p\mathcal{W}(Q_i, \mathbf{Q}_t^*) \Pi(X_{\tau_L}) - K) \right], \end{aligned} \tag{E.11}$$

where the second equality uses (46), (E.9) and (E.10), the inequality uses $\mathbf{Q}_{\tau_L}^* \leq \mathbf{Q}_t^*$. We can show that the inequality in (E.11) is an equality, and[31]

$$V_i(x, \mathbf{Q}_t^*) = J_L(x; Q_i, \mathbf{Q}_t^*) = p\mathcal{W}(Q_i, \mathbf{Q}_t^*) M^* \left(x; \frac{K}{p\mathcal{W}(Q_i, \mathbf{Q}_t^*)} \right), \tag{E.12}$$

where $M^*(x; K)$ is given by

$$M^*(x; K) = (\Pi(\widetilde{x}(K)) - K) \left(\frac{x}{\widetilde{x}(K)} \right)^\beta, \quad x < \widetilde{x}(K), \tag{E.13}$$

$$M^*(x; K) = \Pi(x) - K, \quad x \geq \widetilde{x}(K), \tag{E.14}$$

and $\widetilde{x}(K) = \frac{\beta}{\beta - 1}(r - \mu)K$.

When $\mathbf{Q}_t^* = q > Q_i$, firm i chooses to wait. If Q_{-i} equals the marginal type $\mathbf{Q}_t^* = q > Q_i$, firm $-i$ will enter as Leader at time t. As a result, when firm $-i$ enters as Leader at time $t = \tau_L$, firm i infers $Q_{-i} = \mathbf{Q}_{\tau_L}^*$ and updates its estimate of Q at time $t = \tau_L$ as follows:

$$\mathbb{Q}_i^F = \mathbb{E}_{\tau_L}[Q \mid Q_i, \tau_i > \tau_{-i}] = \mathbb{E}_{\tau_L}\left[\frac{Q_i + Q_{-i}}{2} \mid Q_i, \tau_i > \tau_{-i} \right] = \frac{Q_i + \mathbf{Q}_{\tau_L}^*}{2}. \tag{E.15}$$

Substituting (E.15) into (49)-(50), we obtain that firm i's payoff is equal to $pM\left(\frac{Q_i + \mathbf{Q}_{\tau_L}^*}{2} X_{\tau_L}\right)$ if its competitor enters as Leader at time $t = \tau_L$, where $M(x) = M^*(x; K)$.

Using (52), we can derive the HJB equation for $V_i(x, q)$ as follows:

$$\begin{aligned} rV_i(x, q) =& \frac{\sigma^2 x^2}{2} \frac{\partial^2 V_i(x, q)}{\partial x^2} + \mu x \frac{\partial V_i(x, q)}{\partial x} \\ &+ \Lambda^*(x, q) \left[pM\left(\frac{Q_i + q}{2} x \right) - V_i(x, q) \right] - \frac{\Psi(q)}{\Psi'(q)} \Lambda^*(x, q) \frac{\partial V_i(x, q)}{\partial q} \end{aligned} \tag{E.16}$$

for $q > Q_i$. Because firm i is indifferent between entering and waiting at the entry threshold

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[31] The optimal stopping to the problem $\max_{\tau_L \geq t} \mathbb{E}_t^x \left[e^{-r(\tau_L - t)} (p\mathcal{W}(Q_i, \mathbf{Q}_t^*) \Pi(X_{\tau_L}) - K) \right]$ is $\tau_L^* = \inf \left\{ s \geq t : X_s \geq \widetilde{x}\left(\frac{K}{p\mathcal{W}(Q_i, \mathbf{Q}_t^*)} \right) \right\}$, where $\widetilde{x}(K) = \frac{\beta}{\beta - 1}(r - \mu)K$. Since $\mathbf{Q}_t^* \leq Q_i$, $\mathcal{W}(Q_i, \mathbf{Q}_t^*) \geq \mathcal{W}(\mathbf{Q}_t^*, \mathbf{Q}_t^*) = \mathcal{I}(\mathbf{Q}_t^*)$. When $X_t \geq \widetilde{x}\left(\frac{K}{p\mathcal{W}(Q_i, \mathbf{Q}_t^*)} \right)$, $\tau_L^* = t$ and $\mathbf{Q}_{\tau_L^*}^* = \mathbf{Q}_t^*$. When $X_t < \widetilde{x}\left(\frac{K}{p\mathcal{W}(Q_i, \mathbf{Q}_t^*)} \right)$, $\Lambda^*(X_s, \mathbf{Q}_t^*) = 0$ for any $s \in [t, \tau_L^*)$, which implies $\mathbf{Q}_{\tau_L^*}^* = \mathbf{Q}_t^*$. This proves that the optimal stopping to (E.11) is $\tau_L^* = \inf \left\{ s \geq t : X_s \geq \widetilde{x}\left(\frac{K}{p\mathcal{W}(Q_i, \mathbf{Q}_t^*)} \right) \right\}$, and the inequality in (E.11) holds as an equality.

$q = Q_i$, $V_i(x,q) = J_L(x;Q_i,q)$ satisfies equation (E.16) at $q = Q_i$. Comparing with (51) and (E.10), we see that $\mathcal{W}(Q_i,Q_i) = \mathcal{I}(Q_i)$. Then, in the region $x < \widetilde{x}\Big(\frac{K}{p\mathcal{I}(Q_i)}\Big)$, $V_i(x,Q_i) = J_L(x;Q_i,Q_i) = p\mathcal{I}(Q_i)M^*(x;\frac{K}{p\mathcal{I}(Q_i)})$ satisfies

$$rV_i(x,Q_i) = \frac{\sigma^2 x^2}{2}\frac{\partial^2 V_i(x,Q_i)}{\partial x^2} + \mu x \frac{\partial V_i(x,Q_i)}{\partial x}. \tag{E.17}$$

Substituting the above into (E.16), we then find that $\Lambda^*(x,Q_i)$ is zero in the region $x < \widetilde{x}\Big(\frac{K}{p\mathcal{I}(Q_i)}\Big)$.

Next, we consider the region $x > \widetilde{x}\Big(\frac{K}{p\mathcal{I}(Q_i)}\Big)$. Fix $x > \widetilde{x}\Big(\frac{K}{p\mathcal{I}(Q_i)}\Big)$, for q that is close to Q_i, we have $x > \widetilde{x}\Big(\frac{K}{p\mathcal{W}(Q_i,q)}\Big)$, and $M^*\big(x;\frac{K}{p\mathcal{W}(Q_i,q)}\big) = \Pi(x) - \frac{K}{p\mathcal{W}(Q_i,q)}$, which implies $J_L(x;Q_i,q) = p\mathcal{W}(Q_i,q)\Pi(x) - K$, and

$$\begin{aligned}\frac{\partial V_i(x,q)}{\partial q}\,|_{q=Q_i} = {}& \frac{\partial J_L(x;Q_i,q)}{\partial q}\,|_{q=Q_i} = \Pi(x)p\frac{\partial \mathcal{W}(Q_i,q)}{\partial q}\,|_{q=Q_i} = \Pi(x)p\frac{\partial \int_{\underline{q}}^{q} \frac{Q_i+z}{2}\frac{d\Psi(z)}{\Psi(q)}}{\partial q}\,|_{q=Q_i} \\ ={}& \Pi(x)p\frac{\Psi'(Q_i)}{\Psi(Q_i)}\Big[Q_i - \mathcal{I}(Q_i)\Big].\end{aligned} \tag{E.18}$$

Thus, the coefficient of $\Lambda^*(x,Q_i)$ in (E.16) is given by

$$\begin{aligned}& pM(Q_i x) - J_L(x;Q_i,Q_i) - \frac{\Psi(Q_i)}{\Psi'(Q_i)}\frac{\partial V_i(x,q)}{\partial q}\,|_{q=Q_i} \\ ={}& p[\Pi(Q_i x) - K] - p\mathcal{I}(Q_i)M^*\Big(x;\frac{K}{p\mathcal{I}(Q_i)}\Big) - \Pi(x)p\Big[Q_i - \mathcal{I}(Q_i)\Big] \\ ={}& p[\Pi(Q_i x) - K] - p\mathcal{I}(Q_i)\Big[\Pi(x) - \frac{K}{p\mathcal{I}(Q_i)}\Big] - \Pi(x)p\Big[Q_i - \mathcal{I}(Q_i)\Big] = (1-p)K,\end{aligned} \tag{E.19}$$

where the first equality uses (E.18), $x > \widetilde{x}\Big(\frac{K}{p\mathcal{I}(Q_i)}\Big) > \widetilde{x}\big(\frac{K}{Q_i}\big)$ and $M(x) = M^*(x;K)$ as given by (E.13)-(E.14), the second equality uses (E.14) and $x > \widetilde{x}\Big(\frac{K}{p\mathcal{I}(Q_i)}\Big)$.

In addition, in the region $x > \widetilde{x}\Big(\frac{K}{p\mathcal{I}(Q_i)}\Big)$, using $J_L(x;Q_i,Q_i) = p\mathcal{I}(Q_i)M^*(x;\frac{K}{p\mathcal{I}(Q_i)}) = p\mathcal{I}(Q_i)\Pi(x) - K$, we obtain

$$\Big[\frac{\sigma^2 x^2}{2}\frac{\partial^2}{\partial x^2} + \mu x\frac{\partial}{\partial x} - r\Big]J_L(x;Q_i,Q_i) = rK - p\mathcal{I}(Q_i)x. \tag{E.20}$$

Substituting (E.19) and (E.20) into (E.16), we obtain

$$\Lambda^*(x,Q_i) = \frac{p\mathcal{I}(Q_i)x - rK}{(1-p)K} \tag{E.21}$$

in the region $x > \widetilde{x}\Big(\frac{K}{p\mathcal{I}(Q_i)}\Big)$. Since $\widetilde{x}\Big(\frac{K}{p\mathcal{I}(q)}\Big) = \frac{\beta}{\beta-1}(r-\mu)\frac{K}{p\mathcal{I}(q)} = \overline{x}(q)$, and Q_i takes any value within $(\underline{q},\overline{q})$, we infer $\Lambda^*(x,q) = \frac{p\mathcal{I}(q)x - rK}{(1-p)K}\mathbf{1}_{x>\overline{x}(q)}$ for all $x > 0$ and $q \in (\underline{q},\overline{q})$.

F Extended Information-spillover Model with Different Investment Opportunities

F.1 Model and Solution

In this appendix, we extend the information-spillover model in Section 7 to a setting where two firms differ in their investment opportunities. Specifically, if firm i enters, it captures an industry profit of $N_i\theta Q X_t$ per unit of time. All other assumptions remain the same as in Section 7. We assume that the values of N_a and N_b are public information, with $N_a \geq N_b > 0$.

At the moment of entry ($t = \tau_L$) as Leader, the value of firm i is

$$L^i_{\tau_L}(X_{\tau_L}, Q_i) = \mathbb{E}_{\tau_L}\Big[\int_{\tau_L}^{\infty} e^{-r(s-t)} N_i \theta Q X_s ds \mid Q_i, \tau_i < \tau_{-i}\Big] = pN_i \mathbb{Q}^L_i \frac{X_{\tau_L}}{r-\mu}, \tag{F.1}$$

where $\mathbb{Q}^L_i = \mathbb{E}_{\tau_L}[Q \mid Q_i, \tau_i < \tau_{-i}]$.

Similar to (49)-(50), firm i's value as Follower at τ_L can be written as:

$$F^i_{\tau_L}(X_{\tau_L}, Q_i) = pM\Big(N_i \mathbb{Q}^F_i X_{\tau_L}\Big), \tag{F.2}$$

where $\mathbb{Q}^F_i = \mathbb{E}_{\tau_L}[Q \mid Q_i, \tau_i > \tau_{-i}]$, $M(x) = M^*(x; K)$ is given by (E.13)-(E.14).

We can similarly define $\mathbf{Q}^i_t$ and the Bayesian Nash equilibrium as in Appendix E.1, except that in the continuation value (E.2), $L_{\tau_L}(X_{\tau_L}, Q_i)$ is replaced by $L^i_{\tau_L}(X_{\tau_L}, Q_i)$ in (F.1), and $F_{\tau_L}(X_{\tau_L}, Q_i)$ is replaced by $F^i_{\tau_L}(X_{\tau_L}, Q_i)$ in (F.2).

Next, we summarize the perfect Bayesian Nash equilibrium solution for this game. See Internet Appendix IA.I for technical details.

Theorem 5 *Suppose there exist an increasing function $\zeta(\cdot)$ satisfying*

$$N_a \mathcal{W}(q_a, \zeta(q_a)) = N_b \mathcal{W}(\zeta(q_a), q_a) \tag{F.3}$$

for any $q_a \in [\underline{q}, \overline{q}]$ such that $\zeta(q_a) \in [\underline{q}, \overline{q}]$, where $\mathcal{W}(\cdot,\cdot)$ is given by (E.10)*. Suppose processes $\mathbf{Q}^{a*}_t$, $\mathbf{Q}^{b*}_t$ satisfy $\mathbf{Q}^{a*}_0 = \mathbf{Q}^{b*}_0 = \overline{q}$,*[32]

$$\mathbf{Q}^{a*}_t = \sup\Big\{q_a \in [\zeta^{-1}(\overline{q}), \overline{q}] : X^*_t \leq \widetilde{x}\Big(\frac{K}{pN_a\mathcal{W}(q_a, \overline{q})}\Big)\Big\}, \quad t \in (0, T_a], \tag{F.4}$$

$$d\mathbf{Q}^{a*}_t = \frac{-\Psi(\mathbf{Q}^{a*}_t)}{\Psi'(\mathbf{Q}^{a*}_t)} \Lambda^*_a(X_t, \mathbf{Q}^{a*}_t)dt, \quad t \geq T_a, \text{ and } \quad \mathbf{Q}^{a*}_{T_a} = \zeta^{-1}(\overline{q}), \tag{F.5}$$

[32]We set $\sup\emptyset = \zeta^{-1}(\overline{q})$ in (F.4), which occurs when $X_0 > \widetilde{x}\big(\frac{K}{pN_a\mathcal{W}(\zeta^{-1}(\overline{q}),\overline{q})}\big)$ and $T_a = 0$.

$$d\mathbf{Q}_t^{b*} = \frac{-\Psi(\mathbf{Q}_t^{b*})}{\Psi'(\mathbf{Q}_t^{b*})}\Lambda_b^*(X_t, \mathbf{Q}_t^{b*})dt, \quad t \geq T_a, \text{ and } \quad \mathbf{Q}_t^{b*} = \overline{q}, \quad t \leq T_a, \tag{F.6}$$

and $\mathbf{Q}_t^{b*} = \zeta(\mathbf{Q}_t^{a*})$ *for* $t \geq T_a$*, where* $X_t^* = \max_{s\in[0,t]} X_s$, $\widetilde{x}(K) = \frac{\beta}{\beta-1}(r-\mu)K$, $T_a = \inf\{t \geq 0 : X_t \geq \widetilde{x}\big(\frac{K}{pN_a\mathcal{W}(\zeta^{-1}(\overline{q}),\overline{q})}\big)\}$, $\Lambda_a^*(x, q_a)$ *is given in* (IA.I.10) *and* $\Lambda_b^*(x, q_b)$ *is given in* (IA.I.9). *There exists a perfect Bayesian Nash equilibrium in which firm i's entry time is* $\tau_i^* = \inf\{t \geq 0 : \mathbf{Q}_t^{i*} \leq Q_i\}$, $i = a, b$.

F.2 Empirical Predictions

Next, we discuss economic predictions of Theorem 5. To ease exposition, we assume that $\Psi(q)$ follows a power law: $\Psi(q) = (q/\overline{q})^\alpha$, $q \in [0, \overline{q}]$, where $\overline{q} > 0$ and $\alpha > 0$. Then, $\mathcal{W}(\cdot,\cdot)$ as defined in (E.10) admits the following closed-form expression:

$$\mathcal{W}(Q_i, q) = \frac{Q_i}{2} + \int_0^q \frac{z}{2}\frac{d\Psi(z)}{\Psi(q)} = \frac{Q_i + \frac{\alpha}{\alpha+1}q}{2}. \tag{F.7}$$

Substituting (F.7) into (F.3), we obtain

$$\zeta(q_a) = N_{ab}q_a, \tag{F.8}$$

where $N_{ab} := \frac{N_a - \frac{\alpha}{\alpha+1}N_b}{N_b - \frac{\alpha}{\alpha+1}N_a}$. To ensure $N_{ab} > 0$ and $\zeta(q_a) > 0$ for $q_a > 0$, assume $N_b - \frac{\alpha}{\alpha+1}N_a > 0$.

Without loss of generality, consider the case where $N_a > N_b$. Then, $N_{ab} > 1$, and $T_a = \inf\{t \geq 0 : \mathbf{Q}_t^{a*} \leq \overline{q}/N_{ab}\}$. Using $\frac{\Psi(q)}{\Psi'(q)} = \frac{q}{\alpha}$, we can show $\mathbf{Q}_t^{b*} = N_{ab}\mathbf{Q}_t^{a*} > \mathbf{Q}_t^{a*}$ for $t \geq T_a$.

For $t \leq T_a$, as $\mathbf{Q}_t^{b*} = \overline{q} > Q_b$, firm b chooses to wait and firm a enters as Leader for $t \leq T_a$ if $Q_a \geq \overline{q}/N_{ab}$. If $Q_a < \overline{q}/N_{ab}$, using $\tau_i^* = \inf\{t \geq 0 : \mathbf{Q}_t^{i*} \leq Q_i\}$ and $\mathbf{Q}_t^{b*} = N_{ab}\mathbf{Q}_t^{a*} > \mathbf{Q}_t^{a*}$ for $t \geq T_a$, we can show $\tau_a^* < \tau_b^*$ when Q_a is close to Q_b. Thus, given the same signals, the firm with the higher value of N_i is more likely to become Leader. This prediction aligns with the finding in Décaire and Wittry (2025): "*first-movers tend to receive the strongest signal among their regional peers. Moreover, they are the most likely to benefit from the newly revealed information, as they own the largest number of options in the region ... This allows them to better internalize the information spillover induced by drilling their own wells.*"

Second, we can show that

$$N_a\mathcal{W}(\zeta^{-1}(q_b), q_b) = N_a\mathcal{W}\Big(\frac{q_b}{N_{ab}}, q_b\Big) = N_a\frac{q_b}{2}\Big[\frac{1}{N_{ab}} + \frac{\alpha}{\alpha+1}\Big] = \frac{q_b}{2}N_aN_b\frac{1-\big(\frac{\alpha}{\alpha+1}\big)^2}{N_a - \frac{\alpha}{\alpha+1}N_b}, \tag{F.9}$$

which is decreasing in N_a and increasing in N_b. Thus $\Lambda_b^*(x, q_b)$ given in (IA.I.9) is decreasing in N_a and increasing in N_b. Similarly, we can show that $N_b\mathcal{W}(\zeta(q_a), q_a)$ and $\Lambda_a^*(x, q_a)$ given in

(IA.I.10) are decreasing in N_b and increasing in N_a. Therefore, increasing the value of firm $-i$'s exercise payoff (N_{-i}) leads to a greater investment delay for firm i (i.e., a slower $d\mathbf{Q}_t^{i*}/dt$). This prediction aligns with the finding in Décaire and Wittry (2025): "*an increase in the quantity of information expected to be released by peers increases firms' incentive to delay investment decisions ... increase in the number of nearby peer options reduces the likelihood of project exercise...*"

In summary, our extended information-spillover model generates predictions that are consistent with the empirical findings in Décaire and Wittry (2025).

G Time-varying Second-mover Advantage

In this appendix, we allow Follower's entry cost to depend on $\tau_L - \tau_F$. Intuitively, the longer that Follower waits after Leader enters, the more the former learns from the latter. Moreover, the marginal second-mover advantage, e.g., the marginal benefit of Follower's cost reduction, decreases as Follower's waiting time $\tau_F - \tau_L$ increases. For example, Follower's marginal benefit of learning decreases as it learns from Leader over time. Let $K_F(\tau_F - \tau_L)$ denote Follower's entry cost at the moment it enters at τ_F. We capture these above ideas by using the following entry cost specification for Leader:

$$K_F(\tau_F - \tau_L) = K_L e^{-\zeta(\tau_F - \tau_L)}, \tag{G.1}$$

where $\zeta > 0$ is a parameter that captures time-varying second-mover advantage. Equation (G.1) implies that there is no second-mover advantage if Follower enters immediately after Leader does in that $K_F(0) = K_L$. The specification in (G.1) is more realistic than the one in the main body of our paper, removing firms' incentives to wait just a split second solely for the purpose of enjoying an instant entry-cost reduction: $\Delta K = K_L - K_F$. By making $K_F(\cdot)$ decreasing and convex in the inter-entry time $\tau_F - \tau_L$, we capture the idea that the second mover advantage increases but the marginal second mover advantage decreases as Follower continues to learn from Leader by waiting to enter.

Taking Follower's optimal entry time for (G.3), τ_F^*, as given, Leader's value at τ_L is the same as in Section 2.2 and is given by

$$L(X_{\tau_L}) = \mathbb{E}_{\tau_L}\left[\int_{\tau_L}^{\infty} e^{-r(s-\tau_L)}X_s ds - \int_{\tau_F^*}^{\infty} e^{-r(s-\tau_L)}(1-D)X_s ds\right]. \tag{G.2}$$

Follower's value at $t = \tau_L$ is given by

$$F(X_{\tau_L}) = \max_{\tau_F \geq \tau_L} \mathbb{E}_{\tau_L}\left[\int_{\tau_F}^{\infty} e^{-r(s-\tau_L)}DX_s ds - e^{-r(\tau_F-\tau_L)}K_F(\tau_F - \tau_L)\right]. \tag{G.3}$$

For notational convenience, let $Y_t = e^{\zeta(t-\tau_L)}X_t$, $t \geq \tau_L$. We can show that Y_t is a GBM with drift $\mu + \zeta$ and volatility σ:

$$dY_t = (\mu + \zeta)Y_t dt + \sigma Y_t d\mathcal{Z}_t \quad t \geq \tau_L, \quad Y_{\tau_L} = X_{\tau_L}. \tag{G.4}$$

Using Y_t and (G.1), we can write Follower's value at τ_L as follows:

$$\begin{aligned} F(X_{\tau_L}) &= \max_{\tau_F \geq \tau_L} \mathbb{E}_{\tau_L}\left[e^{-r(\tau_F-\tau_L)}\Big(D\Pi(X_{\tau_F}) - K_L e^{-\zeta(\tau_F-\tau_L)}\Big)\right] \\ &= \max_{\tau_F \geq \tau_L} \mathbb{E}_{\tau_L}\left[e^{-(r+\zeta)(\tau_F-\tau_L)}\Big(D\Pi(Y_{\tau_F}) - K_L\Big)\right], \end{aligned} \tag{G.5}$$

where $\Pi(x) = x/(r-\mu)$. In effect, after Leader enters, Follower solves an optimal entry problem with the same entry cost K_L as Leader, but the payoff being Y rather than X and a discount rate of $r + \zeta$ rather than r.

Follower's optimal entry time is given by $\tau_F^* = \inf\{s \geq \tau_L : Y_s \geq y_F\}$, where y_F is a constant and given by

$$y_F = \frac{1}{D}\frac{\alpha}{\alpha - 1}(r-\mu)K_L. \tag{G.6}$$

In (G.6), $\alpha \in (1, \beta)$ measures optionality for the Y_t process with a discount rate $r + \zeta$:

$$\alpha = \frac{-(\mu + \zeta - \frac{1}{2}\sigma^2) + \sqrt{(\mu + \zeta - \frac{1}{2}\sigma^2)^2 + 2(r+\zeta)\sigma^2}}{\sigma^2}. \tag{G.7}$$

Follower's value $F(X_{\tau_L})$ is thus given by:

$$F(X_{\tau_L}) = (D\Pi(y_F) - K_L)\left(\frac{X_{\tau_L}}{y_F}\right)^{\alpha}, \quad X_{\tau_L} < y_F, \tag{G.8}$$

$$F(X_{\tau_L}) = D\Pi(X_{\tau_L}) - K_L, \quad X_{\tau_L} \geq y_F. \tag{G.9}$$

Solving $L(X_{\tau_L})$ defined in (G.2), we obtain

$$L(X_{\tau_L}) = \Pi(X_{\tau_L}) - (1-D)\Pi(y_F)\left(\frac{X_{\tau_L}}{y_F}\right)^{\alpha}, \quad X_{\tau_L} < y_F, \tag{G.10}$$

$$L(X_{\tau_L}) = D\Pi(X_{\tau_L}), \quad X_{\tau_L} \geq y_F. \tag{G.11}$$

Let $S(x) = F(x) - (L(x) - K_L)$, where $F(\cdot)$ and $L(\cdot)$ are given by (G.8)-(G.9) and (G.10)-

(G.11), respectively. The equation $S(x) = 0$ has a unique root $\widehat{x}$ in the $(0, y_F)$ domain and

$$S(x) = 0\,, \quad x \geq y_F, \tag{G.12}$$

$$S(x) < 0\,, \quad x \in (\widehat{x}, y_F), \tag{G.13}$$

$$S(x) > 0\,, \quad x < \widehat{x}. \tag{G.14}$$

We summarize the symmetric equilibrium strategy in the following theorem.

Theorem 6 *There exists a symmetric Markov perfect equilibrium with the following properties:*

1. *In the $x \geq y_F$ region where y_F is given in (G.6), two firms simultaneously enter and $V_a(x) = V_b(x) = D\Pi(x) - K_L$.*

2. *In the $x \in [\widehat{x}, y_F)$ region, firms compete to become Leader with one firm being randomly selected as Leader and $V_a(x) = V_b(x) = (L(x) - K_L + F(x))/2$.*

3. *In the $x < \widehat{x}$ region, firms wait or play mixed entry strategies. Pre-entry firm values are equal: $V_a(x) = V_b(x) = V_*(x)$, where $V_*(x)$ is the unique solution to the following variational inequality*[33]
$$\max\left\{\frac{\sigma^2 x^2}{2} V_*''(x) + \mu x V_*'(x) - rV_*(x),\ \ (L(x) - K_L) - V_*(x)\right\} = 0, \tag{G.15}$$
and $L(\cdot)$ is given by (G.10)-(G.11). The equilibrium entry rates are $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$, where $\lambda^*(x) = \frac{rL(x) - \left[\frac{\sigma^2 x^2}{2} L''(x) + \mu x L'(x)\right] - rK_L}{S(x)}$ in the probabilistic entry region: $\mathcal{R}^E := \{x < \widehat{x} : V_*(x) = L(x) - K_L\}$, and $\lambda^*(x) = 0$ for all x in the $x < \widehat{x}$ domain but not in $\mathcal{R}^E$, i.e., $x \in (0, \widehat{x}) \setminus \mathcal{R}^E$.

In Figure 7, we plot the value functions in Panel A using the solid line and the equilibrium entry rate $\lambda^*(x)$ in Panel B. The equilibrium solution for this case features four regions. To the left of $\underline{x}$ is the waiting region where $\lambda^*(x) = 0$ and $V_a(x) = V_b(x) = (x/\underline{x})^\beta\,(L(\underline{x}) - K_L)$.[34] In the mixed entry region where $x \in [\underline{x}, \widehat{x})$, firms enter probabilistically and firm values are equal: $V_a(x) = V_b(x) = L(x) - K_L$. Using $\widehat{x} < y_F$ and (G.10), we can pin down the equilibrium entry rate $\lambda^*(x) = \frac{x - (1-D)(\alpha-1)\zeta\Pi(y_F)\left(\frac{x}{y_F}\right)^\alpha - rK_L}{S(x)}$ in this region. In the region where

[33] The boundary conditions are $V_*(0) = 0$ and $V_*(\widehat{x}) = F(\widehat{x})$.

[34] The threshold $\underline{x}$ is determined by the smooth-pasting condition: $V_i'(\underline{x}) = L'(\underline{x})$.

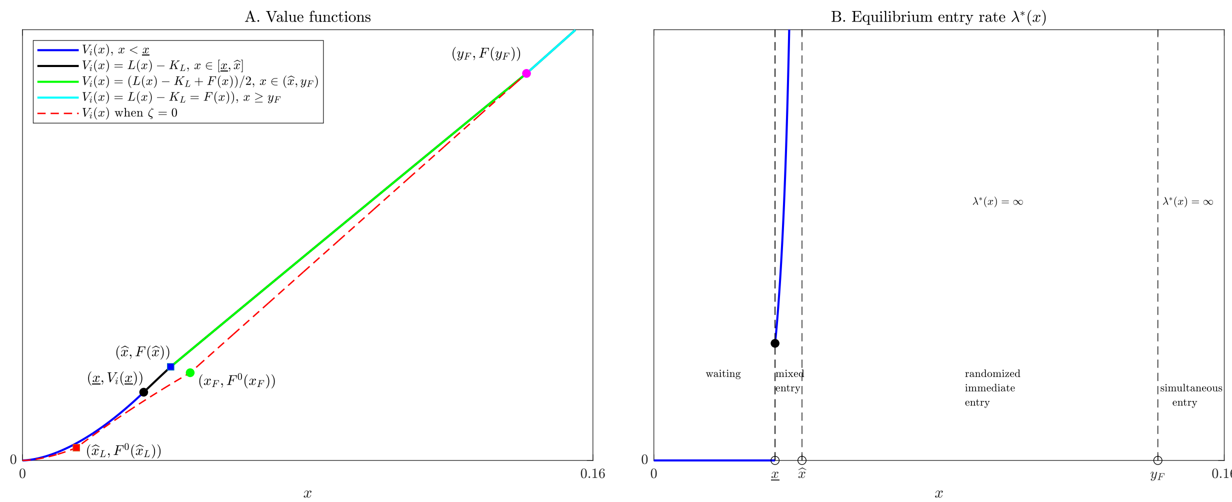

Figure 7: VALUE FUNCTIONS $V_a(x) = V_b(x) = V_*(x)$ AND EQUILIBRIUM PROBABILISTIC ENTRY RATES $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$ WITH A TIME-VARYING SECOND-MOVER ADVANTAGE. The threshold values dividing the total market demand x into the four regions are: $\underline{x}$, $\widehat{x}$, and y_F. The four regions are: 1.) the $x < \underline{x}$ waiting region, 2.) the $x \in [\underline{x}, \widehat{x})$ probabilistic entry (mixed-strategy) region, 3.) the $x \in [\widehat{x}, y_F)$ randomized immediate entry (with Follower voluntarily waiting) and 4.) the $x \geq y_F$ simultaneous entry region. Parameter values are $\zeta = 0.1$, $D = 0.5$, $K_L = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$.

$x \in [\widehat{x}, y_F)$, firms compete to become Leader with one being randomly selected as Leader and $V_a(x) = V_b(x) = (L(x) - K_L + F(x))/2$. Finally, to the right of y_F is the simultaneous entry region where both firms enter instantly as the market demand is high even if two firms split.

As a comparison, we also plot the equilibrium value when $\zeta = 0$ using the red dash line in Panel A of Figure 7. $F^0(x)$ denotes Follower's value when $\zeta = 0$ and is given by (8)-(9). The $\zeta = 0$ result corresponds to Case C with $R = 1$.

A key takeaway is that the equilibrium value $V_i(x)$ is higher when $\zeta > 0$ than when $\zeta = 0$. This is because Follower's stronger incentive to wait when $\zeta > 0$ ($y_F > x_F$) allows Leader to earn more monopoly rents (as the waiting region ($x \leq y_F$) when $\zeta > 0$ is wider than the waiting region ($x \leq x_F$) when $\zeta = 0$). In sum, introducing a time-varying second-mover advantage into the setting where $R = 1$, analyzed in Chapter 9 of Dixit and Pindyck (1994), makes not only Follower but also Leader better off.

Internet Appendices to

"Strategic Investment under Uncertainty with First- and Second-mover Advantages"

IA.A Heuristic Analysis for the Equilibrium in Case A

In this appendix, we first discuss the economic mechanism underlying a firm's entry decision and then we provide a mathematical proof of our equilibrium solution by extending the variational inequality method for a single firm's entry problem to our duopoly setting.

For a given mixed strategy pair $(\lambda_a(x), \lambda_b(x))$, the following HJB equation for firm i's value, $J_i(x) = J_i(x; \lambda_a(x), \lambda_b(x))$, holds:

$$rJ_i(x) = \frac{\sigma^2 x^2}{2} J_i''(x) + \mu x J_i'(x) + \lambda_i(x)[L(x) - K_L - J_i(x)] + \lambda_{-i}(x)[F(x) - J_i(x)]\,, \quad \text{(IA.A.1)}$$

where $L(x)$ is given by (12)-(13) and $F(x)$ is given by (8)-(9). The intuition for the HJB equation (IA.A.1) is as follows. The first two terms on the right side capture the standard diffusion and drift effects of X on $J_i(x)$. The third term describes the effect of firm i's own mixed (entry) strategy on its value and this term equals zero in equilibrium as a rational firm will only mix with strictly positive probabilities between two strategies that yield the same value.[1] The last term in (IA.A.1) describes the effect of the competitor's mixed entry strategy on firm i's value. If the competitor enters, firm i becomes Follower and its value function jumps from $J_i(x)$ to $F(x)$. The firm's optimality requires that the sum of these four terms on the right side equals the annualized firm value $rJ_i(x)$ (Duffie, 2001).

Rewriting (IA.A.1) and using $\lambda_i(x)[L(x) - K_L - J_i(x)] = 0$, we obtain:

$$\lambda_{-i}(x) = \frac{rJ_i(x) - \left[\frac{\sigma^2 x^2}{2} J_i''(x) + \mu x J_i'(x)\right]}{F(x) - J_i(x)}\,. \quad \text{(IA.A.2)}$$

That is, in order to make firm i indifferent between probabilistically entering and waiting, its competitor's entry rate, $\lambda_{-i}(x)$, has to equal the right side of (IA.A.2).

It is worth pointing out that although X is continuous, firm value is discontinuous and jumps when its competitor enters the market. This is an example where strategic interactions generate endogenous uncertainty via an equilibrium jump process because firms play mixed

[1] Otherwise, it is always better for the firm to play the pure strategy (waiting or entering) that yields the higher value: $\max\{L(x) - K_L, J_i(x)\}$. Although in equilibrium this term is zero, we leave it in the HJB equation (IA.A.1) to better understand the economic mechanism.

strategies in equilibrium.

Next, we turn to the symmetric Markov perfect equilibrium. Let $\lambda^*(x) = \lambda_a^*(x) = \lambda_b^*(x)$ denote the symmetric equilibrium Markov perfect mixed strategy. Equation (14) and inequality $S(x) > 0$ together imply the following for $V_i(x)$, firm i's equilibrium value function:

$$L(x) - K_L \leq V_i(x) \leq F(x)\,, \quad x > 0\,. \tag{IA.A.3}$$

That is, *ex ante* firm i's value must be weakly larger than $L(x) - K_L$, Leader's net payoff upon entry at τ_L, and weakly lower than Follower's value $F(x)$ because $S(x) > 0$ for all $x > 0$.

There are two scenarios to consider: 1.) $\lambda^*(x) > 0$ and 2.) $\lambda^*(x) = 0$. When $\lambda^*(x) > 0$, the firm must be indifferent between entering the market (becoming Leader) and waiting. That is, the value functions from the two strategies must equal:

$$V_i(x) = L(x) - K_L \quad \text{if} \quad \lambda^*(x) > 0\,, \tag{IA.A.4}$$

which implies

$$\lambda^*(x) = 0 \quad \text{if} \quad V_i(x) > L(x) - K_L\,. \tag{IA.A.5}$$

Using (IA.A.1) and (IA.A.4), we obtain the following HJB equation for $V_i(x)$:

$$rV_i(x) = \frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) + \lambda^*(x)[F(x) - V_i(x)]\,, \tag{IA.A.6}$$

which holds for both $\lambda^*(x) > 0$ and $\lambda^*(x) = 0$ cases. Re-arranging (IA.A.6) yields the following expression for $\lambda^*(x)$ for all $x > 0$:

$$\lambda^*(x) = \frac{rV_i(x) - \left[\frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x)\right]}{F(x) - V_i(x)}\,. \tag{IA.A.7}$$

When $\lambda^*(x) > 0$, substituting $V_i(x) = L(x) - K_L$ given in (IA.A.4) into (IA.A.7), we obtain

$$\lambda^*(x) = \frac{rL(x) - \left[\frac{\sigma^2 x^2}{2} L''(x) + \mu x L'(x)\right] - rK_L}{F(x) - (L(x) - K_L)}. \tag{IA.A.8}$$

Using (12)-(13), we obtain $rL(x) - \left[\frac{\sigma^2 x^2}{2} L''(x) + \mu x L'(x)\right] = \left(\mathbf{1}_{x<x_F} + D\mathbf{1}_{x\geq x_F}\right)x$. Substituting this into (IA.A.8), we obtain (19).

To complete our model solution, we still need to solve $V_i(x)$ in the $\lambda^*(x) = 0$ region and characterize the $\lambda^*(x) > 0$ region. Next, we show that $V_i(x)$ for $x > 0$ is the unique solution for the following variational inequality (18). The HJB equation (IA.A.6) and the inequality

$V_i(x) \leq F(x)$ given in (IA.A.3) together imply

$$\frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) - r V_i(x) \leq 0\,.$$

Substituting (IA.A.5) into (IA.A.6), we obtain

$$\frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) - r V_i(x) = 0 \quad \text{if} \quad L(x) - K_L < V_i(x).$$

Combining the above with $L(x) - K_L \leq V_i(x)$ given in (IA.A.3), we obtain the variational inequality (18). The boundary conditions are $V_i(0) = 0$ and $\lim_{x\to\infty} V_i(x) - (L(x) - K_L) = 0$. These conditions follow from that $X = 0$ is an absorbing state for a geometric Brownian motion process X, and from the equilibrium result that firms must enter probabilistically when demand is sufficiently high.

Understanding why firms have no incentives to deviate from the equilibrium. A key driving force in our mixed-strategy equilibrium is that the equilibrium firm value, $V_*(x)$, equals Leader's pre-entry net value, $L(x) - K_L$, in the mixed-strategy region where $\lambda^*(x) > 0$. Therefore, the following holds for all $x > 0$:

$$\lambda^*(x)(L(x) - K_L - V_*(x)) = 0\,. \tag{IA.A.9}$$

This result implies that a firm cannot increase its value through any deviation as long as its competitor adopts a mixed strategy (characterized by the equilibrium entry rate $\lambda^*(x)$.) To be precise, the equilibrium value satisfies

$$rV_*(x) = \frac{\sigma^2 x^2}{2} V_*''(x) + \mu x V_*'(x) + \lambda^*(x)(L(x) - K_L - V_*(x)) + \lambda^*(x)(F(x) - V_*(x))\,.$$

Using (IA.A.9) simplifies the preceding equation to:

$$rV_*(x) = \frac{\sigma^2 x^2}{2} V_*''(x) + \mu x V_*'(x) + \lambda^*(x)(F(x) - V_*(x)) \tag{IA.A.10}$$

Next, given firm b's equilibrium strategy $\lambda_b(x) = \lambda^*(x)$, consider firm a's potential deviation strategy $\lambda_a(x)$. The associated firm value, $V_a(x)$, satisfies

$$rV_a(x) = \frac{\sigma^2 x^2}{2} V_a''(x) + \mu x V_a'(x) + \lambda_a(x)(L(x) - K_L - V_a(x)) + \lambda^*(x)(F(x) - V_a(x)).$$

As a firm can always enter as Leader and recall $L(x) - K_L < F(x)$, we thus must have $L(x) - K_L \leq V_a(x) \leq F(x)$. Therefore, $\lambda_a(x)(L(x) - K_L - V_a(x)) \leq 0$. The preceding equation thus implies:

$$rV_a(x) \le \frac{\sigma^2 x^2}{2} V_a''(x) + \mu x V_a'(x) + \lambda^*(x)(F(x) - V_a(x)). \tag{IA.A.11}$$

Comparing (IA.A.10) and (IA.A.11) allows us to conclude $V_a(x) \le V_*(x)$.[2] This explains why firm a has no incentive to deviate from the equilibrium entry rate $\lambda^*(x) > 0$.

The intuition behind these arguments is as follows. If firm a deviates from the equilibrium strategy by lowering its entry rate, the probability that firm a becomes Follower increases. However, the war of attrition will last longer, implying that firm a has to wait longer (in expectation) to collect profits. These two opposing effects turn out to exactly offset each other, leaving firm a no better off by this deviating. Similarly, if firm a deviates from the equilibrium strategy by increasing its entry rate, the war of attrition will end sooner, allowing firm a to collect profits earlier. However, firm a then has a higher probability of becoming Leader, which has to pay a higher entry cost. Again, these two opposing effects exactly offset each other, making firm a no better off.

IA.B Equilibria for Case B: $1 < R < R_{AB}$

IA.B.1 Symmetric Equilibrium for Case B

Next, we summarize the symmetric equilibrium for Case B. In Appendix A.1, we define the equilibrium involving both pure and mixed strategies.

Theorem IA.1 *Let $\widehat{x}_L$ and $\widehat{x}_F$ be the two roots of $S(x) = 0$ in the $(0, x_F)$ region for Case B in Proposition 1.*[3] *Then there exists a symmetric Markov perfect equilibrium with the following properties:*

1. *In the $x \le \widehat{x}_F$ domain, firms only play pure strategies.*
   - *(a) In the $x < \widehat{x}_L$ region, both firms wait and $V_a(x) = V_b(x) = F(x)$.*[4]
   - *(b) In the $x \in [\widehat{x}_L, \widehat{x}_F]$ region, firms compete to become Leader with one firm being randomly selected as Leader and $V_a(x) = V_b(x) = (L(x) - K_L + F(x))/2$.*

[2] Here we use the comparison principle for partial differential equations; see, e.g., Crandall et al. (1992).

[3] If $R = R_{AB}$, the two roots of $S(x) = 0$ are given by $\widehat{x}_L = \widehat{x}_F = \widetilde{x}$, where $\widetilde{x}$ is given in (23).

[4] In the $x < \widehat{x}_L$ region, Leader enters at $\tau_L^* = \inf\{s \ge t : X_s \ge \widehat{x}_L\}$, and $V_i(x)$ equals the present value of the payoff, $L(\widehat{x}_L) - K_L$, at $t = \tau_L^*$. Therefore, we have $V_i(x) = \mathbb{E}_t^x[e^{-r(\tau_L^* - t)}(L(X_{\tau_L^*}) - K_L)] = \left(\frac{x}{\widehat{x}_L}\right)^\beta (L(\widehat{x}_L) - K_L) = \left(\frac{x}{\widehat{x}_L}\right)^\beta F(\widehat{x}_L) = F(x)$, where the third equality uses $L(\widehat{x}_L) - K_L = F(\widehat{x}_L)$.

2. *In the* $x > \widehat{x}_F$ *domain, firms play mixed strategies. Firm value is* $V_a(x) = V_b(x) = V_*(x)$, *where* $V_*(x)$ *is the unique solution to the variational inequality* (18) *in the* $x > \widehat{x}_F$ *domain subject to the boundary conditions:* $V_*(x) = L(x) - K_L$ *as* $x \to \infty$ *and*

$$V_*(\widehat{x}_F) = F(\widehat{x}_F). \tag{IA.B.1}$$

The equilibrium strategy is $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$, *where* $\lambda^*(x) > 0$ *is given by* (19) *in the probabilistic entry region:*

$$\mathcal{R}^E := \{x > \widehat{x}_F : V_*(x) = L(x) - K_L\} \tag{IA.B.2}$$

and $\lambda^*(x) = 0$ *for any* x *in the* $x > \widehat{x}_F$ *domain but not in* $\mathcal{R}^E$, *i.e.,* $x \in (\widehat{x}_F, \infty) \setminus \mathcal{R}^E$.

Intuitively speaking, the cutoff value $\widehat{x}_F$ divides the total market demand x into two domains: (1.) the $x \le \widehat{x}_F$ domain where firms play pure strategies in equilibrium and (2.) the $x > \widehat{x}_F$ domain where firms play mixed strategies as in Case A.

The solution for Case B features at most *five* regions. There are two regions to the left of $\widehat{x}_F$: the $x < \widehat{x}_L$ waiting region and the $x \in [\widehat{x}_L, \widehat{x}_F]$ entry region where firms compete to be Leader and one firm is luckily selected. Theorem IA.1 summarizes the solutions in the $x < \widehat{x}_L$ and $x \in [\widehat{x}_L, \widehat{x}_F]$ regions which apply to both subcases. Next, we summarize the solutions in the $x > \widehat{x}_F$ domain.

Proposition IA.1 *The* $x > \widehat{x}_F$ *domain consists of three regions (*$x \in (\widehat{x}_F, \underline{x}]$, $x \in (\underline{x}, \overline{x})$, *and* $x \ge \overline{x}$*). In the* $x \in (\widehat{x}_F, \underline{x}]$ *region, both firms enter probabilistically at the rate of* $\lambda^*(x) > 0$ *given in* (26), *and firm i's value is given by* $V_i(x) = L(x) - K_L$. *In the* $x \in (\underline{x}, \overline{x})$ *region, both firms wait and firm i's value is* $V_i(x) = \Theta(x; \underline{x}, \overline{x})$, *where* $\Theta(x; a, b)$ *for any* $x \in [a, b]$ *is given by* (29) *and the cutoffs* $\underline{x}$ *and* $\overline{x}$ *are given in Lemma 1. In the* $x \ge \overline{x}$ *region, both firms enter probabilistically at the rate of* $\lambda^*(x) > 0$ *given in* (25) *and firm i's value is* $V_i(x) = D\Pi(x) - K_L$.

For Case B with R that is close to 1,[5] the mixed entry region $(\widehat{x}_F, \underline{x}]$ disappears as the two regions $(\widehat{x}_F, \underline{x}]$ and $(\underline{x}, \overline{x})$ in Proposition IA.1 merge into one waiting region $(\widehat{x}_F, \overline{x})$. In the $x \in (\widehat{x}_F, \overline{x})$ region, both firms wait and firm i's value is $V_i(x) = \Theta(x; \widehat{x}_F, \overline{x})$, where $\Theta(x; a, b)$ for any $x \in [a, b]$ is given by (29), and $\overline{x}$ is given in case (ii) in Lemma 4.

Graphical Illustration. First, by intersecting $L(x) - K_L$ with $F(x)$, we obtain the two regions on the left: (1.) The $x < \widehat{x}_L$ region where firms wait to preserve the option value and (2.) the $x \in [\widehat{x}_L, \widehat{x}_F]$ region where firms compete to enter as Leader as discussed earlier.

[5]More precisely, the degenerate subcase in Case B refers to $1 < R \le R_{B_1B_2}$, where $R_{B_1B_2}$ is defined in Internet Appendix IA.J.

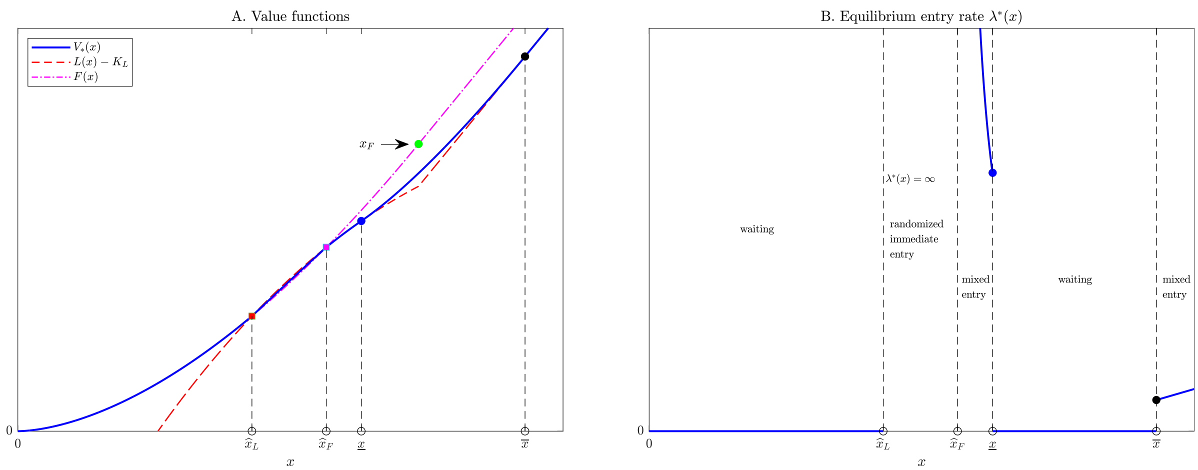

Figure IA-1: VALUE FUNCTIONS AND ENTRY RATES IN THE SYMMETRIC EQUILIBRIA OF CASE B. Parameter values are $D = 0.55$, $K_L = 1.21$, $K_F = 1$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$.

Graphically, we determine the remaining parts of our model solution by smoothly pasting a curve starting from the magenta square at $(\widehat{x}_F, F(\widehat{x}_F))$ onto Leader's net payoff line $L(x) - K_L$. This convex curve is the equilibrium firm value $V_*(x)$ for $x > \widehat{x}_F$. Moreover, the smooth-pasting conditions at $x = \underline{x}$ and $x = \overline{x}$ divides the $x > \widehat{x}_F$ domain into three regions: the $x \in (\widehat{x}_F, \underline{x}]$ and $x \geq \overline{x}$ regions where firms play mixed entry strategies and the $(\underline{x}, \overline{x})$ region where firms wait to lower entry costs. Also, note that $V_*(x)$ is concave in the $x \in (\widehat{x}_F, \underline{x}]$ region, while convex in the $x > \underline{x}$ region.

Panel B of Figure IA-1 plots the equilibrium entry rates. We emphasize that in the last region where $x \geq \overline{x}$, firms probabilistically enter at the rate of $(Dx - rK_L)/(K_L - K_F)$ as Follower immediately enters after Leader does and therefore Leader enjoys no monopoly rents. This is the same as in the $x \geq \overline{x}$ region of Case A. Unlike the probabilistic entry region $x \geq \overline{x}$ where Leader enjoys no monopoly rents, Leader enjoys a (stochastic) period of monopoly rents in the $x \in (\widehat{x}_F, \underline{x}]$ region. The equilibrium entry rate $\lambda^*(x)$ in the $(\widehat{x}_F, \underline{x}]$ region equals $(x - rK_L)/S(x)$.

Summary of Case B: $1 < R \leq R_{AB}$ Case B is the most general case where both the first-mover and second-mover advantages are present. There are five regions: two disconnected waiting regions (one to preserve the option value and the other to lower entry costs), the pure entry strategy region where rents are equalized, the probabilistic entry region where Leader enjoys monopoly rents in equilibrium, and the probabilistic entry region where Leader enjoys no monopoly rents in equilibrium. Finally, we emphasize that the *interaction* between the two types of advantages in our real-option context fundamentally alters how these five regions are determined and connected. For example, firms may enter in one of three different ways: pure strategy, probabilistic entry with or without monopoly rents (of stochastic duration).

Moreover, firm entry is not monotonic as market demand increases. In sum, game-theoretic considerations, when both first- and second-mover advantages are present, fundamentally enrich the equilibrium real-option exercising decisions and firm valuation.

IA.B.2 Asymmetric Pure-strategy Equilibria for Case B

In this appendix, we characterize the asymmetric pure-strategy equilibria for Case B.

Let $\widehat{x}_L$ and $\widehat{x}_F$ denote the two roots of $L(x) - K_L = F(x)$ in the $(0, x_F)$ region for Case B in Proposition 1. In the $x \le \widehat{x}_F$ domain, the pure-strategy equilibrium and value function are the same as those in Part 1 of Theorem IA.1. That is, firms wait if $x < \widehat{x}_L$ and rush to enter as Leader if $x \in [\widehat{x}_L, \widehat{x}_F]$.

In the $x > \widehat{x}_F$ domain, without loss of generality, we analyze the pure-strategy equilibrium where firm a is Leader and firm b is Follower. Firm a solves the following problem:

$$J_L(x) = \max_{\tau \in [t, \widehat{\tau}]} \mathbb{E}_t^x \left[e^{-r(\tau - t)} (L(X_\tau) - K_L) \right] , \tag{IA.B.3}$$

where $\widehat{\tau} = \inf\{s \ge t : X_s \le \widehat{x}_F\}$. Leader's optimal entry time τ_L^* solves the problem defined in (IA.B.3). Now we turn to Follower's problem. Taking Leader's optimal entry time τ_L^* as given, Follower chooses $\tau_F \ge \tau_L^*$ to maximize its value $J_F(X)$ by solving (32).

Next, we summarize the asymmetric pure-strategy equilibria in the $x > \widehat{x}_F$ domain for Case B.

Proposition IA.2 *The asymmetric pure-strategy equilibria in the $x > \widehat{x}_F$ domain for Case B are as follows: Leader's entry time is $\tau_L^* = \inf\{s \ge t : X_s \ge \overline{x}$ or $X_s \le \underline{x}\}$. Leader's value is $J_L(x) = V_i(x)$, where $\underline{x}$, $\overline{x}$, and $V_i(x)$ are given in Proposition IA.1. Follower enters immediately after Leader's entry ($\tau_F^* = \tau_L^*+$) if $X_{\tau_L^*} \ge \overline{x}$ and enters at $\tau_F^* = \inf\{s \ge \tau_L^* : X_s \ge x_F\} > \tau_L^*$ if $X_{\tau_L^*} \le \underline{x}$. Follower's value $J_F(x)$ is given by* (33)*,* (34)*, and* (35) *for $x \in (\widehat{x}_F, \underline{x}]$, $x \in (\underline{x}, \overline{x})$, and $x \ge \overline{x}$, respectively.*

For Case B with R that is close to 1 (see footnote 5), the region $(\widehat{x}_F, \underline{x}]$ disappears as the two regions $(\widehat{x}_F, \underline{x}]$ and $(\underline{x}, \overline{x})$ in Proposition IA.2 merge into one region $(\widehat{x}_F, \overline{x})$. In the $x \in (\widehat{x}_F, \overline{x})$ region, Leader waits and its value is $V_i(x) = \Theta(x; \widehat{x}_F, \overline{x})$, while Follower's values is given by $J_F(x) = \frac{F(\widehat{x}_F)\overline{x}^\gamma - F(\overline{x})\widehat{x}_F^\gamma}{\widehat{x}_F^\beta \overline{x}^\gamma - \widehat{x}_F^\gamma \overline{x}^\beta} x^\beta + \frac{F(\overline{x})\widehat{x}_F^\beta - F(\widehat{x}_F)\overline{x}^\beta}{\widehat{x}_F^\beta \overline{x}^\gamma - \widehat{x}_F^\gamma \overline{x}^\beta} x^\gamma$, where $\Theta(x; a, b)$ for any $x \in [a, b]$ is given by (29), and $\overline{x}$ is given in case (ii) in Lemma 4.

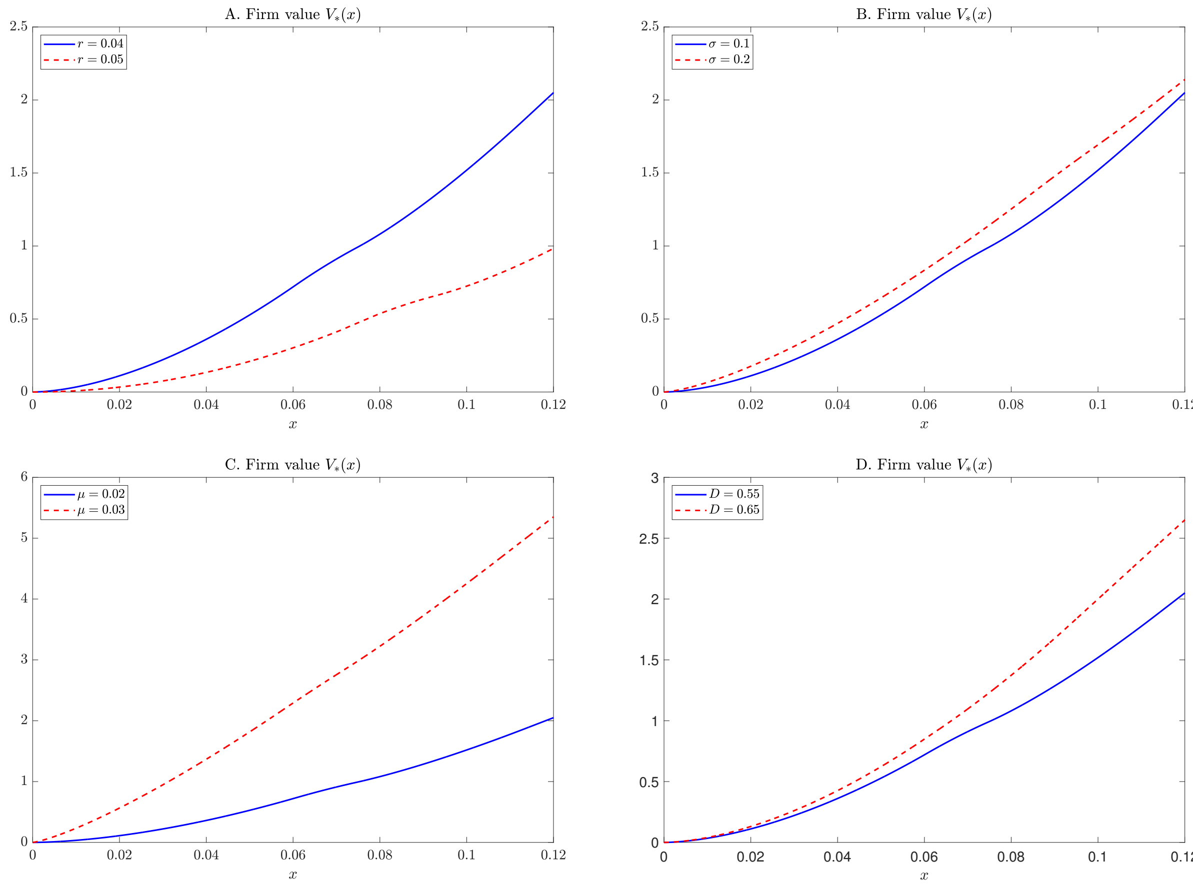

Figure IA-2: Comparative Statics. Parameter values are $D = 0.55$, $K_L = 1.25$, $K_F = 1$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$.

IA.C Comparative Statics

In this Appendix, we explore comparative statics for the equilibrium value (in Case A) in interest rate r, volatility σ, drift μ and D. For the parameter values we have explored, the equilibrium firm value $V_*(x)$ increases with μ, σ and D and decreases with r, as shown in Figure IA-2 of this response document. These are consistent with our intuition.

IA.D Case C

In Case C ($R \leq 1$), the equilibrium is determined by firms' tradeoff between the first-mover advantage and the option value of waiting.

Let $\mathcal{E}_i \subset (0, \infty)$ denote a closed set associated with firm i's entry strategy: firm i enters at t if and only if $X_t \in \mathcal{E}_i$. Let Ψ denote the set of all feasible entry strategies $(\mathcal{E}_a, \mathcal{E}_b)$ and let $J_i(X_t; \mathcal{E}_a, \mathcal{E}_b)$ denote the associated time-t value of firm i defined by (14). Next, we define the pure-strategy equilibrium.

Definition IA.1 A pair of entry strategy $(\mathcal{E}_a^*, \mathcal{E}_b^*)$ is a *pure-strategy equilibrium* if for any $x > 0$ the following conditions hold:

$$J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) \geq J_a(x; \mathcal{E}_a, \mathcal{E}_b^*), \quad \forall\, (\mathcal{E}_a, \mathcal{E}_b^*) \in \Psi, \tag{IA.D.1}$$

$$J_b(x; \mathcal{E}_a^*, \mathcal{E}_b^*) \geq J_b(x; \mathcal{E}_a^*, \mathcal{E}_b), \quad \forall\, (\mathcal{E}_a^*, \mathcal{E}_b) \in \Psi. \tag{IA.D.2}$$

Let $V_i(x)$ denote firm i's equilibrium value function: $V_i(x) = J_i(x; \mathcal{E}_a^*, \mathcal{E}_b^*)$.

Proposition 1 shows that for Case C where $R \leq 1$, $F(x)$ intersects with $L(x) - K_L$ at $\widehat{x}_L$ and $L(x) - K_L \geq F(x)$ for any $x \geq \widehat{x}_L$. Therefore, in the $x \geq \widehat{x}_L$ region, both firms want to enter as Leader but only one firm can be randomly selected (with 50% probability) to be Leader. This is the *rent equalization principle* of Dixit and Pindyck (1994), and Grenadier (1996), which implies that the equilibrium firm value for both firms is:

$$V_i(x) = \frac{L(x) - K_L + F(x)}{2} \tag{IA.D.3}$$

for $x \geq \widehat{x}_L$. In the $x < \widehat{x}_L$ region, firms optimally wait and the equilibrium firm value is: $V_i(x) = F(x)$ as in Dixit and Pindyck (1994).[6] Next, we summarize the solution in the following theorem.

Theorem IA.2 *Consider Case C where $R \leq 1$. Let $\widehat{x}_L$ be the unique root of $L(x) - K_L = F(x)$ in the $(0, x_F)$ region for Case C in Proposition 1. Then there exists a unique pure strategy equilibrium such that firm i's equilibrium value $V_i(x)$ equals $F(x)$, where $F(x)$ is given in* (8) *in the $x < \widehat{x}_L$ region, and $V_i(x)$ is given by* (IA.D.3) *in the $x \geq \widehat{x}_L$ region. Both firms wait in the $x < \widehat{x}_L$ region. In the $x \in [\widehat{x}_L, x_F)$ region, firms compete to enter and one firm is randomly selected to enter immediately as Leader and the other optimally waits until $\tau_F^* = \inf\{s : X_s \geq x_F\}$ to enter as Follower. In the $x \geq x_F$ region, the two firms in effect simultaneously enter with one chosen to be Leader randomly.*

Theorem IA.2 concludes the uniqueness of equilibrium. It is worth noting that simultaneous investment strategies in Pawlina and Kort (2006) and Bustamante (2015) cannot be an equilibrium in our model. This is because our model assumes that firms start with no assets in place and hence have zero cash flows before entry. In contrast, as assets in place generate profits in Pawlina and Kort (2006) and Bustamante (2015) before entering, firms bear additional costs of exercising their growth options. These additional costs of exercising

[6] In the $x < \widehat{x}_L$ region, firms choose to wait solely to preserve their option value.

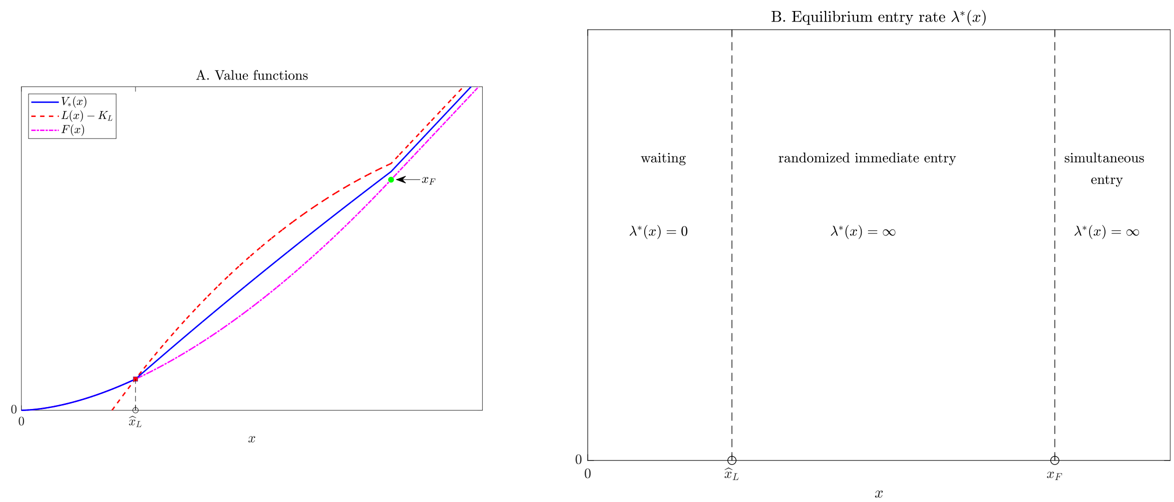

Figure IA-3: Value functions and entry rate in the symmetric equilibrium in Case C. Parameter values are $D = 0.55$, $K_L = 0.9$, $K_F = 1$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$.

entry options allow both firms to invest simultaneously in equilibrium only when demand is sufficiently high.[7]

Graphical Illustration. Next, we use Figure IA-3 to highlight the key results of Case C. We set the entry-cost ratio at $R < 1$.

First, we note that $L(x) - K_L > F(x)$ holds in the $x > \widehat{x}_L$ region, which implies that the first-mover advantage dominates and both firms want to enter first. To select Leader, we need a randomization device while keeping the *ex ante* rents for the two firms equal. The solid (blue) line depicts the value function $V_i(x)$ given in (IA.D.3) in the entry region. To the left of the red square is the $x < \widehat{x}_L$ region, where both firms wait. Note that $\widehat{x}_L < \widetilde{x}$, where $\widetilde{x}$ is given by (23). That is, the option value of waiting is eroded as emphasized in Dixit and Pindyck (1994). Mathematically, we prove $\widehat{x}_L < \widetilde{x}$ for $R \in (0, 1]$ in Lemma 2.

Second, in the $x \geq x_F$ region (recall that $x_F > \widehat{x}_L$ in our example), Follower immediately enters after Leader is randomly chosen. This is the "simultaneous entry" region in Panel B of Figure IA-3. Third, in the $x \in [\widehat{x}_L, x_F)$ region, the (lucky) Leader collects monopoly rents until Follower enters when X_t reaches x_F for the first time. This is the "randomized immediate entry" region in Panel B of Figure IA-3.

In sum, $V_*(x)$ is convex in the $x < \widehat{x}_L$ waiting region, concave in the $x \in [\widehat{x}_L, x_F)$ sequential-entry region, and linear in the $x \geq x_F$ simultaneous-entry region. Note that all entry decisions are pure strategies. Mathematically, there is no smooth-pasting condition involved for Case

[7] Absent second-mover advantage, our model is much simpler than Grenadier (1996), Weeds (2002), Pawlina and Kort (2006), and Bustamante (2015). Thus, our Case C only features a unique leader-follower equilibrium. In Weeds (2002), it is the low arrival rate of a successful innovation that makes both firms willing to wait for demand to rise, which in turn supports simultaneous investment equilibria.

C, as firms compete to be the first mover as soon as Leader's net payoff $L(x) - K_L$ exceeds Follower's value $F(x)$.

Notation-wise, for pure entry strategies, although we do not explicitly refer to equilibrium entry rates, we write $\lambda^*(x) = \infty$ in the entry region and $\lambda^*(x) = 0$ in the waiting region.

IA.E Comparing Grenadier (1996) with Our Model

In this appendix, we highlight how the key mechanism in our model fundamentally differs from that in Grenadier (1996). To do so, we first summarize the solution of a version of Grenadier (1996) that preserves his model's key mechanism.[8]

Grenadier (1996) is in effect an extension of the model of Chapter 9 in Dixit and Pindyck (1994) with the following two features. First, each firm receives a constant (rental) profit Q per period before a firm exercises its real-estate redevelopment option as Leader. Second, Leader's redevelopment lowers its competitor's (rental) profit from Q to zero.[9]

In sum, the profit structure for firms in Grenadier (1996) is as follows: 1.) for $t \leq \tau_L$, both firms receive Q per period; 2.) for $t \in (\tau_L, \tau_F)$, Leader receives the entire market demand $\{X_t\}$ from its new development and Follower receives zero profits; 3.) for $t \geq \tau_F$, both firms receive $\{DX_t\}$.

Given the entry game's structure, firm i solves the following problem in Grenadier (1996):

$$\mathbb{E}_t^x \left[e^{-r(\tau_L - t)} \left(\mathbf{1}_{\tau_i < \tau_{-i}} (L(X_{\tau_i}) - K_L) + \mathbf{1}_{\tau_i > \tau_{-i}} F(X_{\tau_{-i}}) + \mathbf{1}_{\tau_i = \tau_{-i}} \frac{L(X_{\tau_i}) - K_L + F(X_{\tau_i})}{2} \right) \right.$$
$$\left. + \int_t^{\tau_L} e^{-r(s-t)} Q ds \right], \qquad \text{(IA.E.1)}$$

where $L(x)$ is Leader's value given by (12)-(13) and $F(x)$ is Follower's value given by (8)-(9). Compared with the model in Chapter 9 of Dixit and Pindyck (1994), the new term in Grenadier (1996) is the integral on the second line in (IA.E.1), which captures the value of constant rental profits Q that firm i receives before Leader enters at τ_L. Let $V_i(x)$ denote firm i's equilibrium value function for (IA.E.1).

Next, we summarize the solution of (IA.E.1), which features two cases:

[8] To facilitate comparisons between his model and ours, we only provide technical details that are essential for readers to understand the key differences between the two models.

[9] In Grenadier (1996), redevelopment options have other features. First, it takes time for a firm to complete its redevelopment (time to build). Second, Leader's redevelopment lowers Follower's profits from Q to ρQ until Follower exercises its redevelopment option. While realistic, these two features are not necessary to highlight the key differences between our model and Grenadier (1996). To ease comparisons between his model and ours, we set both ρ and time to build to zero.

(i) If the profit flow Q is large, there are four equilibrium regions.

(ii) If Q is small, there are three equilibrium regions as in our Case C.

Because case (ii) in Grenadier (1996) effectively generates the same results as the model in Chapter 9 of Dixit and Pindyck (1994) (which is Case C of our model), we do not discuss the details but simply plot the solution in Panel B of Figure IA-4 for reference.

For our remaining analysis, we focus on case (i) where Q is large. This is the economically more interesting scenario analyzed in Grenadier (1996). There are four equilibrium regions in total: two waiting regions alternate with the two pure-strategy entry regions where firms rush to enter as Leader.

First, when market demand is low: $x \in (0, \widehat{x}_L)$, firms wait to preserve their option values. Second, when market demand is high ($x \geq x_J$), both firms rush to enter, as the value of taking half of the market share is so large that entering simultaneously with the other firm is sufficiently profitable to justify entry. Third, when market demand is medium (i.e., $\widehat{x}_L \leq x \leq x_F$), which is to the immediate right of the first region where $x < \widehat{x}_L$, both firms rush to enter as Leader. As one firm is selected randomly as Leader, the other firm optimally chooses to wait to preserve its option value. This is the case where the total demand is high enough for one firm to be sufficiently profitable but not enough for Follower to enter (once taking its option value of waiting into account).

Finally, when market demand is medium high, i.e., between the two pure-strategy entry regions: $x_F < x < x_J$, both firms decide to wait. This is because had one firm entered, the other would also enter (because the market demand is higher than x_F, the threshold above which Follower optimally enters). Anticipating Follower's optimal response along this path, both firms decide to wait as equally splitting the market demand in the $x_F < x < x_J$ region is not profitable for firms. Intuitively, Follower's immediate entry credibly deters firms from entering first as Leader in this region.

We summarize the model's solution for case (i) in Panel A of Figure IA-4. In the four regions, firms alternate between waiting and entering as Leader with probability one, implying a zero-one binary form of non-monotonic entry strategy in Grenadier (1996). While both our model and Grenadier (1996) predict that firm entry is non-monotonic in market demand, the mechanisms driving the non-monotonicity result in our model and in Grenadier (1996) are quite different. The empirical analysis of Fabrizio and Tsolmon (2014) is consistent with our mechanism but not with the mechanism in Grenadier (1996). In the model of Grenadier (1996), Leader and Follower have the same entry cost ($K_L = K_F$), so the incentive to wait and become Follower is absent. The non-monotonic entry propensity arises in his model from the firms' incentives to continue collecting profits Q from existing assets. By contrast, the non-monotonicity result in our model is driven by a (relatively) small second-mover advantage,

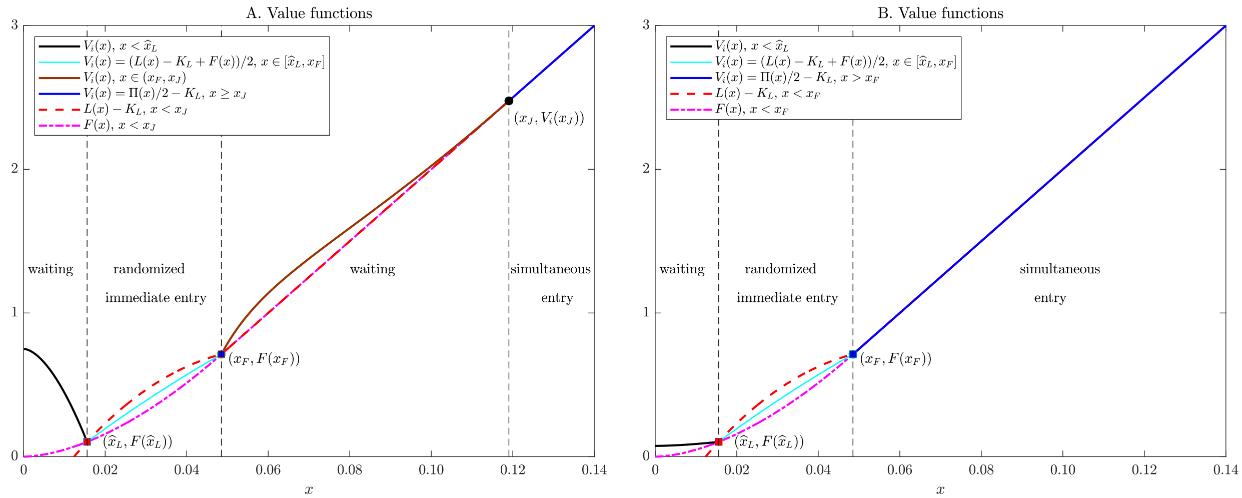

Figure IA-4: Value functions and firm entry strategies in Grenadier (1996). Parameter values are $Q = 0.03$ in Panel A and $Q = 0.003$ in Panel B, and $D = 0.5$, $K_L = K_F = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$ in both panels. The equilibrium cutoff values are: $x_J = 0.1191$ in Panel A and ($\widehat{x}_L = 0.0156$, $x_F = 0.0485$) in both panels.

which corresponds to (relatively) strong patent protection. The intuition is that a firm may still want to innovate when market demand is low, as it expects to collect monopoly rents for a long period under strong patent protection. In terms of time series, even as market demand further decreases, it has a positive effect on Leader, as Follower has even weaker incentives to enter, which in turn further increases the duration of Leader's monopoly rents. The interplay of the two opposing forces (current profits and duration of monopoly rents) renders a firm's entry (innovation) propensity non-monotonic in market demand.

In sum, while both our model and Grenadier (1996) predict that firm entry is non-monotonic in market demand, the predicted entry dynamics and the underlying mechanisms in the two models are quite different, as we have discussed above. Importantly, our model prediction is consistent with Fabrizio and Tsolmon (2014)'s finding that corporate innovations increase with market demand in industries with weaker patent protection, but are less sensitive to market demand in industries with stronger patent protection.

IA.F Different Market Shares for Leader and Follower

In this appendix, we generalize the model in Section 2 by allowing Follower and Leader to have different market shares in the duopoly stage. Specifically, Leader and Follower receive profits $\{D_L X_t; t \geq \tau_F\}$ and $\{D_F X_t; t \geq \tau_F\}$, respectively, where $D_L > 0$ and $D_F > 0$ are constants. The total market demand under duopoly $(D_L + D_F)X_t$ can differ from X_t under monopoly. Moreover, Leader's market share $D_L/(D_L + D_F)$ can differ from Follower's market share $D_F/(D_L + D_F)$.

Follower's value in the (τ_L, τ_F) period is given by:

$$F(x) = \max_{\tau_F \geq t} \mathbb{E}_t^x \left[\int_{\tau_F}^{\infty} e^{-r(s-t)} D_F X_s ds - e^{-r(\tau_F - t)} K_F \right]. \tag{IA.F.1}$$

Let τ_F^* denote Follower's optimal entry time for (IA.F.1). Taking τ_F^* as given, we write Leader's post-entry value function, $L(x)$, for $t \in (\tau_L, \tau_F^*)$ as:

$$L(x) = \mathbb{E}_t^x \left[\int_t^{\tau_F^*} e^{-r(s-t)} X_s ds + \int_{\tau_F^*}^{\infty} e^{-r(s-t)} D_L X_s ds \right]. \tag{IA.F.2}$$

Effectively using the same procedure as in Section 2.2, we obtain Follower's value:

$$F(x) = (D_F \Pi(x_F) - K_F) \left(\frac{x}{x_F} \right)^{\beta}, \quad x < x_F, \tag{IA.F.3}$$

$$F(x) = D_F \Pi(x) - K_F, \quad x \geq x_F, \tag{IA.F.4}$$

where Follower's optimal entry threshold, x_F, is given by

$$x_F = \frac{1}{D_F} \frac{\beta}{\beta - 1} (r - \mu) K_F. \tag{IA.F.5}$$

Solving $L(x)$ defined in (IA.F.2), we obtain

$$L(x) = \Pi(x) - (1 - D_L)\, \Pi(x_F) \left(\frac{x}{x_F} \right)^{\beta}, \quad x < x_F, \tag{IA.F.6}$$

$$L(x) = D_L \Pi(x), \quad x \geq x_F. \tag{IA.F.7}$$

Next, we assume the same entry costs for Leader and Follower, while the second-mover advantage is induced from $D_L < D_F$. As in Definition 1, we can also determine the second-mover advantage via $S(x) > 0$, where $S(x) = F(x) - (L(x) - K_L)$. Here, $F(x)$ and $L(x)$ are given by (IA.F.3)-(IA.F.4) and (IA.F.6)-(IA.F.7), respectively.

Assumption 1 *Market shares satisfy* $1 > D_F > D_L > 0$ *and* $K_L = K_F > 0$.

Under Assumption 1, we can show that $D_F > 1 - \frac{D_F^{1-\beta} - D_F}{\beta}$ and $S(x) > 0$ for all x if and only if $D_L < 1 - \frac{D_F^{1-\beta} - D_F}{\beta}$. Next, we summarize the symmetric equilibrium strategy in the following theorem.

Theorem IA.3 *Suppose Assumption 1 holds.*

1. *When* $D_L < 1 - \frac{D_F^{1-\beta} - D_F}{\beta} < D_F$, *we have* $S(x) > 0$ *for* $x > 0$. *In the symmetric mixed-strategy Markov perfect equilibrium, equilibrium value is* $V_a(x) = V_b(x) = V_*(x)$, *where* $V_*(x)$ *is the unique solution for the following variational-inequality problem in the* $x \geq 0$ *domain:*[10]

$$\max\left\{\frac{\sigma^2 x^2}{2} V_*''(x) + \mu x V_*'(x) - rV_*(x),\ \ (L(x) - K_L) - V_*(x)\right\} = 0, \qquad \text{(IA.F.8)}$$

and $L(x)$ *is given by* (IA.F.6)-(IA.F.7). *The equilibrium entry rates are* $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$, *where* $\lambda^*(x) = 0$ *in the* $V_*(x) > L(x) - K_L$ *region and* $\lambda^*(x)$ *is given below in the* $V_*(x) = L(x) - K_L$ *region:*

$$\lambda^*(x) = \frac{1}{S(x)}\left[\left(\mathbf{1}_{x<x_F} + D_L \mathbf{1}_{x \geq x_F}\right)x - rK_L\right]. \qquad \text{(IA.F.9)}$$

2. *When* $D_L > 1 - \frac{D_F^{1-\beta} - D_F}{\beta}$, $S(x) = 0$ *has two roots,* $\widehat{x}_L$ *and* $\widehat{x}_F$ *satisfying* $S(x) < 0$ *if* $\widehat{x}_L < x < \widehat{x}_F$, *and* $S(x) > 0$ *if* $x < \widehat{x}_L$ *or* $x > \widehat{x}_F$.[11]
   - (a) *In the* $x < \widehat{x}_L$ *region, both firms wait and* $V_a(x) = V_b(x) = F(x)$.
   - (b) *In the* $x \in [\widehat{x}_L, \widehat{x}_F]$ *region, firms compete to become Leader and* $V_a(x) = V_b(x) = (L(x) - K_L + F(x))/2$.
   - (c) *In the* $x > \widehat{x}_F$ *region, firm value is* $V_a(x) = V_b(x) = V_*(x)$, *where* $V_*(x)$ *is the unique solution to the variational inequality* (IA.F.8) *in the* $x > \widehat{x}_F$ *domain.*[12] *The equilibrium strategy is* $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$, *where* $\lambda^*(x) > 0$ *is given by* (IA.F.9) *in the probabilistic entry region* $\mathcal{R}^E := \{x > \widehat{x}_F : V_*(x) = L(x) - K_L\}$, *and* $\lambda^*(x) = 0$ *for any* $x \in (\widehat{x}_F, \infty) \setminus \mathcal{R}^E$.

In Figure IA-5, we plot the value functions in Panel A and the equilibrium entry rate $\lambda^*(x)$ in Panel B, using $D_L = 0.5$, $D_F = 0.8$, and $K_L = K_F = 1$. The equilibrium solution for this simple case features two regions. To the left of $\overline{x} = \frac{1}{D_L}\frac{\beta}{\beta-1}(r-\mu)K_L = 0.097$ is the waiting region where $\lambda^*(x) = 0$ and $V_a(x) = V_b(x) = (x/\overline{x})^\beta (L(\overline{x}) - K_L)$. To the right, where $x \geq \overline{x}$, firms enter probabilistically, and a firm's pre-entry value is thus equal to $L(x) - K_L$, the net payoff upon immediate entry: $V_a(x) = V_b(x) = L(x) - K_L$. Using (IA.F.9) and $\overline{x} > x_F$, we can pin down the equilibrium entry rate $\lambda^*(x) = \frac{D_L x - rK_L}{S(x)}$ for $x \geq \overline{x}$.

[10] The boundary conditions are $V_*(0) = 0$ and $\lim_{x\to\infty} V_*(x) - (L(x) - K_L) = 0$.

[11] If $D_L = 1 + \frac{D_F - D_F^{1-\beta}}{\beta}$, the two roots for the $S(x) = 0$ equation reduce to one root: $\widehat{x}_L = \widehat{x}_F$. For all $x \neq \widehat{x}_L$, $S(x) > 0$.

[12] The boundary conditions is $V_*(x) = L(x) - K_L$ as $x \to \infty$ and $V_*(\widehat{x}_F) = F(\widehat{x}_F)$.

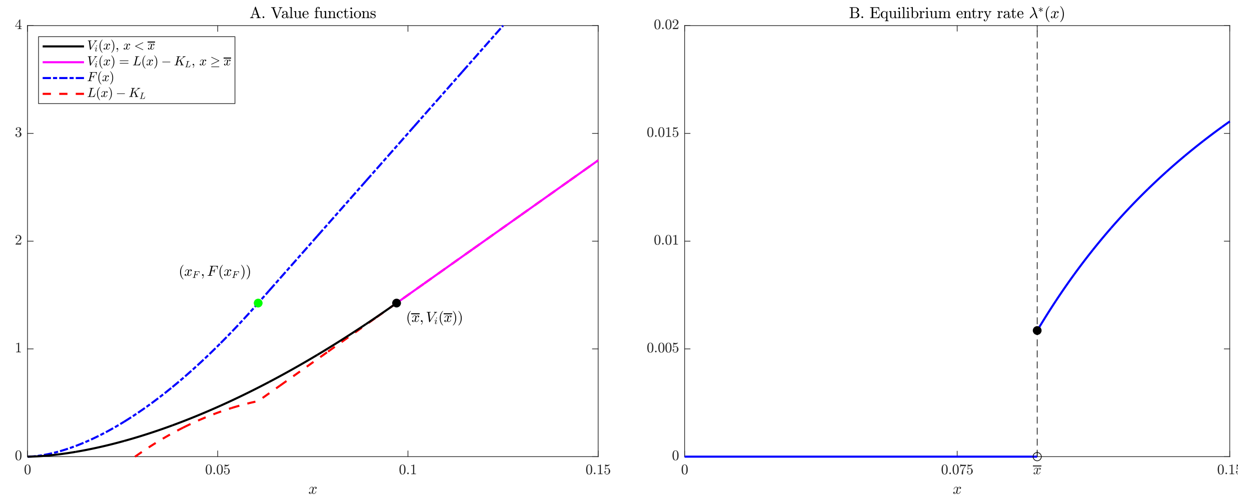

Figure IA-5: VALUE FUNCTIONS $V_a(x) = V_b(x) = V_*(x)$ AND EQUILIBRIUM PROBABILISTIC ENTRY RATES $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$ WHEN $D_L/D_F < 1$ AND $K_L = K_F$. The threshold values dividing the market demand x into the three regions are: $\overline{x} = \frac{1}{D_L}\frac{\beta}{\beta-1}(r-\mu)K_L$. The two regions are: 1.) the $x < \overline{x}$ waiting region, and 2.) the $x \geq \overline{x}$ probabilistic entry (mixed-strategy) region. Parameter values are $D_L = 0.5$, $D_F = 0.8$, $K_L = K_F = 1$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$. The equilibrium thresholds are: $x_F = 0.060$, and $\overline{x} = 0.097$.

In sum, we have purposefully chosen parameter values to obtain a two-region solution for a symmetric equilibrium: (1) a waiting region where $x < \overline{x}$, as in the standard single-firm real-option problem, and (2) a probabilistic-entry region where $x \geq \overline{x}$.

In an even richer setting (by changing parameter values), we expect an even richer solution. For example, we have four-region case as in Figure 1.

For our remaining analysis in this appendix, we make another assumption:

Assumption 2 *Market shares satisfy $D_L/D_F > 1$ with $D_L \in (0,1)$ and $R = K_L/K_F$ is sufficiently large that $S(x) > 0$ for $x \in (0, x_F]$, where x_F is given in (IA.F.5).*

Because Leader has a larger market share than Follower: $D_L/D_F > 1$, we have $S(X_t) < 0$ for sufficiently large market demand X_t. Under Assumption 2, we deliver both first-mover and second-mover advantages on the equilibrium path in our model without getting overly involved in non-essential detailed technical analyses.

Mathematically, there exists a unique root, $\widehat{x}$, for the equation $S(x) = 0$ in the $x > 0$ region, which is a firm's indifference condition between entering as Leader with probability one and being Follower. Solving $S(x) = 0$, we obtain:

$$\widehat{x} = \frac{K_L - K_F}{D_L - D_F}(r - \mu). \tag{IA.F.10}$$

We can verify that the following inequalities hold:

$$S(x) < 0\,, \quad x > \widehat{x}, \qquad \text{(IA.F.11)}$$

$$S(x) > 0\,, \quad 0 < x < \widehat{x}. \qquad \text{(IA.F.12)}$$

That is, a firm is strictly better off by entering first as Leader if and only if $x > \widehat{x}$. Additionally, we can show that $\widehat{x} > x_F$, where x_F is given by (IA.F.5), under Assumption 2.

We summarize the symmetric equilibrium strategy in the following theorem.

Theorem IA.4 *Under Assumption 2, there exists a symmetric Markov perfect equilibrium with the following properties:*

1. *In the $x \geq \widehat{x}$ region where $\widehat{x}$ is given in* (IA.F.10)*, two firms simultaneously enter. One firm is randomly chosen to be Leader and the other firm immediately enters. Pre-entry firm value functions are thus equalized: $V_a(x) = V_b(x) = (L(x) - K_L + F(x))/2$.*

2. *In the $x < \widehat{x}$ region, firms wait or play mixed entry strategies. Pre-entry firm value functions are equal: $V_a(x) = V_b(x) = V_*(x)$, where $V_*(x)$ is the unique solution to the variational inequality* (IA.F.8) *in the $x < \widehat{x}$ domain.*[13] *The equilibrium entry rates are $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$, where $\lambda^*(x) > 0$ is given by* (IA.F.9) *in the probabilistic entry region: $\mathcal{R}^E := \{x < \widehat{x} : V_*(x) = L(x) - K_L\}$, and $\lambda^*(x) = 0$ for all $x \in (0, \widehat{x}) \setminus \mathcal{R}^E$.*

In Figure IA-6, we plot the value functions in Panel A and the equilibrium entry rate $\lambda^*(x)$ in Panel B, using $D_L = 0.58$, $D_F = 0.45$, $K_L = 1.4$, and $K_F = 0.5$. The equilibrium solution for this simple case features three regions. To the left of $\overline{x} = \frac{1}{D_L}\frac{\beta}{\beta-1}(r-\mu)K_L = 0.117$ is the waiting region where $\lambda^*(x) = 0$ and $V_a(x) = V_b(x) = (x/\overline{x})^\beta\,(L(\overline{x}) - K_L)$. In the middle region where $x \in [\overline{x}, \widehat{x})$, firms enter probabilistically and a firm's pre-entry value is thus equal to $L(x) - K_L$, the net payoff upon immediate entry: $V_a(x) = V_b(x) = L(x) - K_L$. Using (IA.F.9) and $\overline{x} > x_F$, we can pin down the equilibrium entry rate $\lambda^*(x) = \frac{D_L x - rK_L}{S(x)}$ in this middle region as $\overline{x} > x_F$. Finally, to the right of $\widehat{x}$ is the simultaneous entry region where firms rush to enter. In this region, Follower immediately enters after Leader is randomly chosen, as the market demand is sufficiently high.

In sum, we have purposefully chosen parameter values to obtain a three-region solution for a symmetric equilibrium: (1) a waiting region where $x < \overline{x}$ as in the standard single-firm real-option problem, (2) a probabilistic-entry region where $x \in [\overline{x}, \widehat{x})$, and (3) a pure-strategy entry region where $x \geq \widehat{x}$.

[13]The boundary conditions is $V_*(0) = 0$ and $V_*(\widehat{x}) = F(\widehat{x})$.

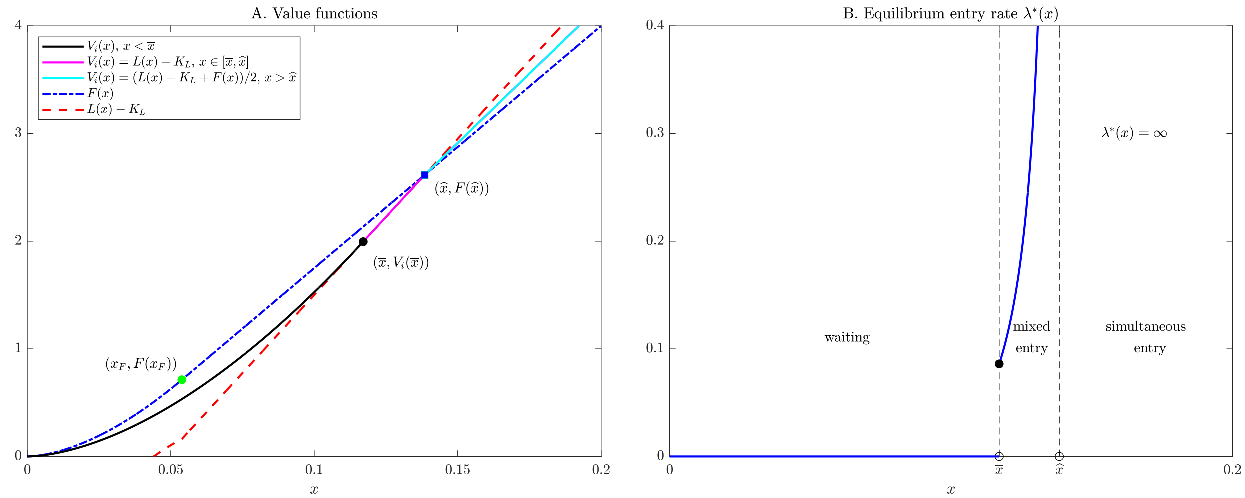

Figure IA-6: VALUE FUNCTIONS $V_a(x) = V_b(x) = V_*(x)$ AND EQUILIBRIUM PROBABILISTIC ENTRY RATES $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$ WHEN $D_L/D_F > 1$. The threshold values dividing the market demand x into the three regions are: $\overline{x} = \frac{1}{D_L}\frac{\beta}{\beta-1}(r-\mu)K_L$ and $\widehat{x} = \frac{K_L-K_F}{D_L-D_F}(r-\mu)$. The three regions are: 1.) the $x < \overline{x}$ waiting region, 2.) the $x \in [\overline{x}, \widehat{x})$ probabilistic entry (mixed-strategy) region, and 3.) the $x \geq \widehat{x}$ simultaneous entry region. Parameter values are $D_L = 0.58$, $D_F = 0.45$, $K_L = 1.4$, $K_F = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$. The equilibrium thresholds are: $x_F = 0.054$, $\overline{x} = 0.117$, and $\widehat{x} = 0.139$.

IA.G Details for Policy Implications in Appendix B

In the cooperative benchmark (B.1), the total surplus is given in the following proposition.

Proposition IA.3 *Let $(\tau_L^{\mathrm{CO}}, \tau_F^{\mathrm{CO}})$ denote the optimal strategies for maximizing (B.1), and let $W^{\mathrm{CO}}(x)$ denote the corresponding optimal total surplus value.*

1. *When $D \leq \frac{1}{2}$, we have*

$$W^{\mathrm{CO}}(x) = [\Pi(x_M) - K_L]\left(\frac{x}{x_M}\right)^{\beta}, \quad x < x_M, \tag{IA.G.1}$$

$$W^{\mathrm{CO}}(x) = \Pi(x) - K_L, \quad x \geq x_M, \tag{IA.G.2}$$

where $x_M = \frac{\beta}{\beta-1}(r-\mu)K_L$. Leader's entry time is $\tau_L^{\mathrm{CO}} = \inf\{s \geq t : X_s \geq x_M\}$ and Follower never enters: $\tau_F^{\mathrm{CO}} = \infty$.

2. *When $D > \frac{1}{2}$ and $K_L < \frac{K_F}{2D-1}$, we have*

$$W^{\mathrm{CO}}(x) = [H(x_M) - K_L]\left(\frac{x}{x_M}\right)^{\beta}, \quad x < x_M\,, \tag{IA.G.3}$$

$$W^{\mathrm{CO}}(x) = H(x) - K_L, \quad x \geq x_M, \tag{IA.G.4}$$

where $x_M = \frac{\beta}{\beta-1}(r-\mu)K_L$, $H(x)$ *is given by*

$$H(x) = \Pi(x) + [(2D-1)\Pi(\widetilde{x}_F) - K_F]\left(\frac{x}{\widetilde{x}_F}\right)^{\beta}, \quad x < \widetilde{x}_F, \tag{IA.G.5}$$

$$H(x) = 2D\Pi(x) - K_F\,, \quad x \geq \widetilde{x}_F, \tag{IA.G.6}$$

and $\widetilde{x}_F = \frac{\beta}{\beta-1}(r-\mu)\frac{K_F}{2D-1}$. *The two firms enter sequentially at* $\tau_L^{\mathrm{CO}} = \inf\{s \geq t : X_s \geq x_M\}$ *and* $\tau_F^{\mathrm{CO}} = \inf\{s \geq t : X_s \geq \widetilde{x}_F\}$, *where* $\widetilde{x}_F > x_M$.

3. *When* $D > \frac{1}{2}$ *and* $K_L \geq \frac{K_F}{2D-1}$, *we have*

$$W^{\mathrm{CO}}(x) = \Big[2D\Pi(x_D) - \big(K_L + K_F\big)\Big]\left(\frac{x}{x_D}\right)^{\beta}, \quad x < x_D, \tag{IA.G.7}$$

$$W^{\mathrm{CO}}(x) = 2D\Pi(x) - \big(K_L + K_F\big), \quad x \geq x_D, \tag{IA.G.8}$$

where $x_D = \frac{\beta}{\beta-1}(r-\mu)\frac{K_L+K_F}{2D} \leq x_M$ *and* $\tau_L^{\mathrm{CO}} = \tau_F^{\mathrm{CO}} = \inf\{s \geq t : X_s \geq x_D\}$.

Using Proposition IA.3, we can compute $\frac{V_i(x)}{W^{\mathrm{CO}}(x)}$ and thus $\Delta(x)$. We summarize the results in the following proposition and (for brevity) focus on the cases where x is small.

Proposition IA.4 *For small* $x > 0$, *the following hold.*

1. *When* $D \leq \frac{1}{2}$, *we have* $\frac{V_i(x)}{W^{\mathrm{CO}}(x)} = D^{\beta}$ *for* $R \geq R_{A_1A_2}$, *and* $\frac{V_i(x)}{W^{\mathrm{CO}}(x)} = 1-(1-D)D^{\beta-1}\beta R^{\beta-1}$ *for* $R \in [R_{AB}, R_{A_1A_2}]$.

2. *When* $D > \frac{1}{2}$ *and* $K_L < \frac{K_F}{2D-1}$, *we have* $\frac{V_i(x)}{W^{\mathrm{CO}}(x)} = D^{\beta}\frac{R}{R+((2D-1)R)^{\beta}}$ *for* $R \geq R_{A_1A_2}$, *and* $\frac{V_i(x)}{W^{\mathrm{CO}}(x)} = \frac{1-(1-D)D^{\beta-1}\beta R^{\beta-1}}{1+R^{\beta-1}(2D-1)^{\beta}}$ *for* $R \in [R_{AB}, R_{A_1A_2}]$.

3. *When* $D > \frac{1}{2}$ *and* $K_L \geq \frac{K_F}{2D-1}$, *we have* $R \geq \frac{1}{2D-1} > R_{A_1A_2}$ *and thus* $\frac{V_i(x)}{W^{\mathrm{CO}}(x)} = \frac{(1+1/R)^{\beta-1}}{2^{\beta}}$.

Proposition IA.5 *For small* $x_0 > 0$, *an optimal subsidy policy,* $\delta_L^{\mathrm{Sub}} \in (0, K_L - K_F R_{AB}]$, *exists for Problem* (B.2).

IA.H Reputation Model with Asymmetric Beliefs

In this appendix, we extend the reputation model of Section 5 to a setting with asymmetric beliefs. Without loss of generality, we assume $1 > \pi_0^a > \pi_0^b > 0$.

Definition IA.2 A Markov entry strategy for firm $i \in \{a, b\}$ in the setting of Section 5 is a pair: $\varphi_i = (\Gamma_i, \lambda_i(x))$, where $\Gamma_i \in [0,1]$ is a constant, and the entry rate $\lambda_i(x)$ is a measurable function from $\mathbb{R}_+$ to $\mathbb{R}_+$. Firm i enters the market with probability Γ_i when X_t first hits the region $\{x : \lambda_i(x) > 0\}$ and then enters the market randomly at an intensity rate of $\lambda_i(X_t)$. A Markov strategy pair $(\varphi_a, \varphi_b) = \{(\Gamma_a, \lambda_a), (\Gamma_b, \lambda_b)\}$ is feasible if and only if $\int_0^t \lambda_i(X_s)ds < \infty$ almost surely for any t. Let Φ denote the set of all feasible entry strategies.

One can see that if we restrict $\Gamma_i = 0$, then Definition IA.2 reduces to Definition 2.[14] The CDF for firm i's entry time is given by

$$G_i(t) = 1 - \left(1 - \Gamma_i \mathbf{1}_{\{\lambda_i(X_s)>0 \text{ for some } s\leq t\}}\right) e^{-\int_0^t \lambda_i(X_u)du}. \tag{IA.H.1}$$

Below we solve the perfect Bayesian Nash equilibrium for the case $1 > \pi_0^a > \pi_0^b > 0$.

Theorem IA.5 *Suppose $R > R_{AB}$ and $1 > \pi_0^a > \pi_0^b > 0$. Then the unique perfect Bayesian Nash equilibrium $\{\varphi_a^*, \varphi_b^*\}$ is given by $\{(\Gamma_a^*, \lambda_a^*), (\Gamma_b^*, \lambda_b^*)\} = \{(0, \lambda^*), (1 - \frac{\pi_0^b}{\pi_0^a}, \lambda^*)\}$, where $\lambda^*(x)$ is given in Theorem 1. In this equilibrium, the revelation time $T^* := \min\{T_a^*, T_b^*\}$ is given by*

$$T^* = \inf\{t \geq 0 : \int_0^t \lambda^*(X_u)du = -\ln \pi_0^a\}. \tag{IA.H.2}$$

Rational firm b's value is $V_b(t,x) = V_*(x)$ for any $t \in [0, T^*)$, where $V_*(x)$ is the unique solution for the variational-inequality problem (18) *in the $x \geq 0$ domain. Rational firm a's value is*

$$\begin{aligned} V_a(t,x) =& \mathbb{E}_t^x \left[e^{-r(\tau_L^* - t)} \left(F(X_{\tau_L^*}) \left(1 - \frac{\pi_0^b}{\pi_0^a}\right) + V_*(X_{\tau_L^*}) \frac{\pi_0^b}{\pi_0^a} \right) \right] \\ =& \left(1 - \frac{\pi_0^b}{\pi_0^a}\right) J_F(x) + \frac{\pi_0^b}{\pi_0^a} V_*(x), \quad t \in [0, \tau_L^*], \end{aligned} \tag{IA.H.3}$$

$$V_a(t,x) = V_*(x), \quad t \in (\tau_L^*, T^*), \tag{IA.H.4}$$

where $\tau_L^* = \inf\{s \geq 0 : X_s \in \mathcal{R}^E\} < T^*$, $J_F(x)$ is given by (A.9)-(A.10) *for $R > R_{A_1A_2}$ and* (33)-(35) *for $R_{AB} < R \leq R_{A_1A_2}$.*

IA.I Technical Details for Appendix F

In this appendix, we characterize a Bayesian Nash equilibrium such that $\mathbf{Q}_t^{b*} = \zeta(\mathbf{Q}_t^{a*})$, where $\zeta(q)$ is a positive and increasing function of q over $[\underline{q}, \overline{q}]$ and satisfies (F.3).

[14] By setting $\Gamma_i = 1$, we get a pure strategy. However, employing arguments similar to those presented below Theorem 2, we can show that there is no pure-strategy equilibrium.

At time $t=\tau_L$, conditional on being Leader, firm i's estimation of Q equals to $\mathcal{W}(Q_i, Q^{-i}_{\tau_L})$, where $\mathcal{W}(Q_i, q)$ is given by (E.10).

Let $V_a(x, q_b)$ denote firm a's equilibrium value function for $X_t = x$ and $\mathbf{Q}^{b*}_t = q_b$. For $\mathbf{Q}^{b*}_t = q_b \le \zeta(Q_a)$, we have $\mathbf{Q}^{a*}_t \le Q_a$, and firm a is willing to enter as Leader by solving the following problem:

$$\begin{aligned} J^a_L(x; Q_a, \mathbf{Q}^{b*}_t) &= \max_{\tau_L \ge t} \ \mathbb{E}^x_t\left[e^{-r(\tau_L - t)}(L^a_{\tau_L}(X_{\tau_L}, Q_a) - K)\right] \\ &= \max_{\tau_L \ge t} \ \mathbb{E}^x_t\left[e^{-r(\tau_L - t)}(pN_a \mathcal{W}(Q_a, \mathbf{Q}^{b*}_{\tau_L})\Pi\big(X_{\tau_L}\big) - K)\right] \\ &\le \max_{\tau_L \ge t} \ \mathbb{E}^x_t\left[e^{-r(\tau_L - t)}(pN_a \mathcal{W}(Q_a, \mathbf{Q}^{b*}_{t})\Pi\big(X_{\tau_L}\big) - K)\right], \end{aligned} \tag{IA.I.1}$$

where the second equality uses (F.1), the inequality uses $\mathbf{Q}^{b*}_{\tau_L} \le \mathbf{Q}^{b*}_t$. We can show that the inequality in (IA.I.1) holds as an equality, and

$$V_a(x, \mathbf{Q}^{b*}_t) = J^a_L(x; Q_a, \mathbf{Q}^{b*}_t) = pN_a \mathcal{W}(Q_a, \mathbf{Q}^{b*}_t) M^*\left(x; \frac{K}{pN_a \mathcal{W}(Q_a, \mathbf{Q}^{b*}_t)}\right), \tag{IA.I.2}$$

where $M^*(x; K)$ is given by (E.13)-(E.14).

When $\mathbf{Q}^{b*}_t = q_b > \zeta(Q_a)$, we have $\mathbf{Q}^{a*}_t > Q_a$ and thus firm a chooses to wait. If $Q_b = \mathbf{Q}^{b*}_t = q_b > \zeta(Q_a)$, firm b will enter as Leader at time t. As a result, firm a infers $Q_b = \mathbf{Q}^{b*}_t$ and updates its estimate of Q according to $\frac{Q_a + \mathbf{Q}^{b*}_t}{2}$. Substituting this into (F.2), we obtain that firm a's payoff as Follower is equal to $pM\big(N_a \frac{Q_a + \mathbf{Q}^{b*}_t}{2} X_t\big)$, where $M(x) = M^*(x; K)$.

Using (E.8) with $\Lambda^b_t = \Lambda^*_b(X_t, \mathbf{Q}^{b*}_t)$, we can derive the HJB equation for $V_a(x, q_b)$ as follows:

$$\begin{aligned} rV_a(x, q_b) = &\frac{\sigma^2 x^2}{2}\frac{\partial^2 V_a(x, q_b)}{\partial x^2} + \mu x \frac{\partial V_a(x, q_b)}{\partial x} \\ &+ \Lambda^*_b(x, q_b)\Big[pM\Big(N_a \frac{Q_a + q_b}{2} x\Big) - V_a(x, q_b)\Big] - \frac{\Psi(q_b)}{\Psi'(q_b)}\Lambda^*_b(x, q_b)\frac{\partial V_a(x, q_b)}{\partial q_b} \end{aligned} \tag{IA.I.3}$$

for $q_b > \zeta(Q_a)$. At time t such that $\mathbf{Q}^{b*}_t = q_b = \zeta(Q_a)$, we have $\mathbf{Q}^{a*}_t = Q_a$, and firm a is indifferent between entering and waiting, which implies that $V_a(x, q_b) = J^a_L(x; Q_a, q_b)$ satisfies equation (IA.I.3) at $q_b = \zeta(Q_a)$. Note that in the region $x < \widetilde{x}\Big(\frac{K}{pN_a \mathcal{W}(Q_a, \zeta(Q_a))}\Big)$, we have $V_a(x, \zeta(Q_a)) = J^a_L(x; Q_a, \zeta(Q_a)) = pN_a \mathcal{W}(Q_a, \zeta(Q_a)) M^*(x; \frac{K}{pN_a \mathcal{W}(Q_a, \zeta(Q_a))})$, and

$$rV_a(x, \zeta(Q_a)) = \frac{\sigma^2 x^2}{2}\frac{\partial^2 V_a(x, \zeta(Q_a))}{\partial x^2} + \mu x \frac{\partial V_a(x, \zeta(Q_a))}{\partial x}. \tag{IA.I.4}$$

Substituting the above into (IA.I.3), we find that $\Lambda^*_b(x, \zeta(Q_a))$ is zero in the region $x < \widetilde{x}\Big(\frac{K}{pN_a \mathcal{W}(Q_a, \zeta(Q_a))}\Big)$.

Next, we consider the region $x > \widetilde{x}\Big(\frac{K}{pN_a\mathcal{W}(Q_a,\zeta(Q_a))}\Big)$. Fix $x > \widetilde{x}\Big(\frac{K}{pN_a\mathcal{W}(Q_a,\zeta(Q_a))}\Big)$, and let q_b be close to $\zeta(Q_a)$. Then $x > \widetilde{x}\Big(\frac{K}{pN_a\mathcal{W}(Q_a,q_b)}\Big)$, and $M^*\big(x;\frac{K}{pN_a\mathcal{W}(Q_a,q_b)}\big) = \Pi(x) - \frac{K}{pN_a\mathcal{W}(Q_a,q_b)}$, which implies $J_L^a(x;Q_a,q_b) = pN_a\mathcal{W}(Q_a,q_b)\Pi(x) - K$, and

$$\begin{aligned}
&\frac{\partial V_a(x,q_b)}{\partial q_b} = \frac{\partial J_L^a(x;Q_a,q_b)}{\partial q_b} = \Pi(x)pN_a\frac{\partial \mathcal{W}(Q_a,q_b)}{\partial q_b} = \Pi(x)pN_a\frac{\partial \int_{\underline{q}}^{q_b}\frac{Q_a+z}{2}\frac{d\Psi(z)}{\Psi(q_b)}}{\partial q_b}\\
=&\Pi(x)pN_a\frac{\Psi'(q_b)}{\Psi(q_b)}\Big[\frac{Q_a+q_b}{2} - \mathcal{W}(Q_a,q_b)\Big]. \qquad \text{(IA.I.5)}
\end{aligned}$$

Thus, the coefficient of $\Lambda_b^*(x,\zeta(Q_a))$ in (IA.I.3) is given by

$$\begin{aligned}
&pM\Big(N_a\frac{Q_a+\zeta(Q_a)}{2}x\Big) - V_a(x,\zeta(Q_a)) - \frac{\Psi(\zeta(Q_a))}{\Psi'(\zeta(Q_a))}\frac{\partial V_a(x,q_b)}{\partial q_b}\Big|_{q_b=\zeta(Q_a)}\\
=&p\Big[\Pi\Big(N_a\frac{Q_a+\zeta(Q_a)}{2}x\Big) - K\Big] - pN_a\mathcal{W}(Q_a,\zeta(Q_a))M^*\Big(x;\frac{K}{pN_a\mathcal{W}(Q_a,\zeta(Q_a))}\Big)\\
&-\Pi(x)pN_a\Big[\frac{Q_a+\zeta(Q_a)}{2} - \mathcal{W}(Q_a,\zeta(Q_a))\Big]\\
=&p\Big[N_a\frac{Q_a+\zeta(Q_a)}{2}\Pi(x) - K\Big] - \Big[pN_a\mathcal{W}(Q_a,\zeta(Q_a))\Pi(x) - K\Big]\\
&-\Pi(x)pN_a\Big[\frac{Q_a+\zeta(Q_a)}{2} - \mathcal{W}(Q_a,\zeta(Q_a))\Big] = (1-p)K, \qquad \text{(IA.I.6)}
\end{aligned}$$

where the first equality uses $x > \widetilde{x}\Big(\frac{K}{pN_a\mathcal{W}(Q_a,\zeta(Q_a))}\Big) > \widetilde{x}\Big(\frac{K}{N_a\frac{Q_a+\zeta(Q_a)}{2}}\Big)$, (IA.I.2), and (IA.I.5), the second equality uses (E.14) and $x > \widetilde{x}\Big(\frac{K}{pN_a\mathcal{W}(Q_a,\zeta(Q_a))}\Big)$.

In the region $x > \widetilde{x}\Big(\frac{K}{pN_a\mathcal{W}(Q_a,\zeta(Q_a))}\Big)$, using $J_L^a(x;Q_a,\zeta(Q_a)) = pN_a\mathcal{W}(Q_a,\zeta(Q_a))\Pi(x) - K$, we obtain

$$\Big[\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2} + \mu x\frac{\partial}{\partial x} - r\Big]J_L^a(x;Q_a,\zeta(Q_a)) = rK - pN_a\mathcal{W}(Q_a,\zeta(Q_a))x. \qquad \text{(IA.I.7)}$$

Substituting (IA.I.6) and (IA.I.7) into (IA.I.3), we obtain

$$\Lambda_b^*(x,\zeta(Q_a)) = \frac{pN_a\mathcal{W}(Q_a,\zeta(Q_a))x - rK}{(1-p)K} \qquad \text{(IA.I.8)}$$

in the region $x > \widetilde{x}\Big(\frac{K}{pN_a\mathcal{W}(Q_a,\zeta(Q_a))}\Big)$. Thus, we have

$$\Lambda_b^*(x,q_b) = \frac{pN_a\mathcal{W}(\zeta^{-1}(q_b),q_b)x - rK}{(1-p)K}\mathbf{1}_{x>\widetilde{x}\big(\frac{K}{pN_a\mathcal{W}(\zeta^{-1}(q_b),q_b)}\big)}, \qquad \text{(IA.I.9)}$$

where $\zeta^{-1}(\cdot)$ is the inverse function of $\zeta(\cdot)$. Similar to (IA.I.9), we can derive

$$\Lambda_a^*(x, q_a) = \frac{pN_b\mathcal{W}(\zeta(q_a), q_a)x - rK}{(1-p)K}\mathbf{1}_{x>\widetilde{x}\left(\frac{K}{pN_b\mathcal{W}(\zeta(q_a),q_a)}\right)}. \tag{IA.I.10}$$

Because $\zeta(\cdot)$ is an increasing function and $\mathbf{Q}_t^{b*} = \zeta(\mathbf{Q}_t^{a*})$, $\mathbf{Q}_t^{a*}$ and $\mathbf{Q}_t^{b*}$ should decrease synchronously and remain constant synchronously. This implies that $\Lambda_b^*(X_t, \mathbf{Q}_t^{b*}) > 0$ if and only if $\Lambda_a^*(X_t, \mathbf{Q}_t^{a*}) > 0$. Thus, we have $N_a\mathcal{W}(\zeta^{-1}(\mathbf{Q}_t^{b*}), \mathbf{Q}_t^{b*}) = N_b\mathcal{W}(\zeta(\mathbf{Q}_t^{a*}), \mathbf{Q}_t^{a*})$. Combining this with $\mathbf{Q}_t^{b*} = \zeta(\mathbf{Q}_t^{a*})$, we conjecture that $\zeta(\cdot)$ should satisfy (F.3) for any $q_a \in [\underline{q}, \overline{q}]$ such that $\zeta(q_a) \in [\underline{q}, \overline{q}]$.

Since $q > \int_{\underline{q}}^{q} z\frac{d\Psi(z)}{\Psi(q)}$, we have $\mathcal{W}(q_1, q_2) > \mathcal{W}(q_2, q_1)$ if $q_1 > q_2 \geq \underline{q}$. When $N_a = N_b$, it follows from (F.3) that $\zeta(q_a) = q_a$. When $N_a > N_b$, it follows from (F.3) that $\mathcal{W}(q_a, \zeta(q_a)) < \mathcal{W}(\zeta(q_a), q_a)$, which implies that $\zeta(q_a) > q_a$.

Next, we summarize the perfect Bayesian Nash equilibrium solution for this game.

IA.J Additional Technical Results

Lemma 2 *Recall $S(x) = F(x) - (L(x) - K_L)$, where $L(x)$ is given by* (12)-(13) *and $F(x)$ is given by* (8)-(9)*.*

(i) For $R \in (1, R_{AB}]$, $\widehat{x}_L$ and $\widehat{x}_F$, the two roots of $S(x) = 0$ in Proposition 1, have the following closed-form expressions:

$$\widehat{x}_F = \overline{\eta}\,(R)\,(r-\mu)K_F \quad \textit{and} \quad \widehat{x}_L = \underline{\eta}\,(R)\,(r-\mu)K_F, \tag{IA.J.1}$$

where $\overline{\eta}(R)$ and $\underline{\eta}(R)$ are given by

$$\overline{\eta}\,(R) := \sup\{y > 0 : y - \left(\frac{1}{D}\frac{\beta}{\beta-1} - 1\right)\left(D\frac{\beta-1}{\beta}\right)^{\beta} y^{\beta} = R\}, \tag{IA.J.2}$$

$$\underline{\eta}\,(R) := \inf\{y > 0 : y - \left(\frac{1}{D}\frac{\beta}{\beta-1} - 1\right)\left(D\frac{\beta-1}{\beta}\right)^{\beta} y^{\beta} = R\}. \tag{IA.J.3}$$

We can further show: $x_F > \widehat{x}_F > \widetilde{x} > \widehat{x}_L$ for $R \in (1, R_{AB})$. For the special case $R = R_{AB}$, $x_F > \widehat{x}_F = \widetilde{x} = \widehat{x}_L$. Finally, $\lim\limits_{R\to 1+} \widehat{x}_F = x_F$.

(ii) For $R \in (0, 1]$, the equation $S(x) = 0$ has only one root in the $x < x_F$ region: $\widehat{x}_L$ given in (IA.J.1)*. Finally, $\widehat{x}_L < \widetilde{x}$.*

(iii) The root $\widehat{x}_L$ increases in $R \in (0, R_{AB}]$ and the root $\widehat{x}_F$ decreases in $R \in (1, R_{AB}]$.

Determining $R_{B_1B_2}$, the cutoff value of R for Subcase B$_1$ and Subcase B$_2$. First, Lemma 1 implies that $\underline{x}$ is continuous in $R \in [1, R_{AB}]$ and satisfies $\widetilde{x} < \underline{x} < x_F$ for any $R \in [1, R_{AB}]$. Second, Lemma 2 implies that $\widehat{x}_F$ is continuous in $R \in (1, R_{AB}]$ and satisfies $\widehat{x}_F \to x_F$ as $R \to 1+$ and $\widehat{x}_F = \widetilde{x}$ for $R = R_{AB}$. Combining these two results, we conclude: $\underline{x} > \widehat{x}_F$ for $R = R_{AB}$, $\underline{x} < \widehat{x}_F$ as $R \to 1+$, $R_{B_1B_2} = \sup\{\ R \in (1, R_{AB}) : \underline{x} \leq \widehat{x}_F\}$ is well-defined, $R_{B_1B_2} \in (1, R_{AB})$, and

$$\underline{x} > \widehat{x}_F, \qquad \text{if } R \in (R_{B_1B_2}, R_{AB}]. \tag{IA.J.4}$$

Lemma 3 *Let $V_*(x)$ be the solution to the variational-inequality problem* (18) *in the $x \geq 0$ region subject to the boundary conditions $V_*(0) = 0$ and $\lim_{x\to\infty} V_*(x) - (L(x) - K_L) = 0$. Let $\mathcal{R}^E$ denote the probabilistic entry region:*

$$\mathcal{R}^E := \{x > 0 : V_*(x) = L(x) - K_L\}. \tag{IA.J.5}$$

(i) For Subcase A_1 where $R > R_{A_1A_2}$, $V_*(x)$ is given by (A.6)-(A.7) *and $\mathcal{R}^E = [\overline{x}, \infty)$, where $\overline{x} = \frac{1}{D}\frac{\beta}{\beta-1}(r-\mu)K_L$.*

(ii) For Subcase A_2 where $R_{AB} < R \leq R_{A_1A_2}$, $V_*(x)$ is given below:

$$V_*(x) = \left(\frac{x}{\widetilde{x}}\right)^{\beta}(L(\widetilde{x}) - K_L)\,, \quad x \in [0, \widetilde{x})\,, \tag{IA.J.6}$$

$$V_*(x) = L(x) - K_L, \quad x \in [\widetilde{x}, \underline{x}]\,, \tag{IA.J.7}$$

$$V_*(x) = \Theta(x; \underline{x}, \overline{x}), \quad x \in (\underline{x}, \overline{x})\,, \tag{IA.J.8}$$

$$V_*(x) = L(x) - K_L = D\Pi(x) - K_L, \quad x \geq \overline{x}\,, \tag{IA.J.9}$$

where $\Theta(x; a, b)$ is given by (29) *and the thresholds, $\underline{x}$ and $\overline{x}$, are given in Lemma 1. Finally, $\mathcal{R}^E = [\widetilde{x}, \underline{x}] \cup [\overline{x}, \infty)$.*

Lemma 4 *Let $V_*(x)$ be the unique solution to the variational-inequality problem* (18) *in the $x > \widehat{x}_F$ region subject to the boundary conditions $\lim_{x\to\infty} V_*(x) - (L(x) - K_L) = 0$ and $V_*(\widehat{x}_F) = F(\widehat{x}_F)$. Let $\Theta(x; a, b)$ for any $x \in [a, b]$ be given by* (29) *and let $\mathcal{R}^E$ denote the probabilistic entry domain defined in* (IA.B.2).

(i) For Subcase B_1 where $R_{B_1B_2} < R \leq R_{AB}$, we have

$$V_*(x) = L(x) - K_L, \quad x \in [\widehat{x}_F, \underline{x}]\,, \tag{IA.J.10}$$

$$V_*(x) = \Theta(x; \underline{x}, \overline{x}), \quad x \in (\underline{x}, \overline{x})\,, \tag{IA.J.11}$$

$$V_*(x) = D\Pi(x) - K_L, \quad x \geq \overline{x}\,, \tag{IA.J.12}$$

where the cutoff, $\widehat{x}_F$, is given in Lemma 2 and the cutoffs, $\underline{x}$ and $\overline{x}$, are given in Lemma 1. The $\mathcal{R}^E$ domain is the union of two disconnected regions: $\mathcal{R}^E = (\widehat{x}_F, \underline{x}] \cup [\overline{x}, \infty)$.

(ii) For Subcase B_2 where $1 < R \leq R_{B_1B_2}$, we have

$$V_*(x) = \Theta(x; \widehat{x}_F, \overline{x}), \quad x \in [\widehat{x}_F, \overline{x}), \tag{IA.J.13}$$

$$V_*(x) = D\Pi(x) - K_L, \quad x \geq \overline{x}, \tag{IA.J.14}$$

where $\widehat{x}_F$ is given in Lemma 2 and $\overline{x}$ is the unique solution of the following equation:

$$\Gamma(\widehat{x}_F, y) = F(\widehat{x}_F) \tag{IA.J.15}$$

in the $y > \widetilde{x}/D$ region. The function $\Gamma(x, y)$ in (IA.J.15) is defined as follows:

$$\Gamma(x,y) = \frac{D(1-\gamma)\Pi(y) + \gamma K_L}{\beta - \gamma} \left(\frac{x}{y}\right)^{\beta} + \frac{D(\beta-1)\Pi(y) - \beta K_L}{\beta - \gamma} \left(\frac{x}{y}\right)^{\gamma} \quad . \tag{IA.J.16}$$

Finally, the probabilistic entry region is given by $\mathcal{R}^E = [\overline{x}, \infty)$.

Lemma 5 *Suppose $R > 1$. The optimal strategy to problem (31) is $\tau_L^* = \inf\{s \geq t : X_s \in \mathcal{R}^E\}$, and the optimal value is $J_L(x) = V_*(x)$, where $V_*(x)$ is the unique solution for the variational-inequality problem (18) in the $x \geq 0$ domain, and $\mathcal{R}^E = \{x > 0 : V_*(x) = L(x) - K_L\}$.*

Lemma 6 *Let $\mathcal{R}^E(K_L) := \{x > 0 : J_L(x) = L(x) - K_L\}$ denote the entry region for problem (31). Then $\mathcal{R}^E(K_L)$ is nonincreasing in $K_L > K_F$ in the sense that $\mathcal{R}^E(K_L) \subseteq \mathcal{R}^E(\widetilde{K}_L)$ for any $\widetilde{K}_L \in (K_F, K_L)$. As a result, the equilibrium entry rate $\lambda^*(x; K_L)$ given in Theorem 1 is nonincreasing in K_L.*

Lemma 7 *Suppose $K_L/K_F \leq 1$. Consider following problem:*

$$W_0(x) = \max_{\tau \geq t} \quad \mathbb{E}_t^x \left[e^{-r(\tau - t)} \frac{L(X_\tau) - K_L + F(X_\tau)}{2} \right]. \tag{IA.J.17}$$

The optimal stopping to problem (IA.J.17) is $\inf\{t \geq 0 : X_t \geq x_M\}$, where $x_M = \frac{\beta}{\beta-1}(r-\mu)K_L$. Moreover, we have

$$W_0(x) = \frac{L(x_M) - K_L + F(x_M)}{2} \left(\frac{x}{x_M}\right)^{\beta}, \quad x < x_M, \tag{IA.J.18}$$

$$W_0(x) = \frac{L(x) - K_L + F(x)}{2}, \quad x \geq x_M. \tag{IA.J.19}$$

As a result, $W_0(x) < L(x) - K_L$ *for any* $x \in [x_M, x_F)$.

IA.K Proofs

In this Internet Appendix, we provide the proofs of all the theorems, propositions, and lemmas presented in the main body of the paper, as well as in the Appendices and Internet Appendices.

Let $\mathcal{A}V(x)$ denote the infinitesimal generator operating on a function $V(x)$:

$$\mathcal{A}V(x) = \frac{\sigma^2}{2}x^2V''(x) + \mu xV'(x) - rV(x)\,. \tag{IA.K.1}$$

Proof of Theorem 1: Let $g(x) := \mathcal{A}V_*(x)$ for $x \geq 0$, where $\mathcal{A}V$ is the infinitesimal generator given in (IA.K.1). Substituting the closed-form expressions for $V_*(x)$ into (IA.K.1)[15], we obtain

$$g(x) = \mathcal{A}V_*(x) = \lambda^*(x)[L(x) - K_L - F(x)], \quad x > 0, \tag{IA.K.2}$$

where $\lambda^*(x)$ is given by (19) for any $x \in \mathcal{R}^E$ and $\lambda^*(x) = 0$ for any $x \in (0,\infty) \setminus \mathcal{R}^E$, $\mathcal{R}^E$ is given in Lemma 3 of Internet Appendix IA.J. Using the expression for $\lambda^*(x)$ given in (19) and $L(x)$ given in (12)-(13), we obtain:

$$g(x) = \mathbf{1}_{x\in\mathcal{R}^E}\left[(rK_L - x)\mathbf{1}_{x<x_F} + (rK_L - Dx)\mathbf{1}_{x>x_F}\right]$$

for any $x > 0$.

Using $\lambda^*(x)$ given by (19) for any $x \in \mathcal{R}^E$, there exist a positive value x' and a positive constant λ', such that $\lambda^*(x) \geq \lambda' > 0$ for all $x > x'$, which further implies:

$$e^{-\int_t^\infty \lambda^*(X_s)ds} = 0, \quad \text{almost surely.} \tag{IA.K.3}$$

Next, we complete our proof in two steps. First, we show that it is suboptimal for firm a to deviate from its equilibrium strategy if firm b does not (Step 1).

Step 1: We prove $V_*(x) \geq J_a(x;\lambda_a,\lambda^*)$ *where* $(\lambda_a,\lambda^*) \in \Phi$.

Let τ_a and τ_b be firm a's and b's stochastic entry time associated with the strategy pair $(\lambda_a,\lambda_b) = (\lambda_a,\lambda^*)$, where $\lambda_a \neq \lambda^*$. Let $\tau := \min\{\tau_a,\tau_b\}$.

Note that $V_*(x)$ is twice continuously differentiable except at finite points and is globally continuously differentiable. Applying Itô's Lemma to $e^{-rs}V_*(X_s)$ for $s \in [t,\tau]$ and taking

[15] According to Lemma 3, the variational-inequality problem (18) admits a unique solution, $V_*(x)$, given by (A.6)-(A.7) for Subcase A_1 and (IA.J.6)-(IA.J.9) for Subcase A_2, respectively.

expectations at time t, we obtain the following expression for $V_*(x)$:

$$V_*(x) = \mathbb{E}_t^x[e^{-r(\tau-t)}V_*(X_\tau)] - \mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}\mathcal{A}V_*(X_s)ds\right]. \tag{IA.K.4}$$

Recall that $V_*(x)$ satisfies the variational inequality (18), we have $V_*(x) \geq L(x) - K_L$, $\forall x > 0$. Substituting it into the right side of (IA.K.4), we obtain the following inequality:

$$V_*(x) \geq \mathbb{E}_t^x[e^{-r(\tau-t)}(L(X_\tau) - K_L)] - \mathbb{E}_t^x[\int_t^\tau e^{-r(s-t)}\mathcal{A}V_*(X_s)ds]. \tag{IA.K.5}$$

Note that

$$\begin{aligned} J_a(x;\lambda_a,\lambda^*) &= \mathbb{E}_t^x\Big[e^{-r(\tau-t)}\left[\mathbf{1}_{\tau_a<\tau_b}(L(X_\tau) - K_L) + \mathbf{1}_{\tau_a>\tau_b}F(X_\tau)\right]\Big] \\ &= \mathbb{E}_t^x[e^{-r(\tau-t)}(L(X_\tau) - K_L)] - \mathbb{E}_t^x\Big[\mathbf{1}_{\tau_a>\tau_b}e^{-r(\tau-t)}(L(X_\tau) - K_L - F(X_\tau))\Big], \end{aligned} \tag{IA.K.6}$$

where the second equality follows from the property: $\mathbf{1}_{\tau_a=\tau_b} = 0$ almost surely. Using (IA.K.6) and (IA.K.5), we obtain

$$J_a(x;\lambda_a,\lambda^*) \leq V_*(x) + \mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}\mathcal{A}V_*(X_s)ds - \mathbf{1}_{\tau_a>\tau_b}e^{-r(\tau-t)}(L(X_\tau) - K_L - F(X_\tau))\right]. \tag{IA.K.7}$$

We can simplify the first term on the right side of (IA.K.7) as follows:

$$\begin{aligned} \mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}\mathcal{A}V_*(X_s)ds\right] =& \mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}g(X_s)ds\right] \\ =& \mathbb{E}_t^x\left[\int_t^\infty \int_t^{\tau_a\wedge z} e^{-r(s-t)}g(X_s)\lambda^*(X_z)e^{-\int_t^z \lambda^*(X_u)du}dsdz\right] \\ =& \mathbb{E}_t^x\left[\int_t^{\tau_a} \int_s^\infty \lambda^*(X_z)e^{-\int_t^z \lambda^*(X_u)du}dze^{-r(s-t)}g(X_s)ds\right] \\ =& \mathbb{E}_t^x\left[\int_t^{\tau_a} e^{-\int_t^s (r+\lambda^*(X_u))du}g(X_s)ds\right] \\ =& \mathbb{E}_t^x\left[\int_t^{\tau_a} e^{-\int_t^s (r+\lambda^*(X_u))du}\lambda^*(X_s)[L(X_s) - K_L - F(X_s)]ds\right] \\ =& \mathbb{E}_t^x\left[\mathbf{1}_{\tau_a>\tau_b}e^{-r(\tau_b-t)}[L(X_{\tau_b}) - K_L - F(X_{\tau_b})]\right] \end{aligned} \tag{IA.K.8}$$

using (IA.K.2), Tonelli's Theorem (to interchange the integration order in the third equality as $g(x) \leq 0$ and $\lambda^*(x) \geq 0$ for any $x > 0$), integration by parts, and (IA.K.3). Combining (IA.K.7) and (IA.K.8) yields $J_a(x;\lambda_a,\lambda^*) \leq V_*(x)$.

Step 2: We prove $V_*(x) = J_a(x; \lambda^*, \lambda^*)$.

Recall that τ_a^* and τ_b^* are firms a's and b's stochastic entry time, respectively, associated with strategy $(\lambda_a(x), \lambda_b(x)) = (\lambda^*(x), \lambda^*(x))$ and $\tau^* := \min\{\tau_a^*, \tau_b^*\}$. Because $\lambda^*(x) = 0$ for any $x \in (0, \infty) \setminus \mathcal{R}^E$, we have $X_{\tau^*} \in \mathcal{R}^E$, which implies $V_*(X_{\tau^*}) = L(X_{\tau^*}) - K_L$. Therefore, we can see that (IA.K.5)-(IA.K.7) hold with equality if λ_a, τ_a, τ_b and τ therein are set to λ^*, τ_a^*, τ_b^* and τ^*, respectively. We have thus shown $V_*(x) = J_a(x; \lambda^*, \lambda^*)$.

In sum, combining our analyses in Steps 1 and 2, we obtain $J_a(x; \lambda^*, \lambda^*) \geq J_a(x; \lambda_a, \lambda^*)$. By symmetry, we also have $J_b(x; \lambda^*, \lambda^*) \geq J_b(x; \lambda^*, \lambda_b)$ for $(\lambda^*, \lambda_b) \in \Phi$. $\square$

Proof of the Existence of Equilibrium in Theorem 2: We denote $\varphi_a^* = \varphi_b^* = \varphi^* = (\emptyset, \lambda^*(x))$ in the following proof. Using (40), and $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$, we can see $T^* = T_a^* = T_b^*$ and $e^{-\int_0^{T^*} \lambda^*(X_u)du} = \pi_0$.

Step 1: Prove $V_*(x) \geq J_a(x; \varphi_a, \varphi_b^*, T^*)$ for any $x > 0$, where $(\varphi_a, \varphi_b^*) \in \Phi$ and $T^* = T^{\varphi_a^*, \varphi_b^*}$.

Let $\widehat{\tau}_a$ denote the entry time of the rational firm a associated with φ_a, and let τ_b denote the entry time of firm b from firm a's perspective, associated with φ_b^*. Let $\tau_L := \min\{\widehat{\tau}_a, \tau_b\}$.

Applying Itô's Lemma to $e^{-rs}V_*(X_s)$ for $s \in [t, \tau_L \wedge T^*]$ and taking expectations at time $t < T^*$, we obtain the following expression for $V_*(x)$:

$$\begin{aligned} &V_*(x) \\ =&\mathbb{E}_t^x\left[e^{-r(\tau_L \wedge T^* - t)} V_*(X_{\tau_L \wedge T^*})\right] - \mathbb{E}_t^x\left[\int_t^{\tau_L \wedge T^*} e^{-r(s-t)} \mathcal{A}V_*(X_s) ds\right] \\ =&\mathbb{E}_t^x\left[e^{-r(\tau_L - t)} \mathbf{1}_{\tau_L \leq T^*} V_*(X_{\tau_L}) + e^{-r(T^*-t)} \mathbf{1}_{\tau_L > T^*} J_L(X_T)\right] - \mathbb{E}_t^x\left[\int_t^{\tau_L \wedge T^*} e^{-r(s-t)} \mathcal{A}V_*(X_s) ds\right], \end{aligned} \tag{IA.K.9}$$

where we have used $V_*(x) = J_L(x)$ in the second equality, $\mathcal{A}V$ is the infinitesimal generator given in (IA.K.1). Combining (C.2) with (IA.K.9), we obtain

$$\begin{aligned} &J_a(x; \varphi_a, \varphi_b^*, T^*) - V_*(x) \\ =& \mathbb{E}_t^x\Big[e^{-r(\tau_L - t)} \mathbf{1}_{\tau_L \leq T^*} \left[\mathbf{1}_{\widehat{\tau}_a < \tau_b}(L(X_{\tau_L}) - K_L) + \mathbf{1}_{\widehat{\tau}_a > \tau_b} F(X_{\tau_L}) - V_*(X_{\tau_L})\right]\Big] \\ &+ \mathbb{E}_t^x\left[\int_t^{\tau_L \wedge T^*} e^{-r(s-t)} \mathcal{A}V_*(X_s) ds\right] \\ =& \mathbb{E}_t^x\Big[e^{-r(\tau_L - t)} \mathbf{1}_{\tau_L \leq T^*}(L(X_{\tau_L}) - K_L - V_*(X_{\tau_L}))\Big] \\ &- \mathbb{E}_t^x\Big[\mathbf{1}_{\tau_L \leq T^*} \mathbf{1}_{\widehat{\tau}_a > \tau_b} e^{-r(\tau_L - t)}\big(L(X_{\tau_L}) - K_L - F(X_{\tau_L})\big)\Big] + \mathbb{E}_t^x\left[\int_t^{\tau_L \wedge T^*} e^{-r(s-t)} \mathcal{A}V_*(X_s) ds\right] \end{aligned}$$

$$\leq \mathbb{E}_t^x \left[\int_t^{\tau_L \wedge T^*} e^{-r(s-t)} \mathcal{A} V_*(X_s) ds - \mathbf{1}_{\tau_L \leq T^*} \mathbf{1}_{\widehat{\tau}_a > \tau_b} e^{-r(\tau_L - t)} \big(L(X_\tau) - K_L - F(X_{\tau_L}) \big) \right], \tag{IA.K.10}$$

where the first and second equalities follow from the property that $\mathbf{1}_{\widehat{\tau}_a = \tau_b} = 0$ almost surely, and the inequality is due to $L(x) - K_L \leq V_*(x)$ for all x. We can simplify the first term on the right side of (IA.K.10) as follows:

$$\begin{aligned}
&\mathbb{E}_t^x \left[\int_t^{\tau_L \wedge T^*} e^{-r(s-t)} \mathcal{A} V_*(X_s) ds \right] \\
=&\mathbb{E}_t^x \left[\int_t^{\infty} \int_t^{(\widehat{\tau}_a \wedge z) \wedge T^*} e^{-r(s-t)} \mathcal{A} V_*(X_s) \lambda^*(X_z) e^{-\int_t^z \lambda^*(X_u) du} ds dz \right] \\
=&\mathbb{E}_t^x \left[\int_t^{\widehat{\tau}_a \wedge T^*} \int_s^{\infty} \lambda^*(X_z) e^{-\int_t^z \lambda^*(X_u) du} dz e^{-r(s-t)} \mathcal{A} V_*(X_s) ds \right] \\
=&\mathbb{E}_t^x \left[\int_t^{\widehat{\tau}_a \wedge T^*} e^{-\int_t^s (r + \lambda^*(X_u)) du} \lambda^*(X_s) [L(X_s) - K_L - F(X_s)] ds \right] \\
=&\mathbb{E}_t^x \left[\mathbf{1}_{\tau_b \leq \widehat{\tau}_a \wedge T^*} e^{-r(\tau_b - t)} [L(X_{\tau_b}) - K_L - F(X_{\tau_b})] \right] \\
=&\mathbb{E}_t^x \left[\mathbf{1}_{\tau_L \leq T^*} \mathbf{1}_{\widehat{\tau}_a > \tau_b} e^{-r(\tau_b - t)} [L(X_{\tau_b}) - K_L - F(X_{\tau_b})] \right]
\end{aligned} \tag{IA.K.11}$$

where we have used $\mathcal{A} V_*(x) = \lambda^*(x)[L(x) - K_L - F(x)]$, T^* is a constant given $\{X_s : s \geq 0\}$, Tonelli's Theorem (to interchange the integration order in the second equality as $\mathcal{A} V_*(x) \leq 0$ and $\lambda^*(x) \geq 0$ for any $x > 0$), integration by parts, and (IA.K.3). Combining (IA.K.10) and (IA.K.11) yields $V_*(x) \geq J_a(x; \varphi_a, \varphi_b^*, T^*)$.

Step 2: Prove $V_*(x) = J_a(x; \varphi_a^*, \varphi_b^*, T^*)$ for $x > 0$.

Let τ_i^* be firm i's unconditional entry time from firm $-i$'s perspective, and let $\widehat{\tau}_i^*$ be the rational firm i's entry time associated with strategy $(\varphi_a^*, \varphi_b^*)$. Define $\tau_L^* := \min\{\widehat{\tau}_a^*, \tau_b^*\}$. Because $\lambda^*(x) = 0$ for any $x \notin \mathcal{R}^E$, we have $X_{\tau_L^*} \in \mathcal{R}^E$, where $\mathcal{R}^E = \{x > 0 : V_*(x) = L(x) - K_L\}$ is given in the proof of Theorem 1. It follows that $V_*(X_{\tau_L^*}) = L(X_{\tau_L^*}) - K_L$. We can then see that (IA.K.10) holds with equality if $\widehat{\tau}_a$, τ_b, and τ_L therein are set to $\widehat{\tau}_a^*$, τ_b^*, and τ_L^*, respectively. We have thus shown $V_*(x) = J_a(x; \varphi_a^*, \varphi_b^*, T^*)$.

In sum, we have proven (C.3) via Steps 1 and 2. Similar to Steps 1 and 2, we can also prove (C.4). □

Proof of the Uniqueness of Equilibrium in Theorem 2: We will prove the uniqueness of equilibrium strategy under Definition 4, with additional regularity requirement that $\lambda_i(x)$ is right-continuous in x. We use $(\varphi_a^*, \varphi_b^*) \in \Phi$ to denote an equilibrium strategy, where $\varphi_i^* = (\mathcal{E}_i^*, \lambda_i^*(x))$. We will show that $\mathcal{E}_i^* = \emptyset$ and $\lambda_i^*(x)$ is the same as $\lambda^*(x)$ given in Theorem 1. The proof is completed in three steps.

Step 1: We prove that $J_a(x;\varphi_a^*,\varphi_b^*,T^*) \geq V_*(x)$ *for any equilibrium strategy* $(\varphi_a^*,\varphi_b^*) \in \Phi$*, where* $T^* = T^{\varphi_a^*,\varphi_b^*}$.

Using $L(x) - K_L < F(x)$ for all $x > 0$ and that $V_*(x)$ is the equilibrium value in Theorem 1, we have

$$V_*(x) \leq F(x). \tag{IA.K.12}$$

Denote $\mathcal{R}^E = \{x > 0 : L(x) - K_L = V_*(x)\}$. Consider a deviation strategy of rational firm a: $\varphi_a = (\mathcal{E}_a, \lambda_a)$ with $\mathcal{E}_a = \mathcal{R}^E$ and $\lambda_a(x) \equiv 0$. Define $\widehat{\tau}_a = \inf\{s : X_s \in \mathcal{R}^E\}$. Using (C.2), $L(X_{\widehat{\tau}_a}) - K_L = V_*(X_{\widehat{\tau}_a})$, inequality (IA.K.12), and $J_L(x) = V_*(x)$ for any $x > 0$, we can see

$$\begin{aligned} J_a(x;\varphi_a,\varphi_b^*,T^*) &\geq \mathbb{E}_t^x\left[e^{-r(\tau_L - t)}V_*(X_{\tau_L})\right] \\ &= V_*(x) + \mathbb{E}_t^x\left[\int_t^{\tau_L} e^{-r(s-t)}\mathcal{A}V_*(X_s)ds\right] = V_*(x), \end{aligned} \tag{IA.K.13}$$

where τ_L is Leader's entry time under strategy (φ_a, φ_b^*), and the second equality uses $\mathcal{A}V_*(x) = 0$ for $x \notin \mathcal{R}^E$ and $X_s \notin \mathcal{R}^E$ for any $s < \widehat{\tau}_a$.

Using (C.3) and (IA.K.13), we have $J_a(x;\varphi_a^*,\varphi_b^*,T^*) \geq V_*(x)$.

Step 2: We prove that $T_a = T_b$ *for any equilibrium strategy* $(\varphi_a^*,\varphi_b^*) \in \Phi$*, where* $T_i = T_i^{\varphi_a^*,\varphi_b^*}$.

We first prove that $\mathcal{E}_i^* \subseteq \mathcal{R}^E$ and $\lambda_i^*(x) = 0$ for almost all $x \notin \mathcal{R}^E$. For the sake of contradiction, assume there exists an $\widehat{x} \in \mathcal{E}_i^* \cap (\mathcal{R}^E)^c$, or that $\lambda_i^*(x) > 0$ for any $x \in \mathbb{D}$, where $\mathbb{D}$ has positive measure and $\mathbb{D} \cap \mathcal{R}^E = \emptyset$. In the former case, we have $J_i(\widehat{x};\varphi_a^*,\varphi_b^*,T^*) = L(\widehat{x}) - K_L$. In the latter case, there exists an interval $(x_1, x_2) \subseteq \mathbb{D}$ such that $\lambda_i^*(x) > 0$ for any $x \in (x_1, x_2)$, implying that firm i is indifferent between entering and waiting at any $\widehat{x} \in (x_1, x_2)$ and $J_i(\widehat{x};\varphi_a^*,\varphi_b^*,T^*) = L(\widehat{x}) - K_L$. According to Lemma 3, we have $V_*(x) > L(x) - K_L$ for any $x \notin \mathcal{R}^E$. Combining this with $J_i(\widehat{x};\varphi_a^*,\varphi_b^*,T^*) = L(\widehat{x}) - K_L$, we have $V_*(\widehat{x}) > J_i(\widehat{x};\varphi_a^*,\varphi_b^*,T^*)$, which contradicts the result proved in Step 1. Hence, $\mathcal{E}_i^* \subseteq \mathcal{R}^E$ and $\lambda_i^*(x) = 0$ for almost all $x \notin \mathcal{R}^E$.

Next, we prove that either T_a or T_b is finite. For the sake of contradiction, assume there exists $x_1 > \overline{x}$, such that $x_1 \notin \mathcal{E}_i^*$ and $\lambda_{-i}^*(x_1) < \lambda^*(x_1)$. Since $\mathcal{E}_i^*$ is a closed set and both $\lambda_{-i}^*(x)$ and $\lambda^*(x)$ are right-continuous, there exists an interval $[x_1, x_2]$ with $x_2 > x_1$, such that $x \notin \mathcal{E}_i^*$ and $\lambda_{-i}^*(x) < \lambda^*(x)$ for any $x \in [x_1, x_2]$. Then, for any $x \in (x_1, x_2)$, firm i prefers to wait or enter probabilistically. In this region, the waiting benefit $\lambda_{-i}^*(x)(K_L - K_F)$ must be weakly higher than the waiting cost $Dx - rK_L$, which implies that $\lambda_{-i}^*(x) \geq \frac{Dx - rK_L}{K_L - K_F} = \lambda^*(x)$ for any $x \in (x_1, x_2)$. Then we arrive at a contradiction. Thus, we have $x \in \mathcal{E}_i^*$ or $\lambda_{-i}^*(x) \geq$

$\lambda^*(x) > \frac{D\overline{x}-rK_L}{K_L-K_F} > 0$ for any $x > \overline{x}$. If $\mathcal{E}_i^* \neq \emptyset$, then $\inf\{t \geq 0 : X_t \in \mathcal{E}_i^*\} < \infty$, and thus $T_i < \infty$. Thus, we only need to consider the case $\mathcal{E}_a^* = \emptyset$ and $\mathcal{E}_b^* = \emptyset$. In this case, we have $\lambda_{-i}^*(x) > \frac{D\overline{x}-rK_L}{K_L-K_F} > 0$ for any $x > \overline{x}$. It follows that $\int_0^t \lambda_{-i}^*(X_s)ds \geq \frac{D\overline{x}-rK_L}{K_L-K_F} \int_0^t \mathbf{1}_{X_s \geq \overline{x}} ds$, which together with $\int_0^\infty \mathbf{1}_{X_s \geq \overline{x}} ds = \infty$ implies that $T_{-i} = \inf\{t \geq 0 : \int_0^t \lambda_{-i}^*(X_u)du = -\ln \pi_0\}$ is finite almost surely. This proves that either T_a or T_b is finite.

For the sake of contradiction, assume $T_i < T_{-i}$. Then T_i is finite. Because $\mathcal{E}_i^* \subseteq \mathcal{R}^E$ and $\lambda_i^*(x) = 0$ for almost all $x \notin \mathcal{R}^E$, we derive from (A.2), definition of T_i, and the continuity of X_t that $X_{T_i} \in \mathcal{R}^E$. At time $t = T_i$, firm $-i$ has the belief that firm i is crazy for sure and will enter by solving the problem (31). Because $X_{T_i} \in \mathcal{R}^E$, a rational firm $-i$ will enter the market immediately at time T_i. This implies $\pi_{T_i}^{-i} = 1$ and thus $T_{-i} = \inf\{t \geq 0 : \pi_t^{-i} = 1\} \leq T_i$, which contradicts the assumption that $T_i < T_{-i}$. Therefore, we have $T_a = T_b < \infty$.

Step 3: Proof of uniqueness of equilibrium.

We first show that $\mathcal{E}_i^* = \emptyset$. For the sake of contradiction, assume $\mathcal{E}_a^* \neq \emptyset$ and consider an initial state $x \in \mathcal{E}_a^*$. Then we have $T_a = 0$. Using $T_a = T_b$, we have $T_b = 0$. Thus, two rational firms compete to enter the market at time 0. However, as $F(x) > L(x) - K_L$, firm i strictly prefers to become Follower and will deviate to wait for a while to be Follower. This leads to a contradiction. Therefore, $\mathcal{E}_a^* = \emptyset$. Similarly, we can prove $\mathcal{E}_b^* = \emptyset$.

Next, we prove $\lambda_a^*(x) = \lambda_b^*(x)$. For the sake of contradiction, assume $\{x > 0 : \lambda_a^*(x) > \lambda_b^*(x)\}$ have positive measure. Then, there exists an interval $(\widehat{x}_1, \widehat{x}_2)$, such that $\lambda_a^*(x) > \lambda_b^*(x)$ for any $x \in (\widehat{x}_1, \widehat{x}_2)$. Using the fact that $\lambda_i^*(x)$ is right-continuous in $x \in (\widehat{x}_1, \widehat{x}_2)$, there exists an $\epsilon > 0$ and an interval $(x_1, x_2) \subseteq (\widehat{x}_1, \widehat{x}_2)$, such that

$$\lambda_a^*(x) > \epsilon + \lambda_b^*(x), \quad x \in (x_1, x_2). \tag{IA.K.14}$$

Note that $\mathbb{P}\Big(X_t \in (x_1, x_2),\ \forall t \in [0, t_1]\Big) > 0$ for any constant $t_1 > 0$. Using $T_i = \inf\{t \geq 0 : \pi_t^i = 1\}$, (A.2), and (IA.K.14), we conclude that $T_a < T_b$ with positive probability, which contradicts the result proved in Step 2. Hence, $\{x > 0 : \lambda_a^*(x) > \lambda_b^*(x)\}$ has Lebesgue measure zero. Similarly, we can show that $\{x > 0 : \lambda_a^*(x) < \lambda_b^*(x)\}$ has Lebesgue measure zero. Hence, we have $\lambda_a^*(x) = \lambda_b^*(x)$ for almost every $x > 0$. Using the right-continuity of $\lambda_i^*(x)$, we conclude that $\lambda_a^*(x) = \lambda_b^*(x)$ for any $x > 0$.

Following the same proof in Appendix IA.A, we can show that rational firm i's equilibrium value $V_i(x) = V_*(x)$, is the unique solution to the variational inequality (18), and the equilibrium strategy $\lambda_i^*(x)$ is uniquely determined as follows: $\lambda_i^*(x) = 0$ when $V_*(x) > L(x) - K_L$ and $\lambda_i^*(x)$ is given in (19) when $V_*(x) = L(x) - K_L$. □

Proof of Theorem 3: Since we consider symmetric equilibrium, we use $J_i(x, k, K_L^i; \varphi_a, \varphi_b)$ to denote firm i's continuation value function at time t, as defined in (D.2), for a given

$X_t = x > 0$, $K_t = K_t^a = K_t^b = k$, and a feasible Markov strategy pair (φ_a, φ_b). In the following, we let $\varphi^*(k, \cdot) = \inf\{t \geq 0 : K_t^* \geq k\}$, where K_t^* satisfies (42).

We complete our proof in two steps.

Step 1: We prove that $V_a(x,k) \geq J_a(x,k,K_L^a;\varphi_a,\varphi^*)$ *where* $(\varphi_a,\varphi^*) \in \Phi$, $x = X_t$, $k = K_t^* = K_t^{a,\varphi_a^*,\varphi_b^*} = K_t^{b,\varphi_a^*,\varphi_b^*}$, *and* $V_i(x,k)$ *is characterized in Appendix E.3.*

Let τ_a and τ_b be firm a's and b's stochastic entry times associated with the strategy pair (φ_a, φ^*), respectively. Define $\tau := \min\{\tau_a, \tau_b\}$. Let $\mathcal{B}V(x,k)$ denote the infinitesimal generator operating on a function $V(x,k)$:

$$\mathcal{B}V(x,k) = \frac{\sigma^2 x^2}{2}\frac{\partial^2 V(x,k)}{\partial x^2} + \mu x \frac{\partial V(x,k)}{\partial x} - rV(x,k) + \frac{1-\Psi(k)}{\Psi'(k)}\Lambda^*(x,k)\frac{\partial V(x,k)}{\partial k}, \tag{IA.K.15}$$

where $\Lambda^*(x,k) = \lambda^*(x;k)$ is the same as the equilibrium entry rate in Theorem 1 but with K_L replaced by k. Using (D.10), we can see that for $k \geq K_L^i$, we have

$$\begin{aligned}&\mathcal{B}V_i(x,k) + \Lambda^*(x,k)[F(x) - V_i(x,k)] = \mathcal{B}V_*(x;K_L^i) + \lambda^*(x;k)[F(x) - V_*(x;K_L^i)]\\ =&[\lambda^*(x;k) - \lambda^*(x;K_L^i)][F(x) - V_*(x;K_L^i)] \leq 0,\end{aligned} \tag{IA.K.16}$$

where the second equality uses $\lambda^*(x;K_L^i) = \frac{rV_*(x;K_L^i) - \left[\frac{\sigma^2 x^2}{2}\frac{\partial^2 V_*(x;K_L^i)}{\partial x^2} + \mu x \frac{\partial V_*(x;K_L^i)}{\partial x}\right]}{F(x) - V_*(x;K_L^i)}$ and $\frac{\partial V_*(x;K_L^i)}{\partial k} = 0$, and the inequality follows from the fact that $\lambda^*(x;k)$ is decreasing in k (see Lemma 6 in Internet Appendix IA.J).

Applying Itô's Lemma to $e^{-rs}V_a(X_s, K_s^*)$ for $s \in [t, \tau]$ and taking expectations at time t with $X_t = x$ and $K_t^* = k$, we obtain the following expression for $V_a(x,k)$:

$$V_a(x,k) = \mathbb{E}_t^{x,k}[e^{-r(\tau-t)}V_a(X_\tau, K_\tau^*)] - \mathbb{E}_t^{x,k}\left[\int_t^\tau e^{-r(s-t)}\mathcal{B}V_a(X_s, K_s^*)ds\right], \tag{IA.K.17}$$

where $\mathcal{B}V_a(x,k)$ is defined in (IA.K.15). Given that no firm enters before time t, the CDF of τ_b from firm a's perspective is given by

$$\begin{aligned}&\mathbb{P}_s(\tau_b \leq s \mid \tau_b \geq t) = \frac{\mathbb{P}_s\big(\tau_b \in [t,s]\big)}{\mathbb{P}_s(\tau_b \geq t)}\\ =&\frac{e^{-\int_0^t \Lambda^*(X_u,K_u^*)du} - e^{-\int_0^s \Lambda^*(X_u,K_u^*)du}}{e^{-\int_0^t \Lambda^*(X_u,K_u^*)du}} = 1 - e^{-\int_t^s \Lambda^*(X_u,K_u^*)du}\end{aligned} \tag{IA.K.18}$$

for $s \geq t$, where $\mathbb{P}_s$ is time-s probability that only depends on $\{X_z\}_{z\in[0,s]}$, the second equality uss (43). Then, we can simplify the second term on the right side of (IA.K.17) as follows:

$$\begin{aligned}
&\mathbb{E}_t^{x,k}\left[\int_t^{\tau} e^{-r(s-t)}\mathcal{B}V_a(X_s,K_s^*)ds\right]\\
=&\mathbb{E}_t^{x,k}\left[\int_t^{\infty}\int_t^{\tau_a\wedge z} e^{-r(s-t)}\mathcal{B}V_a(X_s,K_s^*)\Lambda^*(X_z,K_z^*)e^{-\int_t^z \Lambda^*(X_u,K_u^*)du}dsdz\right]\\
=&\mathbb{E}_t^{x,k}\left[\int_t^{\tau_a}\int_s^{\infty} \Lambda^*(X_z,K_z^*)e^{-\int_t^z \Lambda^*(X_u,K_u^*)du}dze^{-r(s-t)}\mathcal{B}V_a(X_s,K_s^*)ds\right]\\
=&\mathbb{E}_t^{x,k}\left[\int_t^{\tau_a} e^{-\int_t^s(r+\Lambda^*(X_u,K_u^*))du}\mathcal{B}V_a(X_s,K_s^*)ds\right]\\
\leq&\mathbb{E}_t^{x,k}\left[\int_t^{\tau_a} e^{-\int_t^s(r+\Lambda^*(X_u,K_u^*))du}\Lambda^*(X_s,K_s^*)[V_a(X_s,K_s^*)-F(X_s)]ds\right]\\
=&\mathbb{E}_t^{x,k}\left[\mathbf{1}_{\tau_a>\tau_b}e^{-r(\tau_b-t)}[V_a(X_{\tau_b},K_{\tau_b}^*)-F(X_{\tau_b})]\right],
\end{aligned}\tag{IA.K.19}$$

where the inequality follows from the fact that $\mathcal{B}V_a(x,k)+\Lambda^*(x,k)[F(x)-V_a(x,k)]\leq 0$, as implied by equality (D.11) for $k<K_L^i$ and inequality (IA.K.16) for $k\geq K_L^i$. Plugging (IA.K.19) into the right side of (IA.K.17) and using $\mathbf{1}_{\tau_a=\tau_b}=0$ almost surely, we have

$$V_a(x,k)\geq\mathbb{E}_t^{x,k}[e^{-r(\tau-t)}V_a(X_{\tau_a},K_{\tau_a}^*)\mathbf{1}_{\tau_a<\tau_b}+e^{-r(\tau-t)}F(X_{\tau_b})\mathbf{1}_{\tau_a>\tau_b}].\tag{IA.K.20}$$

Since it is suboptimal for a firm to enter as Leader, we have $V_a(x,k)\geq L(x)-K_L^a$, $\forall x>0$, $k\in[\underline{k},\overline{k}]$. Substituting $V_a(X_{\tau_a},K_{\tau_a}^*)\geq L(X_{\tau_a})-K_L^a$ into the right side of (IA.K.20), we obtain

$$V_a(x,k)\geq\mathbb{E}_t^{x,k}[e^{-r(\tau-t)}(L(X_{\tau_a})-K_L^a)\mathbf{1}_{\tau_a<\tau_b}+e^{-r(\tau-t)}F(X_{\tau_b})\mathbf{1}_{\tau_a>\tau_b}]=J_a(x,k,K_L^a;\varphi_a,\varphi^*),\tag{IA.K.21}$$

where the equality follows from the property that $\mathbf{1}_{\tau_a=\tau_b}=0$ almost surely.

Step 2: We prove that $V_a(x,k)=J_a(x,k,K_L^a;\varphi^*,\varphi^*)$.

Recall that τ_a^* and τ_b^* are the stochastic entry times of firms a and b, respectively, associated with strategy (φ^*,φ^*), and $\tau^*=\min\{\tau_a^*,\tau_b^*\}$. Because $\mathcal{B}V_a(x,k)+\Lambda^*(x,k)[F(x)-V_a(x,k)]=0$ for $k<K_L^a$, and noting that $K_s^*<K_L^a$ for $s<\tau_a^*$, we can see that both (IA.K.19) and (IA.K.20) hold with equality if τ_a, τ_b, and τ are set to τ_a^*, τ_b^*, and τ^*, respectively. Since $\Lambda^*(x,k)=0$ for any $x\notin\mathcal{R}^E(k)$, where $\mathcal{R}^E(k)$ is the mixed entry region given in Theorem 1 but with K_L replaced by k, we have $dK_t^*=0$ for $X_t\notin\mathcal{R}^E(K_t^*)$. Hence, we conclude from (D.5) and the continuity of (X_t,K_t^*) in t that $X_{\tau_a^*}\in\mathcal{R}^E(K_L^a)$ and $K_{\tau_a^*}^*=K_L^a$, which implies that $V_a(X_{\tau_a^*},K_{\tau_a^*}^*)=V_a(X_{\tau_a^*},K_L^a)=L(X_{\tau_a^*})-K_L^a$. Therefore, we can see that (IA.K.21) holds with equality if τ_a, τ_b, and τ are set to τ_a^*, τ_b^*, and τ^*, respectively. We have thus shown that $V_a(x,k)=J_a(x,k,K_L^a;\varphi^*,\varphi^*)$.

Combining our analyses in Steps 1 and 2, we obtain $J_a(x, k, K_L^a; \varphi^*, \varphi^*) \geq J_a(x, k, K_L^a; \varphi_a, \varphi^*)$. By symmetry, we also have $J_b(x, k, K_L^b; \varphi^*, \varphi^*) \geq J_b(x, k, K_L^b; \varphi^*, \varphi_b)$ for $(\varphi^*, \varphi_b) \in \Phi$. $\square$

Proof of Theorem 4: Since we consider symmetric equilibrium, we use $J_i(x, q, Q_i; \varphi_a, \varphi_b)$ to denote firm i's continuation value function at time t, as defined in (E.2), for a given $X_t = x > 0$, $\mathbf{Q}_t^a = \mathbf{Q}_t^b = q$, and a feasible Markov strategy pair (φ_a, φ_b). In the following, we let $\varphi_i^*(Q_i, \cdot) = \inf\{t \geq 0 : Q_t^* \leq Q_i\}$, where $\mathbf{Q}_t^*$ satisfies (52) and $\Lambda^*(x, q)$ is given in Theorem 4.

We complete our proof in two steps.

Step 1: We prove $V_a(x, q) \geq J_a(x, q, Q_a; \varphi_a, \varphi_b^*)$ where $(\varphi_a, \varphi_b^*) \in \Phi$, $x = X_t$, $q = \mathbf{Q}_t^* = Q_t^{a,\varphi_a^*,\varphi_b^*} = Q_t^{b,\varphi_a^*,\varphi_b^*}$, $V_i(x, q)$ is characterized in Appendix E.

Let τ_a and τ_b be firm a's and b's stochastic entry time associated with the strategy pair (φ_a, φ_b^*), and let $\tau_L := \min\{\tau_a, \tau_b\}$.

Let $\mathcal{B}V(x, q)$ denote the infinitesimal generator operating on a function $V(x, q)$:

$$\mathcal{B}V(x, q) = \mathcal{A}V(x, q) - \frac{\Psi(q)}{\Psi'(q)}\Lambda^*(x, q)\frac{\partial V(x, q)}{\partial q}, \qquad \text{(IA.K.22)}$$

where $\mathcal{A}V(x, q) = \frac{\sigma^2 x^2}{2}\frac{\partial^2 V(x,q)}{\partial x^2} + \mu x \frac{\partial V(x,q)}{\partial x} - rV(x, q)$, and $\Lambda^*(x, q)$ is given in Theorem 4. From equation (E.16), we have

$$\mathcal{B}V_i(x, q) + \Lambda^*(x, q)\Big[pM\Big(\frac{Q_i + q}{2}x\Big) - V_i(x, q)\Big] = 0 \qquad \text{(IA.K.23)}$$

for any $q > Q_i$ and $x > 0$.

Because $\Lambda^*(x, q) = 0$ for $x < \frac{\beta}{\beta-1}(r - \mu)\frac{K}{p\mathcal{I}(q)}$, and $\mathcal{I}(q) < \mathcal{W}(Q_i, q)$ for $q < Q_i$, it follows that for any $x < \frac{\beta}{\beta-1}(r - \mu)\frac{K}{p\mathcal{W}(Q_i,q)}$ and $q < Q_i$, we have $\Lambda^*(x, q) = 0$, and

$$\begin{aligned}&\mathcal{B}V_i(x, q) + \Lambda^*(x, q)\Big[pM\Big(\frac{Q_i + q}{2}x\Big) - V_i(x, q)\Big]\\ =&\mathcal{A}J_L(x; Q_i, q) = p\mathcal{W}(Q_i, q)\mathcal{A}M^*\left(x; \frac{K}{p\mathcal{W}(Q_i, q)}\right) = 0,\end{aligned} \qquad \text{(IA.K.24)}$$

where the first equality uses (E.12), the second equality uses (E.12), the last equality uses (E.13).

For any $x > \frac{\beta}{\beta-1}(r - \mu)\frac{K}{p\mathcal{W}(Q_i,q)}$ and $q < Q_i$,

$$\begin{aligned}&\mathcal{B}V_i(x, q) + \Lambda^*(x, q)\Big[pM\Big(\frac{Q_i + q}{2}x\Big) - V_i(x, q)\Big]\\ =&\mathcal{A}(p\mathcal{W}(Q_i, q)\Pi(x) - K) - \frac{\Psi(q)}{\Psi'(q)}\Lambda^*(x, q)p\Pi(x)\frac{\partial \mathcal{W}(Q_i, q)}{\partial q}\end{aligned}$$

$$
\begin{aligned}
&+ \Lambda^*(x,q)\Big[pM\Big(\frac{Q_i+q}{2}x\Big) - V_i(x,q)\Big] \\
=&rK - p\mathcal{W}(Q_i,q)x - \Lambda^*(x,q)p\Pi(x)\Big[\frac{Q_i+q}{2} - \mathcal{W}(Q_i,q)\Big] \\
&+ \Lambda^*(x,q)\Big[p\Big(\frac{Q_i+q}{2}\Pi(x) - K\Big) - (p\mathcal{W}(Q_i,q)\Pi(x) - K)\Big] \\
=&rK - p\mathcal{W}(Q_i,q)x + \Lambda^*(x,q)(1-p)K,
\end{aligned} \tag{IA.K.25}
$$

where the first equality uses (E.12) and (E.14), the second equality uses $\frac{\partial \mathcal{W}(Q_i,q)}{\partial q} = \frac{\partial \int_{\underline{q}}^{q} \frac{Q_i+z}{2}\frac{d\Psi(z)}{\Psi(q)}}{\partial q} = \frac{\Psi'(q)}{\Psi(q)}\Big[\frac{Q_i+q}{2} - \mathcal{W}(Q_i,q)\Big]$, $\frac{Q_i+q}{2} > \mathcal{W}(Q_i,q)$, $M(x) = M^*(x;K)$ and (E.14).

Since $q < Q_i$, we have $\mathcal{I}(q) < \mathcal{W}(Q_i,q)$ and $\frac{\beta}{\beta-1}(r-\mu)\frac{K}{p\mathcal{W}(Q_i,q)} < \frac{\beta}{\beta-1}(r-\mu)\frac{K}{p\mathcal{I}(q)}$. For $x \in (\frac{\beta}{\beta-1}(r-\mu)\frac{K}{p\mathcal{W}(Q_i,q)}, \frac{\beta}{\beta-1}(r-\mu)\frac{K}{p\mathcal{I}(q)})$, $\Lambda^*(x,q) = 0$ and (IA.K.25) is given by

$$
rK - p\mathcal{W}(Q_i,q)x + \Lambda^*(x,q)(1-p)K = rK - p\mathcal{W}(Q_i,q)x \leq rK - \frac{\beta}{\beta-1}(r-\mu)K < 0. \tag{IA.K.26}
$$

For $x \geq \frac{\beta}{\beta-1}(r-\mu)\frac{K}{p\mathcal{I}(q)}$, $\Lambda^*(x,q) = \frac{p\mathcal{I}(q)x - rK}{(1-p)K}$ and (IA.K.25) is given by

$$
px[\mathcal{I}(q) - \mathcal{W}(Q_i,q)] < 0. \tag{IA.K.27}
$$

Combining (IA.K.24), (IA.K.26) and (IA.K.27), we obtain

$$
\mathcal{B}V_i(x,q) + \Lambda^*(x,q)\Big[pM\Big(\frac{Q_i+q}{2}x\Big) - V_i(x,q)\Big] \leq 0 \tag{IA.K.28}
$$

for any $q < Q_i$.

Applying Itô's Lemma to $e^{-rs}V_a(X_s, Q_s^*)$ for $s \in [t, \tau_L]$ and taking expectations at time t with $X_t = x$ and $\mathbf{Q}_t^* = q$, we obtain the following expression for $V_a(x,q)$:

$$
V_a(x,q) = \mathbb{E}_t^{x,q}[e^{-r(\tau_L - t)}V_a(X_{\tau_L}, \mathbf{Q}_{\tau_L}^*)] - \mathbb{E}_t^{x,q}\left[\int_t^{\tau_L} e^{-r(s-t)}\mathcal{B}V_a(X_s, Q_s^*)ds\right]. \tag{IA.K.29}
$$

Given that no firm enters before time t, the CDF of τ_b from firm a's perspective is given by

$$
\begin{aligned}
&\mathbb{P}_s(\tau_b \leq s \mid \tau_b \geq t) = \frac{\mathbb{P}_s\big(\tau_b \in [t,s]\big)}{\mathbb{P}_s(\tau_b \geq t)} \\
=&\frac{e^{-\int_0^t \Lambda^*(X_u,Q_u^*)du} - e^{-\int_0^s \Lambda^*(X_u,Q_u^*)du}}{e^{-\int_0^t \Lambda^*(X_u,Q_u^*)du}} = 1 - e^{-\int_t^s \Lambda^*(X_u,Q_u^*)du}
\end{aligned} \tag{IA.K.30}
$$

for $s \geq t$, where $\mathbb{P}_s$ is time-s probability that only depends on $\{X_z\}_{z\in[0,s]}$.

Using (IA.K.30), we can simplify the second term on the right side of (IA.K.29) as follows:

$$
\begin{aligned}
&\mathbb{E}_t^{x,q}\left[\int_t^{\tau_L} e^{-r(s-t)}\mathcal{B}V_a(X_s,Q_s^*)ds\right]\\
=&\mathbb{E}_t^{x,q}\left[\int_t^{\infty}\int_t^{\tau_a\wedge z} e^{-r(s-t)}\mathcal{B}V_a(X_s,Q_s^*)\Lambda^*(X_z,Q_z^*)e^{-\int_t^z\Lambda^*(X_u,Q_u^*)du}dsdz\right]\\
=&\mathbb{E}_t^{x,q}\left[\int_t^{\tau_a}\int_s^{\infty}\Lambda^*(X_z,Q_z^*)e^{-\int_t^z\Lambda^*(X_u,Q_u^*)du}dze^{-r(s-t)}\mathcal{B}V_a(X_s,Q_s^*)ds\right]\\
=&\mathbb{E}_t^{x,q}\left[\int_t^{\tau_a} e^{-\int_t^s(r+\Lambda^*(X_u,Q_u^*))du}\mathcal{B}V_a(X_s,Q_s^*)ds\right]\\
\leq&\mathbb{E}_t^{x,q}\left[\int_t^{\tau_a} e^{-\int_t^s(r+\Lambda^*(X_u,Q_u^*))du}\Lambda^*(X_s,Q_s^*)\Big[V_a(X_s,Q_s^*)-pM\Big(\frac{Q_a+Q_s^*}{2}X_s\Big)\Big]ds\right]\\
=&\mathbb{E}_t^{x,q}\left[\mathbf{1}_{\tau_a>\tau_b}e^{-r(\tau_b-t)}[V_a(X_{\tau_b},Q_{\tau_b}^*)-F_{\tau_b}(X_{\tau_b},Q_a)]\right], \qquad \text{(IA.K.31)}
\end{aligned}
$$

where the inequality follows from that $\mathcal{B}V_a(x,q)+\Lambda^*(x,q)\big[pM\big(\frac{Q_i+q}{2}x\big)-V_a(x,q)\big]\leq 0$, as implied by equality (IA.K.23) for $q>Q_i$ and inequality (IA.K.28) for $q<Q_i$, the last equality uses (49) and (E.15). Plugging (IA.K.31) into the right side of (IA.K.29) and using $\mathbf{1}_{\tau_a=\tau_b}=0$ almost surely, we have

$$V_a(x,q)\geq\mathbb{E}_t^{x,q}[e^{-r(\tau_L-t)}V_a(X_{\tau_L},Q_{\tau_L}^*)\mathbf{1}_{\tau_a<\tau_b}+e^{-r(\tau_L-t)}F_{\tau_L}(X_{\tau_L},Q_a)\mathbf{1}_{\tau_a>\tau_b}]. \qquad \text{(IA.K.32)}$$

As it is suboptimal for firm to enter as Leader, we have $V_a(X_{\tau_L},Q_{\tau_L}^*)\geq J_L(X_{\tau_L};Q_a,Q_{\tau_L}^*)\geq L_{\tau_L}(X_{\tau_L},Q_a)-K$. Substituting it into the right side of (IA.K.32), we obtain the following inequality:

$$
\begin{aligned}
V_a(x,q)&\geq\mathbb{E}_t^{x,q}\Big[e^{-r(\tau_L-t)}\Big(L_{\tau_L}(X_{\tau_L},Q_a)-K\Big)\mathbf{1}_{\tau_a<\tau_b}+e^{-r(\tau_L-t)}F_{\tau_L}(X_{\tau_L},Q_a)\mathbf{1}_{\tau_a>\tau_b}\Big]\\
&=J_a(x,q,Q_a;\varphi_a,\varphi_b^*), \qquad \text{(IA.K.33)}
\end{aligned}
$$

where equality follows from the property: $\mathbf{1}_{\tau_a=\tau_b}=0$ almost surely.

Step 2: We prove $V_a(x,q)=J_a(x,q,Q_a;\varphi_a^*,\varphi_b^*)$.

Recall that τ_a^* and τ_b^* are firms a's and b's stochastic entry time, respectively, associated with strategy $(\varphi_a^*,\varphi_b^*)$, and $\tau_L^*:=\min\{\tau_a^*,\tau_b^*\}$. By applying (IA.K.23) for $i=a$ and $q>Q_a$, and noting that $Q_s^*>Q_a$ for $s<\tau_a^*$, we can see that both of (IA.K.31), (IA.K.32) hold with equality if τ_a, τ_b and τ_L therein are set to τ_a^*, τ_b^* and τ_L^*. Because $\Lambda^*(x,q)=0$ for any $x<\overline{x}(q)=\frac{\beta}{\beta-1}(r-\mu)\frac{K}{p\mathcal{I}(q)}$, we have $d\mathbf{Q}_t^*=0$ for $X_t<\overline{x}(\mathbf{Q}_t^*)$. Hence, we conclude from $\tau_i^*=\inf\{t:\mathbf{Q}_t^*\leq Q_i\}$ and the continuity of $(X_t,\mathbf{Q}_t^*)$ in t that $X_{\tau_a^*}\geq\overline{x}(Q_{\tau_a^*}^*)$ and $Q_a=Q_{\tau_a^*}^*>Q_b$ if $\tau_a^*<\tau_b^*$. It follows that $V_a(X_{\tau_a^*},Q_{\tau_a^*}^*)=V_a(X_{\tau_a^*},Q_a)=J_L(X_{\tau_a^*};Q_a,Q_a)=L_{\tau_L^*}(X_{\tau_L^*},Q_a)-K$ if $\tau_a^*<\tau_b^*$. Therefore, we can see that (IA.K.33) holds with equality if τ_a, τ_b and τ_L therein are set to τ_a^*, τ_b^* and τ_L^*, respectively. We have thus shown $V_a(x,q) = J_a(x,q,Q_a;\varphi_a^*,\varphi_b^*)$.

In sum, combining our analyses in Steps 1 and 2, we can obtain $J_a(x,q,Q_a;\varphi_a^*,\varphi_b^*) \geq J_a(x,q,Q_a;\varphi_a,\varphi_b^*)$. By symmetry, we also have $J_b(x,q,Q_b;\varphi_a^*,\varphi_b^*) \geq J_b(x,q,Q_b;\varphi_a^*,\varphi_b)$ for $(\varphi_a^*,\varphi_b) \in \Phi$. $\square$

Proof of Theorem 5: Let $J_i(x,q_{-i},Q_i;\varphi_a,\varphi_b)$ denote firm i's continuation value function at time t for a given $X_t = x > 0$, $\mathbf{Q}_t^{-i*} = q_{-i}$, and a feasible Markov strategy pair (φ_a,φ_b). In the following, we let $\varphi_i^*(Q_i,\cdot) = \inf\{t \geq 0 : \mathbf{Q}_t^i \leq Q_i\}$, where $\mathbf{Q}_t^a$ satisfies (F.4)-(F.5) and $\mathbf{Q}_t^b$ satisfies (F.6).

We complete our proof in two steps.

Step 1: We prove $V_a(x,q_b) \geq J_a(x,q_b,Q_a;\varphi_a,\varphi_b^*)$ where $(\varphi_a,\varphi_b^*) \in \Phi$, $x = X_t$, $q_b = \mathbf{Q}_t^b$, $Q_a < \zeta^{-1}(\overline{q})$, $V_a(x,q_b)$ is characterized in Appendix F.

Let τ_a and τ_b be firm a's and b's stochastic entry time associated with the strategy pair (φ_a,φ_b^*), and let $\tau_L := \min\{\tau_a,\tau_b\}$.

Let $\mathcal{B}_iV(x,q)$ denote the infinitesimal generator operating on a function $V(x,q)$:

$$\mathcal{B}_iV(x,q) = \mathcal{A}V(x,q) - \frac{\Psi(q)}{\Psi'(q)}\Lambda_{-i}^*(x,q)\frac{\partial V(x,q)}{\partial q}, \tag{IA.K.34}$$

where $\mathcal{A}V(x,q) = \frac{\sigma^2x^2}{2}\frac{\partial^2 V(x,q)}{\partial x^2} + \mu x\frac{\partial V(x,q)}{\partial x} - rV(x,q)$, and $\Lambda_a^*(x,q_a)$ and $\Lambda_b^*(x,q_b)$ are given in (IA.I.10) and (IA.I.9), respectively. Since $Q_0^b = \overline{q}$, we have $X_t < \widetilde{x}\big(\frac{K}{pN_a\mathcal{W}(\zeta^{-1}(\overline{q}),\overline{q})}\big)$ and $\Lambda_b^*(X_t,\overline{q}) = 0$, for any $t < T_a$. Thus, (F.6) is equivalent to

$$d\mathbf{Q}_t^b = \frac{-\Psi(\mathbf{Q}_t^b)}{\Psi'(\mathbf{Q}_t^b)}\Lambda_b^*(X_t,\mathbf{Q}_t^b)dt, \quad t \geq 0, \text{ and } \quad Q_0^b = \overline{q}. \tag{IA.K.35}$$

Then equation (IA.I.3) holds for any $q_b > \zeta(Q_a)$ and $x > 0$, and thus

$$\mathcal{B}_aV_a(x,q_b) + \Lambda_b^*(x,q_b)\Big[pM\Big(N_a\frac{Q_a+q_b}{2}x\Big) - V_a(x,q_b)\Big] = 0 \tag{IA.K.36}$$

for any $q_b > \zeta(Q_a)$ and $x > 0$.

Because $\Lambda_b^*(x,q_b) = 0$ for $x < \widetilde{x}\big(\frac{K}{pN_a\mathcal{W}(\zeta^{-1}(q_b),q_b)}\big)$, and $\mathcal{W}(\zeta^{-1}(q_b),q_b) < \mathcal{W}(Q_a,q_b)$ for $q_b < \zeta(Q_a)$, it follows that for any $x < \widetilde{x}\big(\frac{K}{pN_a\mathcal{W}(Q_a,q_b)}\big)$ and $q_b < \zeta(Q_a)$, we have $\Lambda^*(x,q_b) = 0$, and

$$\begin{aligned}&\mathcal{B}_aV_a(x,q_b) + \Lambda_b^*(x,q_b)\Big[pM\Big(N_a\frac{Q_a+q_b}{2}x\Big) - V_a(x,q_b)\Big]\\ =&\mathcal{A}J_L^a(x;Q_a,q_b) = pN_a\mathcal{W}(Q_a,q_b)\mathcal{A}M^*\left(x;\frac{K}{pN_a\mathcal{W}(Q_a,q_b)}\right) = 0,\end{aligned} \tag{IA.K.37}$$

where the first and second equalities uses (IA.I.2), the last equality uses (E.13).

For any $x > \widetilde{x}\big(\frac{K}{pN_a\mathcal{W}(Q_a,q_b)}\big)$ and $q_b < \zeta(Q_a)$,

$$\begin{aligned}
&\mathcal{B}_a V_a(x,q_b) + \Lambda_b^*(x,q_b)\Big[pM\Big(N_a\frac{Q_a+q_b}{2}x\Big) - V_a(x,q_b)\Big]\\
=&\mathcal{A}(pN_a\mathcal{W}(Q_a,q_b)\Pi(x)-K) - \frac{\Psi(q_b)}{\Psi'(q_b)}\Lambda_b^*(x,q_b)pN_a\Pi(x)\frac{\partial\mathcal{W}(Q_a,q_b)}{\partial q_b}\\
&+\Lambda_b^*(x,q_b)\Big[pM\Big(N_a\frac{Q_a+q_b}{2}x\Big) - V_a(x,q_b)\Big]\\
=&rK - pN_a\mathcal{W}(Q_a,q_b)x - \Lambda_b^*(x,q_b)pN_a\Pi(x)\Big[\frac{Q_a+q_b}{2} - \mathcal{W}(Q_a,q_b)\Big]\\
&+\Lambda_b^*(x,q_b)\Big[p\Big(N_a\frac{Q_a+q_b}{2}\Pi(x)-K\Big) - (pN_a\mathcal{W}(Q_a,q_b)\Pi(x)-K)\Big]\\
=&rK - pN_a\mathcal{W}(Q_a,q_b)x + \Lambda_b^*(x,q_b)(1-p)K, \qquad \text{(IA.K.38)}
\end{aligned}$$

where the first equality uses (IA.I.2) and (E.14), and the second equality uses $\frac{\partial\mathcal{W}(Q_a,q)}{\partial q} = \frac{\partial\int_{\underline{q}}^{q}\frac{Q_a+z}{2}\frac{d\Psi(z)}{\Psi(q)}}{\partial q} = \frac{\Psi'(q)}{\Psi(q)}\Big[\frac{Q_a+q}{2} - \mathcal{W}(Q_a,q)\Big]$, $\frac{Q_a+q_b}{2} > \mathcal{W}(Q_a,q_b)$.

As $q_b < \zeta(Q_a)$, we have $\mathcal{W}(\zeta^{-1}(q_b),q_b) < \mathcal{W}(Q_a,q_b)$ and $\widetilde{x}\big(\frac{K}{pN_a\mathcal{W}(Q_a,q_b)}\big) < \widetilde{x}\big(\frac{K}{pN_a\mathcal{W}(\zeta^{-1}(q_b),q_b)}\big)$. When $\widetilde{x}\big(\frac{K}{pN_a\mathcal{W}(Q_a,q_b)}\big) < x < \widetilde{x}\big(\frac{K}{pN_a\mathcal{W}(\zeta^{-1}(q_b),q_b)}\big)$, $\Lambda_b^*(x,q_b)=0$ and (IA.K.38) becomes

$$rK - pN_a\mathcal{W}(Q_a,q_b)x \le rK - \frac{\beta}{\beta-1}(r-\mu)K < 0. \qquad \text{(IA.K.39)}$$

For $x \ge \widetilde{x}\big(\frac{K}{pN_a\mathcal{W}(\zeta^{-1}(q_b),q_b)}\big)$, $\Lambda_b^*(x,q_b) = \frac{pN_a\mathcal{W}(\zeta^{-1}(q_b),q_b)x-rK}{(1-p)K}$ and (IA.K.38) becomes

$$pN_a x[\mathcal{W}(\zeta^{-1}(q_b),q_b) - \mathcal{W}(Q_a,q_b)] < 0. \qquad \text{(IA.K.40)}$$

Combining (IA.K.37), (IA.K.39) and (IA.K.40), we obtain

$$\mathcal{B}_a V_a(x,q_b) + \Lambda_b^*(x,q_b)\Big[pM\Big(N_a\frac{Q_a+q_b}{2}x\Big) - V_a(x,q_b)\Big] \le 0 \qquad \text{(IA.K.41)}$$

for any $q_b < \zeta(Q_a)$. Following the proofs of (IA.K.29)-(IA.K.33), using (IA.K.36), (IA.K.41), and $V_a(X_{\tau_L}, Q^a_{\tau_L}) \ge J^a_L(X_{\tau_L}; Q_a, Q^b_{\tau_L}) \ge L^a_{\tau_L}(X_{\tau_L}, Q_a) - K$, we can obtain

$$\begin{aligned}
V_a(x,q_b) &\ge \mathbb{E}_t^{x,q_b}\Big[e^{-r(\tau_L-t)}\Big(L^a_{\tau_L}(X_{\tau_L},Q_a)-K\Big)\mathbf{1}_{\tau_a<\tau_b} + e^{-r(\tau_L-t)}F^a_{\tau_L}(X_{\tau_L},Q_a)\mathbf{1}_{\tau_a>\tau_b}\Big]\\
&= J_a(x,q_b,Q_a;\varphi_a,\varphi_b^*), \qquad \text{(IA.K.42)}
\end{aligned}$$

In addition, following Step 2 in the proof of Theorem 4, we can show that $V_a(x,q_b) = J_a(x,q_b,Q_a;\varphi_a^*,\varphi_b^*)$.

Step 2: We prove $V_a(x, q_b) \geq J_a(x, q_b, Q_a; \varphi_a, \varphi_b^*)$ where $(\varphi_a, \varphi_b^*) \in \Phi$, $x = X_t$, $q_b = \mathbf{Q}_t^b$, $Q_a \in [\zeta^{-1}(\overline{q}), \overline{q}]$.

Since $Q_0^b = \overline{q}$, we have $X_t < \widetilde{x}\big(\frac{K}{pN_a\mathcal{W}(\zeta^{-1}(\overline{q}),\overline{q})}\big)$ and $\Lambda_b^*(X_t, \overline{q}) = 0$, for any $t < T_a$. It follows that $dQ_t^b = 0$ and $\mathbf{Q}_t^b = \overline{q}$ for any $t < T_a$. Thus, firm a knows that firm b will wait during the period $t < T_a$. Moreover, since $Q_a \geq \zeta^{-1}(\overline{q}) \geq \zeta^{-1}(\mathbf{Q}_t^{b*})$, and following the argument in Step 1, we can show it is optimal for firm a to enter as Leader by solving (IA.I.1) at time $t \geq T_a$, and equilibrium value

$$J_a(x, q_b, Q_a; \varphi_a^*, \varphi_b^*) = V_a(x, q_b) = J_L^a(x; Q_a, \mathbf{Q}_t^{b*}) \geq J_a(x, q_b, Q_a; \varphi_a, \varphi_b^*) \qquad \text{(IA.K.43)}$$

for $q_b < \overline{q}$ and $Q_a \in [\zeta^{-1}(\overline{q}), \overline{q}]$, where $J_L^a(x; Q_a, q_b)$ is given in (IA.I.1).

Since $\mathbf{Q}_t^b$ is non-increasing in t and firm a will enter as Leader at time $t \geq T_a$, it is optimal for firm a to enter as Leader by solving (IA.I.1) at time $t < T_a$. Solving (IA.I.1) with $\mathbf{Q}_t^b = \overline{q}$ at time $t < T_a$, we can obtain the optimal stopping

$$\tau_L^* = \inf\Big\{s \geq t : X_t \geq \widetilde{x}\Big(\frac{K}{pN_a\mathcal{W}(Q_a, \overline{q})}\Big)\Big\}.$$

This optimal stopping is equivalent to $\tau_a^* = \inf\{t \geq 0 : \mathbf{Q}_t^a \leq Q_a\}$ under (F.4). It follows that the equilibrium value $J_a(x, q_b, Q_a; \varphi_a^*, \varphi_b^*) = V_a(x, \overline{q})$ as given in (IA.I.2) with $\mathbf{Q}_t^b = \overline{q}$ satisfies (IA.K.43) for $q_b = \overline{q}$ and $Q_a \in [\zeta^{-1}(\overline{q}), \overline{q}]$.

In summary, combining our analyses in Steps 1 and 2, we obtain $J_a(x, q_b, Q_a; \varphi_a^*, \varphi_b^*) \geq J_a(x, q_b, Q_a; \varphi_a, \varphi_b^*)$ for any $Q_a, q_b \in [\underline{q}, \overline{q}]$. Similarly, we also have $J_b(x, q_a, Q_b; \varphi_a^*, \varphi_b^*) \geq J_b(x, q_a, Q_b; \varphi_a^*, \varphi_b)$ for any $Q_b, q_a \in [\underline{q}, \overline{q}]$, and $(\varphi_a^*, \varphi_b) \in \Phi$.

□

Proof of Lemma 1: We first consider the case where $R = R_{A_1A_2}$. We can verify that $V_*(x)$ given in (A.6)-(A.7) satisfies:

$$V_*(\overline{x}) = D\Pi(\overline{x}) - K_L, \quad V_*'(\overline{x}) = D\Pi'(\overline{x}), \qquad \text{(IA.K.44)}$$

and

$$\frac{L(\widetilde{x}) - K_L}{\widetilde{x}^\beta} = \frac{L(\overline{x}) - K_L}{(\overline{x})^\beta}, \qquad \text{(IA.K.45)}$$

where $\overline{x} = \frac{1}{D}\frac{\beta}{\beta-1}(r-\mu)K_L$. Therefore for any $x \leq \overline{x} = \frac{1}{D}\frac{\beta}{\beta-1}(r-\mu)K_L$, we have

$$V_*(x) = \left(\frac{x}{\overline{x}}\right)^\beta (L(\overline{x}) - K_L) = \left(\frac{x}{\widetilde{\overline{x}}}\right)^\beta (L(\widetilde{x}) - K_L), \qquad \text{(IA.K.46)}$$

where the second equality follows from (IA.K.45). We can then verify the following value-matching and smooth-pasting conditions:

$$V_*(\widetilde{x}) = L(\widetilde{x}) - K_L \quad \text{and} \quad V_*'(\widetilde{x}) = L'(\widetilde{x}). \tag{IA.K.47}$$

Using (IA.K.45) and formulas for $\theta_1(a,b)$ and $\theta_2(a,b)$ given in (A.12), we obtain:

$$\theta_1(\widetilde{x},\overline{x}) = \left(\frac{1}{\widetilde{x}}\right)^{\beta}(L(\widetilde{x}) - K_L)$$

and $\theta_2(\widetilde{x},\overline{x}) = 0$, which together with (IA.K.46) imply $\Theta(x;\widetilde{x},\overline{x}) = V_*(x)$ for any $x \in [\widetilde{x},\overline{x}]$. Equations (IA.K.44) and (IA.K.47) imply $\underline{x} = \widetilde{x}$ and $\overline{x} = \frac{1}{D}\frac{\beta}{\beta-1}(r-\mu)K_L$ in (A.13)-(A.14).

Next, we turn to the case where $R \in [1, R_{A_1A_2})$. First, we prove the existence of the pair $(\underline{x},\overline{x})$ satisfying (A.13)-(A.14).

Proof of the existence of the pair $(\underline{x},\overline{x})$ for the $R \in [1, R_{A_1A_2})$ case. First, we can verify that for any $y > 0$, $\Gamma(x,y)$ defined in (IA.J.16) has the following properties:

$$\Gamma(y,y) = D\Pi(y) - K_L \quad \text{and} \quad \Gamma_x(y,y) = D\Pi'(y). \tag{IA.K.48}$$

Let $\widehat{L}(x) := \Pi(x) - (1-D)\,\Pi(x_F)\left(\frac{x}{x_F}\right)^{\beta} - K_L$, where $x > 0$. Next, we show that there exists a pair $(\underline{x},\overline{x})$ satisfying $\underline{x} > \widetilde{x}$, $\overline{x} > 2\widetilde{x}$,

$$\Gamma(\underline{x},\overline{x}) = \widehat{L}(\underline{x}) \quad \text{and} \quad \Gamma_x(\underline{x},\overline{x}) = \widehat{L}'(\underline{x}). \tag{IA.K.49}$$

Let h_1 and h_2 denote the following two constants:

$$h_1 := \frac{1}{\overline{x}^{\beta}}\frac{D(1-\gamma)\Pi(\overline{x}) + \gamma K_L}{\beta - \gamma}, \tag{IA.K.50}$$

$$h_2 := \frac{1}{\overline{x}^{\gamma}}\frac{D(\beta-1)\Pi(\overline{x}) - \beta K_L}{\beta - \gamma}. \tag{IA.K.51}$$

Then we can show that (IA.K.49) are equivalent to

$$h_1\underline{x}^{\beta} + h_2\underline{x}^{\gamma} = \Pi(\underline{x}) - (1-D)\,\Pi(x_F)\left(\frac{\underline{x}}{x_F}\right)^{\beta} - K_L, \tag{IA.K.52}$$

$$h_1\beta\underline{x}^{\beta} + h_2\gamma\underline{x}^{\gamma} = \Pi(\underline{x}) - \beta\,(1-D)\,\Pi(x_F)\left(\frac{\underline{x}}{x_F}\right)^{\beta}. \tag{IA.K.53}$$

Equations (IA.K.52)–(IA.K.53) hold if and only if h_1 and h_2 are respectively given by

$$h_1 = \frac{1}{\underline{x}^\beta}\left[\frac{(1-\gamma)\Pi(\underline{x})+\gamma K_L}{\beta-\gamma} - (1-D)\,\Pi(x_F)\left(\frac{\underline{x}}{x_F}\right)^\beta\right], \tag{IA.K.54}$$

$$h_2 = \frac{1}{\underline{x}^\gamma}\frac{(\beta-1)\Pi(\underline{x})-\beta K_L}{\beta-\gamma}. \tag{IA.K.55}$$

Using (IA.K.50) and (IA.K.51), we can see that (IA.K.54) and (IA.K.55) are equivalent to

$$\frac{D(1-\gamma)\Pi(\overline{x})+\gamma K_L}{\beta-\gamma} = \left(\frac{\overline{x}}{\underline{x}}\right)^\beta\left[\frac{(1-\gamma)\Pi(\underline{x})+\gamma K_L}{\beta-\gamma} - (1-D)\,\Pi(x_F)\left(\frac{\underline{x}}{x_F}\right)^\beta\right], \tag{IA.K.56}$$

$$[D(\beta-1)\Pi(\overline{x})-\beta K_L] = \left(\frac{\overline{x}}{\underline{x}}\right)^\gamma[(\beta-1)\Pi(\underline{x})-\beta K_L], \tag{IA.K.57}$$

respectively. We introduce the following change of variables:

$$U := \frac{D\Pi(\overline{x})(\beta-1)}{\beta K_L} - 1, \tag{IA.K.58}$$

$$u := \frac{\Pi(\underline{x})(\beta-1)}{\beta K_L} - 1. \tag{IA.K.59}$$

Then, we can show that $D(\beta-1)\Pi(\overline{x}) = \beta K_L(1+U)$, $(\beta-1)\Pi(\underline{x}) = \beta K_L(1+u)$, $\overline{x}/\underline{x} = \frac{1}{D}\frac{1+U}{1+u}$, and $\underline{x}/x_F = \Pi(\underline{x})/\Pi(x_F) = (1+u)RD$. Therefore, (IA.K.57) is equivalent to (A.16) and (IA.K.56) is equivalent to (A.17). Using these calculations, we conclude that a pair $(\underline{x}, \overline{x})$ satisfies $\underline{x} > \widetilde{x}$, $\overline{x} > \frac{1}{D}\widetilde{x}$ and (IA.K.49) if and only if (A.15)-(A.17) hold for some $u > 0$ and $U > 0$.

Next, we show that there exist a pair $(u > 0, U > 0)$ such that (A.16)-(A.17) hold. For the $H(\cdot)$ function defined in (A.18), $H(0) = 1/(\beta-1) > 0$, $\lim_{z\to\infty} H(z) = 0$, and its derivative is

$$H'(z) = -\frac{\beta[(1-\gamma)(1+z)+\gamma]}{(\beta-\gamma)(1+z)^{\beta+1}} \leq -\frac{\beta}{(\beta-\gamma)(1+z)^{\beta+1}} < 0, \quad z \geq 0. \tag{IA.K.60}$$

Since $1 \leq R < \left(\frac{\left(\frac{1}{D}\right)^\beta - 1}{\left(\frac{1}{D}-1\right)\beta}\right)^{\frac{1}{\beta-1}} = R_{A_1A_2}$, we have $\left(\frac{1}{D}\right)^\beta H(0) - \left(\frac{1}{D}-1\right)\frac{\beta}{\beta-1}R^{\beta-1} > \left(\frac{1}{D}\right)^\beta \frac{1}{\beta-1} - \frac{\left(\frac{1}{D}\right)^\beta - 1}{\beta-1} = \frac{1}{\beta-1}$. Then, the continuity and monotonicity of $H(z)$ imply that there exists a pair

(u_1, u_2) satisfying $0 < u_1 < u_2 < +\infty$, such that

$$\left(\frac{1}{D}\right)^{\beta} H(u_1) - \left(\frac{1}{D} - 1\right)\frac{\beta}{\beta - 1}R^{\beta-1} = \frac{1}{\beta - 1} \quad \text{and} \quad \left(\frac{1}{D}\right)^{\beta} H(u_2) - \left(\frac{1}{D} - 1\right)\frac{\beta}{\beta - 1}R^{\beta-1} = 0\,. \tag{IA.K.61}$$

Additionally, $\left(\frac{1}{D}\right)^{\beta} H(z) - \left(\frac{1}{D} - 1\right)\frac{\beta}{\beta-1}R^{\beta-1} \in (0, \frac{1}{\beta-1}]$ for any $z \in [u_1, u_2)$. Since $H(z)$ is smooth and decreases from $\frac{1}{\beta-1}$ to 0 as z increases from 0 to ∞, we can define a smooth and increasing function $g(z)$ for $z \in [u_1, u_2)$ as follows:

$$g(z) := H^{-1}\left(\left(\frac{1}{D}\right)^{\beta} H(z) - \left(\frac{1}{D} - 1\right)\frac{\beta}{\beta - 1}R^{\beta-1}\right), \quad z \in [u_1, u_2), \tag{IA.K.62}$$

where $H^{-1}(\,\cdot\,)$ is the inverse function of $H(\,\cdot\,)$. We have $g(u_1) = 0$ and $\lim_{u\uparrow u_2} g(u) = \infty$.

Let $\widetilde{H}(z) := z(1 + z)^{-\gamma}$ for $z \geq 0$, which is smooth and increasing in $z \geq 0$ with the properties of $\widetilde{H}(0) = 0$ and $\lim_{z\to\infty} \widetilde{H}(z) = \infty$. Then we can define another smooth and increasing function, $\widetilde{g}(z)$ for $z \geq 0$ as follows:

$$\widetilde{g}(z) := \widetilde{H}^{-1}\left(\left(\frac{1}{D}\right)^{\gamma} \widetilde{H}(z)\right), \quad z \geq 0, \tag{IA.K.63}$$

where $\widetilde{H}^{-1}(\,\cdot\,)$ is the inverse function of $\widetilde{H}(\,\cdot\,)$. Note that $\widetilde{g}(0) = 0$, $\widetilde{g}(z) < \widetilde{H}^{-1}\left(\widetilde{H}(z)\right) = z$ for any $z > 0$ as $\gamma < 0$, $D \in (0, 1)$, $\widetilde{g}(u_1) - g(u_1) = \widetilde{g}(u_1) > \widetilde{g}(0) = 0$, and $\widetilde{g}(u_2 - \delta) - g(u_2 - \delta) \leq u_2 - g_1(u_2 - \delta) < 0$ for sufficiently small $\delta > 0$ as $\lim_{z\uparrow u_2} g(z) = \infty$. Because $g(z)$ and $\widetilde{g}(z)$ are continuous on $[u_1, u_2)$, there exists a value $u \in (u_1, u_2)$ such that $\widetilde{g}(u) = g(u)$. By setting $U = \widetilde{g}(u) = g(u) > 0$ and using (IA.K.62)-(IA.K.63), we conclude that the pair (u, U) satisfy (A.16)-(A.17). Using (IA.K.58)-(IA.K.59), $u > 0$ and $U > 0$, we obtain $\underline{x}$ and $\overline{x}$ as given in (A.15), and they satisfy $\overline{x} > \frac{1}{D}\widetilde{x}$ and $\underline{x} > \widetilde{x}$. We further obtain (IA.K.49), using $u > 0$, $U > 0$ and (A.15)-(A.17).

Next, we prove $\underline{x} < x_F$ by contradiction. If $\underline{x} \geq x_F$ were to hold, then

$$\Gamma(\underline{x}, \overline{x}) = \widehat{L}(\underline{x}) = \Pi(\underline{x}) - (1 - D)\,\Pi(x_F)\left(\frac{\underline{x}}{x_F}\right)^{\beta} - K_L \leq D\Pi(\underline{x}) - K_L, \tag{IA.K.64}$$

where the first equality follows from (IA.K.49) and the inequality follows from $\underline{x} \geq x_F$. For any $y \geq \frac{1}{D}\widetilde{x}$, we can show $D(\beta - 1)\Pi(y) - \beta K_L \geq 0$ and

$$D(1 - \gamma)\Pi(y) + \gamma K_L \geq K_L[\frac{\beta}{\beta - 1}(1 - \gamma) + \gamma] = K_L\frac{\beta - \gamma}{\beta - 1} > 0.$$

Therefore, for any $y \geq \frac{1}{D}\widetilde{x} > x > 0$, we have

$$\Gamma_{xx}(x,y) \\ = \frac{D(1-\gamma)\Pi(y)+\gamma K_L}{(\beta-\gamma)x^2}\beta(\beta-1)\left(\frac{x}{y}\right)^{\beta} + \frac{D(\beta-1)\Pi(y)-\beta K_L}{(\beta-\gamma)x^2}\gamma(\gamma-1)\left(\frac{x}{y}\right)^{\gamma} > 0. \tag{IA.K.65}$$

Using (IA.K.65) and evaluating (IA.K.48) at $y = \overline{x}$, we obtain:

$$\Gamma(x,\overline{x}) > D\Pi(x) - K_L, \quad x \in (0,\overline{x}) \cup (\overline{x}, +\infty). \tag{IA.K.66}$$

Because the inequality in (IA.K.64) is strict if $\underline{x} > x_F$, we obtain $\underline{x} = \overline{x} = x_F$ using (IA.K.66), which contradicts $\overline{x} > \frac{1}{D}\widetilde{x} \geq x_F$. Therefore, $\underline{x} < x_F$ has to hold.

Finally, we prove that (A.13)-(A.14) hold for the pair $(\underline{x}, \overline{x})$ that we have just obtained. Using (IA.K.49), $\underline{x} < x_F$, and $L(x) = \widehat{L}(x)$ in the $x < x_F$ region, we obtain:

$$\Gamma(\underline{x},\overline{x}) = L(\underline{x}) - K_L \quad \text{and} \quad \Gamma_x(\underline{x},\overline{x}) = L'(\underline{x}). \tag{IA.K.67}$$

Using (IA.K.50)-(IA.K.51), we obtain $\Gamma(x,\overline{x}) = h_1 x^{\beta} + h_2 x^{\gamma}$. Using (IA.K.67) and $\underline{x} < x_F \leq \frac{1}{D}\widetilde{x} < \overline{x}$, we obtain

$$h_1 \underline{x}^{\beta} + h_2 \underline{x}^{\gamma} = L(\underline{x}) - K_L, \tag{IA.K.68}$$

$$h_1 \overline{x}^{\beta} + h_2 \overline{x}^{\gamma} = L(\overline{x}) - K_L. \tag{IA.K.69}$$

Using (IA.K.68)-(IA.K.69), $\theta_1(\underline{x},\overline{x})\underline{x}^{\beta} + \theta_2(\underline{x},\overline{x})\underline{x}^{\gamma} = L(\underline{x}) - K_L$, and $\theta_1(\underline{x},\overline{x})\overline{x}^{\beta} + \theta_2(\underline{x},\overline{x})\overline{x}^{\gamma} = L(\overline{x}) - K_L$, we obtain $h_1 = \theta_1(\underline{x},\overline{x})$, $h_2 = \theta_2(\underline{x},\overline{x})$, and

$$\Gamma(x,\overline{x}) = \Theta(x;\underline{x},\overline{x}), \quad x \in [\underline{x},\overline{x}]. \tag{IA.K.70}$$

Using (IA.K.48) and (IA.K.67), we conclude that (A.13) and (A.14) hold.

In addition, we conclude from (IA.K.65) and (IA.K.70) that $\Theta_{xx}(x;\underline{x},\overline{x}) > 0$ for any $x \in [\underline{x},\overline{x}]$.

Proof of the uniqueness of the pair $(\underline{x}, \overline{x})$ for the $R \in [1, R_{A_1A_2})$ case. Using $h_1 = \theta_1(\underline{x}, \overline{x})$, $h_2 = \theta_2(\underline{x}, \overline{x})$, (IA.K.50)-(IA.K.51) and (IA.K.54)-(IA.K.55), we obtain:

$$\theta_1(\underline{x}, \overline{x}) = \frac{1}{\overline{x}^\beta} \frac{D(1-\gamma)\Pi(\overline{x}) + \gamma K_L}{\beta - \gamma} = \frac{1}{\underline{x}^\beta} \left[\frac{(1-\gamma)\Pi(\underline{x}) + \gamma K_L}{\beta - \gamma} - (1-D)\,\Pi(x_F) \left(\frac{\underline{x}}{x_F} \right)^\beta \right], \tag{IA.K.71}$$

$$\theta_2(\underline{x}, \overline{x}) = \frac{1}{\overline{x}^\gamma} \frac{D(\beta-1)\Pi(\overline{x}) - \beta K_L}{\beta - \gamma} = \frac{1}{\underline{x}^\gamma} \frac{(\beta-1)\Pi(\underline{x}) - \beta K_L}{\beta - \gamma}. \tag{IA.K.72}$$

Let (a, b) denote a pair in the domain $(\widetilde{x}, x_F) \times (\frac{1}{D}\widetilde{x}, \infty)$ satisfying (A.13) and (A.14):

$$\Theta(a; a, b) = L(a) - K_L, \quad \Theta_x(a; a, b) = L'(a), \tag{IA.K.73}$$

$$\Theta(b; a, b) = D\Pi(b) - K_L, \quad \Theta_x(b; a, b) = D\Pi'(b). \tag{IA.K.74}$$

We show that (a, b) has to be the same as the $(\underline{x}, \overline{x})$ that we have just characterized.

Using the function $\Gamma(\,\cdot\,,\,\cdot\,)$ given by (IA.J.16), we obtain

$$\Gamma(b, b) = D\Pi(b) - K_L \quad \text{and} \quad \Gamma_x(b, b) = D\Pi'(b). \tag{IA.K.75}$$

Using $\Theta(x; a, b) = \theta_1(a, b)x^\beta + \theta_2(a, b)x^\gamma$, (IA.K.74)-(IA.K.75), and $\Gamma(x, b) = h_1 x^\beta + h_2 x^\gamma$, where $h_1 = \frac{1}{b^\beta} \frac{D(1-\gamma)\Pi(b) + \gamma K_L}{\beta - \gamma}$ and $h_2 = \frac{1}{b^\gamma} \frac{D(\beta-1)\Pi(b) - \beta K_L}{\beta - \gamma}$, we obtain $h_1 = \theta_1(a, b)$ and $h_2 = \theta_2(a, b)$ and thus $\Theta(x; a, b) = \Gamma(x, b)$. Then, using (IA.K.73), we obtain

$$\Gamma(a, b) = L(a) - K_L \quad \text{and} \quad \Gamma_x(a, b) = L'(a)\,. \tag{IA.K.76}$$

Using the function $\Gamma(\,\cdot\,,\,\cdot\,)$ given by (IA.J.16), we obtain

$$\Gamma_y(x, y) = \frac{(1-\beta)(1-\gamma)D\Pi(y) - \beta\gamma K_L}{(\beta - \gamma)y} \left[\left(\frac{x}{y}\right)^\beta - \left(\frac{x}{y}\right)^\gamma \right]. \tag{IA.K.77}$$

For any $y \geq \frac{1}{D}\widetilde{x}$, we can show

$$(1-\beta)(1-\gamma)D\Pi(y) - \beta\gamma K_L \leq (1-\beta)(1-\gamma)\Pi(\widetilde{x}) - \beta\gamma K_L = -\beta K_L < 0$$

and then derive from (IA.K.77) that

$$\Gamma_y(x, y) > 0, \quad 0 < x < y,\ y \geq \frac{1}{D}\widetilde{x}. \tag{IA.K.78}$$

Using (IA.K.75)-(IA.K.76) and $\Gamma(x,b) = \theta_1(a,b)x^\beta + \theta_2(a,b)x^\gamma$, we obtain

$$\theta_2(a,b) = \frac{1}{b^\gamma}\frac{D(\beta-1)\Pi(b) - \beta K_L}{\beta-\gamma} = \frac{1}{a^\gamma}\frac{(\beta-1)\Pi(a) - \beta K_L}{\beta-\gamma}. \tag{IA.K.79}$$

Since $[D(\beta-1)\Pi(x) - \beta K_L]\, x^{-\gamma}$ is increasing in $x \geq \frac{1}{D}\widetilde{x}$, using (IA.K.72) and (IA.K.79), we conclude $b = \overline{x}$ if $a = \underline{x}$.

Therefore, we only need to prove that $a = \underline{x}$ holds. Again, we proceed with a proof by contradiction. Suppose $a \neq \underline{x}$.

Using (IA.K.65), (IA.K.67), and the concavity of $L(x)$ in the $x \in (0, x_F)$ region ($L''(x) < 0$), we can show that $\Gamma(x,\overline{x}) > L(x) - K_L$ holds for all $x \in (0,\underline{x}) \cup (\underline{x}, x_F]$. Since $a \neq \underline{x}$ and $a < x_F$, the following inequality has to hold:

$$\Gamma(a,\overline{x}) > L(a) - K_L = \Gamma(a,b). \tag{IA.K.80}$$

Using $\overline{x} > \frac{1}{D}\widetilde{x}$, $b > \frac{1}{D}\widetilde{x}$, $a < x_F \leq \frac{1}{D}\widetilde{x}$, (IA.K.78), and (IA.K.80), we can see that $\overline{x} > b$.

Equations (IA.K.72) and (IA.K.79) imply that

$$\begin{aligned} &[(\beta-1)\Pi(a) - \beta K_L]\, a^{-\gamma} = [D(\beta-1)\Pi(b) - \beta K_L]\, b^{-\gamma} \\ &< [D(\beta-1)\Pi(\overline{x}) - \beta K_L]\, \overline{x}^{-\gamma} = [(\beta-1)\Pi(\underline{x}) - \beta K_L]\, \underline{x}^{-\gamma}, \end{aligned} \tag{IA.K.81}$$

where the inequality uses $\overline{x} > b \geq \frac{1}{D}\widetilde{x}$ and the property that $[D(\beta-1)\Pi(x) - \beta K_L]\, x^{-\gamma}$ increases with x in the $x \geq \frac{1}{D}\widetilde{x}$ region. Using $a > \widetilde{x}$, $\underline{x} > \widetilde{x}$, (IA.K.81), and the property that $[(\beta-1)\Pi(x) - \beta K_L]\, x^{-\gamma}$ increases with x in the $x \geq \widetilde{x}$ region, we obtain $a < \underline{x}$.

Therefore, we have

$$\Gamma(\underline{x}, b) < \Gamma(\underline{x}, \overline{x}) = L(\underline{x}) - K_L \leq \Gamma(\underline{x}, b), \tag{IA.K.82}$$

where the first inequality follows from (IA.K.78) and $\underline{x} < x_F \leq \frac{1}{D}\widetilde{x} < b < \overline{x}$ and the second inequality follows from $a < \underline{x} < x_F$, $L''(x) < 0$ in the $x \in (0, x_F)$ region, (IA.K.65), and (IA.K.76). We thus obtain a contradiction and complete our proof for the uniqueness of the pair $(\underline{x}, \overline{x})$ in the domain $(\widetilde{x}, x_F) \times (\frac{1}{D}\widetilde{x}, \infty)$ satisfying (A.13)-(A.14).

The uniqueness of the pair $(\underline{x}, \overline{x})$ implies that the system of equations (A.16)-(A.17) admits a unique solution pair (u, U) in the domain $(0, \frac{1}{DR} - 1) \times (0, \infty)$. As we have shown, if $u \in [u_1, u_2)$ is a root of $g(u) - \widetilde{g}(u) = 0$, then the pair $(u, U) = (u, g(u))$ is one solution to the system of equations (A.16)-(A.17). Because the solution of the system (A.16)-(A.17) is unique in the domain $(0, \frac{1}{DR} - 1) \times (0, \infty)$, the equation $g(u) - \widetilde{g}(u) = 0$ must have a unique root in the domain $[u_1, u_2) \cap (0, \frac{1}{DR} - 1)$. Since $g(\,\cdot\,)$ is smooth in R and $\widetilde{g}(\,\cdot\,)$ is independent of R, u

being the unique root of $g(u) - \widetilde{g}(u) = 0$ is continuously differentiable in R. Therefore, using (A.15), we have shown that both $\underline{x}$ and $\overline{x}$ are continuously differentiable in $R \in [1, R_{A_1 A_2})$. $\square$

Proof of Lemma 2: Denote $\psi(y) := y - \left(\frac{1}{D}\frac{\beta}{\beta-1} - 1\right)\left(D\frac{\beta-1}{\beta}\right)^{\beta} y^{\beta}$ where $y > 0$. We can verify that $L(x) - F(x) = \psi\left(\frac{x}{(r-\mu)K_F}\right) K_F$ for $x \le x_F$ and $L(x) - F(x) = K_F = \psi\left(\frac{1}{D}\frac{\beta}{\beta-1}\right) K_F$ for $x \ge x_F$. Therefore, we can write $L(x) - K_L - F(x)$ as:

$$L(x) - K_L - F(x) = K_F\left[\psi\left(\frac{x}{(r-\mu)K_F}\right) - R\right], \qquad x \le x_F, \tag{IA.K.83}$$

$$L(x) - K_L - F(x) = K_F\left[\psi\left(\frac{1}{D}\frac{\beta}{\beta-1}\right) - R\right], \qquad x \ge x_F, \tag{IA.K.84}$$

where $R = K_L/K_F$.

We can verify that $\psi(y)$ is increasing ($\psi'(y) > 0$) for $y \in (0, y^*)$ and decreasing ($\psi'(y) < 0$) for $y > y^*$, where

$$y^* = \frac{1}{D}\frac{\beta}{\beta-1}\left(\frac{\frac{1}{D}}{\frac{1}{D}\beta - (\beta-1)}\right)^{1/(\beta-1)}. \tag{IA.K.85}$$

We can show $y^* < \frac{1}{D}\frac{\beta}{\beta-1}$, $\psi(0) = 0$, $\psi(y^*) = R_{AB}$, and $\psi(\frac{1}{D}\frac{\beta}{\beta-1}) = 1$. Using these properties, we can show that for any fixed $R \in (1, R_{AB})$, the $\psi(y) = R$ equation has two positive roots: $\widehat{y}_L$ and $\widehat{y}_F$, satisfying $0 < \widehat{y}_L < y^* < \widehat{y}_F$ and the $\psi(y) = R_{AB}$ equation has one positive root: $y = y^* > 0$, where y^* is given in (IA.K.85). Therefore, (IA.J.2) and (IA.J.3) are well defined, where $\overline{\eta}(R) = \widehat{y}_F$, $\underline{\eta}(R) = \widehat{y}_L$, $\overline{\eta}(R)$ is decreasing in $R \in (1, R_{AB}]$, and $\underline{\eta}(R)$ is increasing in $R \in (1, R_{AB}]$. Since $\psi(y^*) = R_{AB}$, we have $\overline{\eta}(R_{AB}) = \underline{\eta}(R_{AB}) = y^* = \frac{\beta}{\beta-1}R_{AB}$. Together with the closed-form expressions of $\widehat{x}_L$ and $\widehat{x}_F$ in (IA.J.1), we obtain $\widehat{x}_F = \widetilde{x} = \widehat{x}_L$ for $R = R_{AB}$. Since $\psi(\frac{1}{D}\frac{\beta}{\beta-1}) = 1$, we have $\overline{\eta}(1) = \frac{1}{D}\frac{\beta}{\beta-1}$. Together with the closed-form expression of $\widehat{x}_F$ given in (IA.J.1), we obtain $\lim_{R\to 1+} \widehat{x}_F = x_F$.

Note that $\psi(y)$ is increasing in $y \in [0, y^*]$ and decreasing in $y \ge y^*$, and for any $R \in (1, R_{AB})$, we have $\psi(\overline{\eta}(R)) = \psi(\underline{\eta}(R)) = R$, $0 < \underline{\eta}(R) < y^* < \overline{\eta}(R) \le \frac{1}{D}\frac{\beta}{\beta-1}$. We then infer

$$\psi(y) < R, \quad y \in (0, \underline{\eta}(R)) \cup (\overline{\eta}(R), +\infty), \tag{IA.K.86}$$

$$\psi(y) > R, \quad y \in (\underline{\eta}(R), \overline{\eta}(R)). \tag{IA.K.87}$$

Using (IA.K.83)-(IA.K.84), (IA.K.86)-(IA.K.87), and (IA.J.1), for any $R \in (1, R_{AB})$ we obtain

$$L(x) - K_L < F(x), \quad x < \widehat{x}_L,$$

$$L(x) - K_L = F(x), \quad x = \widehat{x}_L,$$

$$
\begin{aligned}
L(x) - K_L &> F(x), \quad \widehat{x}_L < x < \widehat{x}_F, \\
L(x) - K_L &= F(x), \quad x = \widehat{x}_F, \\
L(x) - K_L &< F(x), \quad x > \widehat{x}_F.
\end{aligned} \tag{IA.K.88}
$$

When $R \in (1, R_{AB})$, we have

$$
\widetilde{x} = \frac{\beta}{\beta-1}(r-\mu)K_L < R_{AB}\frac{\beta}{\beta-1}(r-\mu)K_F = y^*(r-\mu)K_F < \overline{\eta}(R)(r-\mu)K_F = \widehat{x}_F.
$$

Note that $\overline{\eta}(R)$ is decreasing in $R \in [1, R_{AB}]$ and $\overline{\eta}\,(1) = \frac{1}{D}\frac{\beta}{\beta-1}$. Then, when $R \in (1, R_{AB})$, we obtain $\overline{\eta}\,(R) < \frac{1}{D}\frac{\beta}{\beta-1}$, which in turn implies $x_F > \widehat{x}_F$ using (IA.J.1).

Next, we consider the $R \in (0,1)$ case.[16] Note that $\psi(y)$ is increasing in $y \in [0, y^*]$ and decreasing in $y \geq y^*$. Also for any $R \in (0,1)$, we have $\psi(\overline{\eta}(R)) = \psi(\underline{\eta}(R)) = R$ and $0 < \underline{\eta}(R) < y^* < \frac{1}{D}\frac{\beta}{\beta-1} < \overline{\eta}(R)$. We then obtain that $\psi(y) < R$ for any $y \in (0, \underline{\eta}(R))$ and $\psi(y) > R$ for any $y \in (\underline{\eta}(R), \frac{1}{D}\frac{\beta}{\beta-1}]$. Using these results and (IA.K.83)-(IA.K.84), for $R \in (0,1)$ we obtain

$$
\begin{aligned}
L(x) - K_L &< F(x), \quad x < \widehat{x}_L, \\
L(x) - K_L &= F(x), \quad x = \widehat{x}_L, \\
L(x) - K_L &> F(x), \quad x > \widehat{x}_L.
\end{aligned} \tag{IA.K.89}
$$

Next, we show that $\widehat{x}_L < \widetilde{x}$ for any $R \in (0, R_{AB})$. For any $R \in (0, R_{AB})$,

$$
\begin{aligned}
\psi\left(\frac{\beta}{\beta-1}R\right) &= R\left[\frac{\beta}{\beta-1} - \Big(\frac{1}{D}\frac{\beta}{\beta-1} - 1\Big)D^{\beta}R^{\beta-1}\right] \\
>& R\left[\frac{\beta}{\beta-1} - \Big(\frac{1}{D}\frac{\beta}{\beta-1} - 1\Big)D^{\beta}R_{AB}^{\beta-1}\right] = R,
\end{aligned}
$$

where the inequality uses $D \in (0,1)$ and $\beta > 1$, the last equality follows from $R_{AB} = D^{-1}\big[\beta\big(1-D\big)+D\big]^{-\frac{1}{\beta-1}}$. Since $\psi(y)$ is increasing in $y \in [0, y^*]$ and decreasing in $y \geq y^*$ and $\psi(\underline{\eta}(R)) = \psi(\overline{\eta}(R)) = R$, we have $\underline{\eta}(R) < \frac{\beta}{\beta-1}R$. We then derive $\widehat{x}_L < \widetilde{x}$ from (IA.J.1). □

Proof of Proposition 1: Let $K_{AB} = R_{AB}K_F$. We can find that K_{AB} is the unique root of the following equation for $K_L > K_F$: $\frac{L(\widetilde{x})-K_L}{\widetilde{x}^{\beta}} = \frac{F(x_F)}{x_F^{\beta}}$, where $\widetilde{x}$ is given in (23).[17] Using (12)-(13) for $L(x)$ and (8)-(9) for $F(x)$, we can verify $K_{AB} = \max_{x>0}\left[L(x) - F(x)\right]$. Then,

[16] The $R = 1$ case can be analyzed in the same way.

[17] If $R = R_{AB}$, the two roots for the $S(x) = 0$ equation reduce to one root: $\widehat{x}_L = \widehat{x}_F = \widetilde{x}$, where $\widetilde{x}$ is given in (23). For all $x \neq \widetilde{x}$, $S(x) > 0$.

for any $x > 0$, we have

$$L(x) - K_L - F(x) \leq \max_{x>0} [L(x) - F(x)] - K_L = K_{AB} - K_L = K_F(R_{AB} - R) < 0$$

for $R > R_{AB}$. We thus have proven Case A in Proposition 1.

The results in Case B in Proposition 1 are the same as those given in (IA.K.88) and the results in Case C in Proposition 1 are the same as those given in (IA.K.89). We have completed our proofs for Case B and Case C of Proposition 1. In particular, when $R = 1$, $S(x) = 0$ has a unique root $\widehat{x}_L \in (0, x_F)$, and $S(x) = 0$ for $x \geq x_F$, $S(x) < 0$ for $x \in (\widehat{x}_L, x_F)$, $S(x) > 0$ for $x < \widehat{x}_L$. $\square$

Proof of Proposition 2: Using similar arguments as those for the proofs of Proposition 4, and Theorems 1, we can prove that $\mathcal{E}_a^* = [\widetilde{x}, \underline{x}] \cup [\overline{x}, \infty)$ and $\mathcal{E}_b^* = \emptyset$ form an asymmetric pure-strategy equilibrium.

Next, we show that Follower's value $J_F(x)$ is given by (33)-(35). Using (32), we obtain

$$\begin{aligned} J_F(x) =& \ \mathbb{E}_t^x \left[e^{-r(\tau_L^* - t)} \max_{\tau_F \geq \tau_L^*} \mathbb{E}_{\tau_L^*} \left[e^{-r(\tau_F - \tau_L^*)} \Big(D\Pi(X_{\tau_F}) - K_F \Big) \right] \right] \\ =& \ \mathbb{E}_t^x \left[e^{-r(\tau_L^* - t)} F(X_{\tau_L^*}) \right], \end{aligned} \tag{IA.K.90}$$

where the second equality follows from the definition of $F(x)$ given in (6).

Finally, we derive explicit solutions for the three regions. First, for $x \leq \underline{x}$, we have $\tau_L^* = \inf\{s \geq t : X_s \in [\widetilde{x}, \underline{x}]\}$ and $X_s \leq \underline{x} < x_F$ for any $s \in [t, \tau_L^*]$. Then, using (IA.K.90) and $\mathcal{A}F(x) = 0$ for $x < x_F$, we conclude that (33) holds, where the generator $\mathcal{A}$ is defined in (IA.K.1). Second, for $x \geq \overline{x}$, Leader enters immediately so that $\tau_L^* = t$. Using this result, (IA.K.90), and (9), we obtain (35). Third, for $x \in (\underline{x}, \overline{x})$, $\tau_L^* = \inf\{s \geq t : X_s = \underline{x} \text{ or } \overline{x}\}$. Using this result and (IA.K.90), we obtain (34). $\square$

Proof of Proposition 3: This is implied by Theorem 1 and part (i) in Lemma 3. $\square$

Proof of Proposition IA.5: We prove the existence of optimal subsidy policy δ_L^{Sub} in two steps.

Step 1: We prove for any $\delta_L > K_L - K_F R_{AB}$,

$$2V_i(x_0; K_F R_{AB}, K_F) - \mathbb{E}[e^{-r\tau_L^*}(K_L - K_F R_{AB})] > 2V_i(x_0; K_L - \delta_L, K_F) - \mathbb{E}[e^{-r\tau_L}\delta_L], \tag{IA.K.91}$$

where τ_L^* is Leader's entry time under entry costs $(K_F R_{AB}, K_F)$, and τ_L is Leader's entry time under entry costs $(K_L - \delta_L, K_F)$.

When x_0 is sufficiently low and $\frac{K_L-\delta_L}{K_F} \leq R_{AB}$, we derive from Theorem IA.1 that $V_i(x_0; K_L-\delta_L, K_F) = (D\Pi(x_F) - K_F)\left(\frac{x_0}{x_F}\right)^\beta$, where $x_F = \frac{1}{D}\frac{\beta}{\beta-1}(r-\mu)K_F$. Hence, $V_i(x_0; K_L-\delta_L, K_F)$ is independent of $\delta_L \geq K_L - K_F R_{AB}$. In particular, $V_i(x_0; K_F R_{AB}, K_F) = V_i(x_0; K_L - \delta_L, K_F)$.

Using $\delta_L > K_L - K_F R_{AB}$, $\mathbb{E}[e^{-r\tau_L^*}] = \left(\frac{x_0}{\underline{\eta}(R_{AB})(r-\mu)K_F}\right)^\beta$ and $\mathbb{E}[e^{-r\tau_L}] = \left(\frac{x_0}{\underline{\eta}(R)(r-\mu)K_F}\right)^\beta$, where $\underline{\eta}(R)$ is given in Lemma 2, $R = \frac{K_L-\delta_L}{K_F} < R_{AB}$, we have $\mathbb{E}[e^{-r\tau_L}] > \mathbb{E}[e^{-r\tau_L^*}]$. Since $\delta_L > K_L - K_F R_{AB} > 0$, we have $\mathbb{E}[e^{-r\tau_L}\delta_L] > \mathbb{E}[e^{-r\tau_L^*}(K_L - K_F R_{AB})]$. This proves (IA.K.91).

Step 2: We prove for any $\delta_L \leq 0$,

$$2V_i(x_0; K_L, K_F) > 2V_i(x_0; K_L - \delta_L, K_F) - \mathbb{E}[e^{-r\tau_L}\delta_L]. \tag{IA.K.92}$$

When x_0 is sufficiently low and $\delta_L < K_L - K_F R_{AB}$, we derive from Lemma 3 that

$$V_i(x_0; K_L - \delta_L, K_F) = \left(\frac{x_0}{\widetilde{x}}\right)^\beta (L(\widetilde{x}) - (K_L - \delta_L)), \tag{IA.K.93}$$

where $\widetilde{x} = \frac{\beta}{\beta-1}(r-\mu)(K_L-\delta_L)$ for $\frac{K_L-\delta_L}{K_F} \in [R_{AB}, R_{A_1A_2}]$, and $\widetilde{x} = \frac{1}{D}\frac{\beta}{\beta-1}(r-\mu)(K_L-\delta_L)$ for $\frac{K_L-\delta_L}{K_F} > R_{A_1A_2}$. We can verify that

$$\frac{\partial}{\partial x}\frac{L(x) - (K_L - \delta_L)}{x^\beta} \mid_{x=\widetilde{x}} = 0,$$

which implies that $\frac{\partial}{\partial \delta_L}\left(\frac{x_0}{\widetilde{x}}\right)^\beta (L(\widetilde{x}) - (K_L - \delta_L)) = \left(\frac{x_0}{\widetilde{x}}\right)^\beta$. Hence,

$$\frac{\partial}{\partial \delta_L}2V_i(x_0; K_L - \delta_L, K_F) - \mathbb{E}[e^{-r\tau_L}]\frac{\partial}{\partial \delta_L}\delta_L = 2\left(\frac{x_0}{\widetilde{x}}\right)^\beta - \mathbb{E}[e^{-r\tau_L}] > 0, \tag{IA.K.94}$$

where the inequality uses $\left(\frac{x_0}{\widetilde{x}}\right)^\beta = \mathbb{E}[e^{-r\widetilde{\tau}_L}] > \mathbb{E}[e^{-r\tau_L}]$, where $\widetilde{\tau}_L = \inf\{s : X_s \geq \widetilde{x}\} < \tau_L$. In addition, Lemma 6 implies that $\mathbb{E}[e^{-r\tau_L}]$ is non-decreasing in δ_L. Combining this with (IA.K.94), Then we conclude that $2V_i(x_0; K_L - \delta_L, K_F) - \mathbb{E}[e^{-r\tau_L}\delta_L]$ is increasing in $\delta_L \leq 0$. This proves (IA.K.92).

Combining (IA.K.91) and (IA.K.92), we show that $2V_i(x_0; K_L - \delta_L, K_F) - \mathbb{E}[e^{-r\tau_L}\delta_L]$ achieves the maximum at some $\delta_L^{\text{Sub}} \in [0, K_L - K_F R_{AB}]$. Using (IA.K.94) with $\delta_L = 0$, we know $2V_i(x_0; K_L - \delta_L, K_F) - \mathbb{E}[e^{-r\tau_L}\delta_L]$ is increasing in $\delta \in [0, \epsilon]$ for some small $\epsilon > 0$. This proves that $\delta_L^{\text{Sub}} \in (\epsilon, K_L - K_F R_{AB}]$.

□

Proof of Lemma 3: We first prove this lemma for Subcase A_1.

Subcase $\mathbf{A}_1$: $R > R_{A_1A_2}$**.** For any $x > 0$, let

$$\psi(x,R) := \left(\frac{x}{\overline{x}(R)}\right)^{\beta} [D\Pi(\overline{x}(R)) - K_F R] - (L(x) - K_F R),$$

where $\overline{x}(R) = \frac{1}{D}\frac{\beta}{\beta-1}(r-\mu)K_F R$.

First, we can show that $\psi_{xx}(x,R) > 0$ for any $x \in (0, x_F) \cup (x_F, \infty)$ by using $D\Pi(\overline{x}(R)) - K_F R = K_F R/(\beta - 1) > 0$, $L''(x) < 0$ for $x < x_F$ and $L(x) = D\Pi(x)$ for $x > x_F$. Using $\overline{x}(R) > x_F$, $\psi(\overline{x}(R), R) = 0$, $\psi_x(x,R)|_{x=\overline{x}(R)} = 0$, and $\psi_{xx}(x,R) > 0$ $\forall x > x_F$, we obtain

$$\psi(x,R) > 0, \quad \text{for} \quad x \in [x_F, \overline{x}(R)),\ R > 1. \tag{IA.K.95}$$

Let $\widehat{x}_M := \frac{\beta}{\beta-1}(r-\mu)K_F R_{A_1A_2}$. Using (IA.K.45), we obtain

$$\psi(x, R_{A_1A_2}) = \left(\frac{x}{\widehat{x}_M}\right)^{\beta} [L(\widehat{x}_M) - K_{A_1A_2}] - (L(x) - K_{A_1A_2}),$$

where $K_{A_1A_2} = K_F R_{A_1A_2}$. Using $\psi(\widehat{x}_M, R_{A_1A_2}) = 0$, $\psi_x(\widehat{x}_M, R_{A_1A_2}) = 0$ and $\psi_{xx}(x, R_{A_1A_2}) > 0$ $\forall x \in (0, x_F)$, we obtain

$$\psi(x, R_{A_1A_2}) \geq 0, \quad x \in (0, x_F), \tag{IA.K.96}$$

where the equality holds if and only if $x = \widehat{x}_M$. Using $\psi_R(x,R) = K_F\left[1 - \left(\frac{x}{\overline{x}(R)}\right)^{\beta}\right] > 0$ for any $x < \overline{x}(R)$ and using $x_F < \overline{x}(R)$ for $R > R_{A_1A_2} > 1$, we obtain

$$\psi(x,R) > \psi(x, R_{A_1A_2}) \geq 0, \quad x \in (0, x_F], \quad R > R_{A_1A_2}. \tag{IA.K.97}$$

Using $V_*(x)$ given in (A.6)-(A.7), (IA.K.97), and (IA.K.95), we obtain

$$V_*(x) - (L(x) - K_L) = \psi(x,R) > 0, \quad x < \overline{x}. \tag{IA.K.98}$$

Finally, we conclude our proof that $V_*(x)$ given in (A.6)-(A.7) solves problem (18) in the $x \geq 0$ region by using (IA.K.98) and the following two properties: 1.) for any $x < \overline{x}$, $\mathcal{A}V_*(x) = 0$, where the generator $\mathcal{A}$ is given in (IA.K.1) and 2.) for any $x \geq \overline{x}$, $V_*(x) = D\Pi(x) - K_L = L(x) - K_L$ and

$$\mathcal{A}\left(D\Pi(x) - K_L\right) = rK_L - Dx \leq rK_L - D\overline{x} = rK_L\left(1 - \frac{\beta}{\beta-1}\frac{r-\mu}{r}\right) < 0\,, \tag{IA.K.99}$$

where the last inequality follows from $\frac{\beta}{\beta-1}\frac{r-\mu}{r} > 1$.[18]

Subcase A$_2$: $R_{AB} < R \le R_{A_1A_2}$. Below we provide a proof for the case where $R = K_L/K_F$ satisfies $R_{AB} < R < R_{A_1A_2}$.[19] First, we can show that $\mathcal{A}V_*(x) = \mathcal{A}\Theta(x; \underline{x}, \overline{x}) = 0$ holds in the $x \in (\underline{x}, \overline{x})$ region, where $\mathcal{A}$ is given in (IA.K.1).

Second, using (IA.K.67), (IA.K.65), and $L''(x) < 0$, we can show that $\Gamma(x, \overline{x}) > L(x) - K_L$ for any $x \in (\underline{x}, x_F]$. Combining this result with $\overline{x} > x_F$, (IA.K.66), and $D\Pi(x) - K_L = L(x) - K_L$ for any $x \ge x_F$, we obtain

$$V_*(x) = \Theta(x; \underline{x}, \overline{x}) = \Gamma(x, \overline{x}) > L(x) - K_L, \quad x \in (\underline{x}, \overline{x}),$$

where the second equality follows from (IA.K.70).

Third, for any $x \ge \overline{x} \ge \frac{1}{D}\widetilde{x}$, $V_*(x) = D\Pi(x) - K_L = L(x) - K_L$ and (IA.K.99) holds. For any $x \in [\widetilde{x}, \underline{x}]$, we have $V_*(x) = L(x) - K_L$ and

$$\mathcal{A}(L(x) - K_L) = rK_L - x \le rK_L - \widetilde{x} = rK_L\left(1 - \frac{\beta}{\beta-1}\frac{r-\mu}{r}\right) < 0\,.$$

Finally, for any $x \in (0, \widetilde{x})$, using $V_*(\widetilde{x}) = L(\widetilde{x}) - K_L$, $V_*'(\widetilde{x}) = L'(\widetilde{x})$, and $V_*''(x) > 0 > L''(x)$, we obtain $V_*(x) > L(x) - K_L$. In addition, we have $\mathcal{A}V_*(x) = 0$ for any $x \in (0, \widetilde{x})$.

In sum, we have proven that $V_*(x)$ given in (IA.J.6)-(IA.J.9) solves (18) for $x \ge 0$. □

Proof of Lemma 4: We first prove this lemma for Subcase B$_1$.

Subcase B$_1$: $R_{B_1B_2} < R \le R_{AB}$. Using (IA.J.4), we can follow essentially the same proof procedure as that for Subcase A$_2$ to show that $V_*(x)$ satisfies (18) for any $x \ge \underline{x}$. For any $x \in [\widehat{x}_F, \underline{x}]$, we have $V_*(x) = L(x) - K_L$ and

$$\mathcal{A}(L(x) - K_L) = rK_L - x \le rK_L - \widetilde{x} = rK_L\left(1 - \frac{\beta}{\beta-1}\frac{r-\mu}{r}\right) < 0$$

where the first inequality follows from $\widehat{x}_F > \widetilde{x}$ (see Lemma 2). As $F(x) = L(x) - K_L$ for $x = \widehat{x}_F$ (see Proposition 1), the boundary condition (IA.B.1) holds.

[18]Recall that β is the positive root of the (fundamental) quadratic equation: $g(z) = 0$ where $g(z) = \frac{1}{2}\sigma^2 z(z-1) + \mu z - r$. Using $g(1) = \mu - r < 0$ and $g(z) \to \infty$ as $z \to \infty$, we obtain $\beta > 1$. The inequality $\frac{\beta}{\beta-1}\frac{r-\mu}{r} > 1$ is the same as $r > \beta\mu$. This result apparently holds when $\mu \le 0$. When $\mu > 0$, $g(r/\mu) = \frac{1}{2}\sigma^2(r/\mu)(r/\mu - 1) > 0$ and $g(\beta) = 0$ imply $r > \beta\mu$.

[19]For the special case where $R = R_{A_1A_2}$, the proof is as follows. Using (IA.K.45) and Lemma 1 and simplifying (IA.J.6)-(IA.J.9), we obtain $\underline{x} = \widetilde{x}$, $\overline{x} = \frac{1}{D}\widetilde{x}$, and $V_*(x)$ given in (A.6)-(A.7).

Subcase B$_2$: $1 < R \le R_{B_1B_2}$. First, note that

$$\begin{aligned}&\Gamma(\widehat{x}_F, \widetilde{x}/D) - F(\widehat{x}_F)\\ &= \frac{K_L}{\beta-1}\left(\frac{\widehat{x}_F}{\widetilde{x}/D}\right)^{\beta} - \frac{K_F}{\beta-1}\left(\frac{\widehat{x}_F}{x_F}\right)^{\beta} = \frac{K_L}{\beta-1}\left(\frac{\widehat{x}_F}{\widetilde{x}/D}\right)^{\beta}\left[1-\left(\frac{K_L}{K_F}\right)^{\beta-1}\right] < 0,\end{aligned}$$

using (IA.J.16), $\frac{D(1-\gamma)\Pi\left(\frac{1}{D}\widetilde{x}\right)+\gamma K_L}{\beta-\gamma} = \frac{K_L}{\beta-1}$ and $F(\widehat{x}_F) = \frac{K_F}{\beta-1}\left(\frac{\widehat{x}_F}{x_F}\right)^{\beta}$. Combining the above result with $\lim_{y\to+\infty}\Gamma(\widehat{x}_F, y) = \infty$ and (IA.K.78), we can show that equation (IA.J.15) admits a unique solution in the $y > \widetilde{x}/D$ region. Let $\overline{y}$ denote the unique solution of equation (IA.J.15) in the $y > \widetilde{x}/D$ region.

Note that both $\Gamma(x, \overline{y})$ and $\Theta(x; \widehat{x}_F, \overline{y})$ functions take the form of $\theta_1 x^{\beta} + \theta_2 x^{\gamma}$ in the region $x \in [\widehat{x}_F, \overline{y}]$, where θ_1 and θ_2 are constant parameters to be determined by $\Gamma(x, \overline{y}) = L(x) - K_L = \Theta(x; \widehat{x}_F, \overline{y})$ at $x = \widehat{x}_F$ and $x = \widehat{x}_F = \overline{y}$. We thus conclude that $\Gamma(x, \overline{y}) = \Theta(x; \widehat{x}_F, \overline{y})$ holds for any $x \in [\widehat{x}_F, \overline{y}]$. Using $\Gamma(\widehat{x}_F, \overline{y}) = F(\widehat{x}_F)$, $V_*(x)$ given in (IA.J.13)-(IA.J.14) satisfies the boundary condition (IA.B.1).

Using Lemma 2, we obtain $\widehat{x}_F < x_F$ when $R \in (1, R_{B_1B_2}]$. Recall $(\underline{x}, \overline{x})$ as given in Lemma 1. Using Lemmas 2-1, one can show that $\widehat{x}_F \ge \underline{x}$. Using (IA.K.65), (IA.K.67), and $L''(x) < 0$ for any $x \in (0, x_F)$, we obtain $\Gamma(x, \overline{x}) \ge L(x) - K_L$ for any $x \in (0, x_F)$. Combining this inequality result with $\underline{x} \le \widehat{x}_F < x_F$, we obtain

$$\Gamma(\widehat{x}_F, \overline{x}) \ge L(\widehat{x}_F) - K_L = F(\widehat{x}_F) = \Gamma(\widehat{x}_F, \overline{y}), \tag{IA.K.100}$$

where the first equality follows from Proposition 1 and the second equality follows from (IA.J.15). Using $\overline{x} > \widetilde{x}/D$, $\overline{y} > \widetilde{x}/D$, $\widehat{x}_F \le x_F < \widetilde{x}/D$, (IA.K.78), and (IA.K.100), we obtain $\overline{x} \ge \overline{y}$. Differentiating (IA.K.77) with respect to x, we obtain

$$\Gamma_{xy}(x, y) = \frac{(1-\beta)(1-\gamma)D\Pi(y) - \beta\gamma K_L}{\beta-\gamma}\left(\beta\frac{x^{\beta-1}}{y^{\beta+1}} - \gamma\frac{x^{\gamma-1}}{y^{\gamma+1}}\right) < 0, \quad 0 < x < y,\ y \ge \frac{1}{D}\widetilde{x}, \tag{IA.K.101}$$

where the inequality uses

$$(1-\beta)(1-\gamma)D\Pi(y) - \beta\gamma K_L \le (1-\beta)(1-\gamma)\Pi(\widetilde{x}) - \beta\gamma K_L = -\beta K_L < 0.$$

We thus have obtained:

$$L'(\widehat{x}_F) \le L'(\underline{x}) = \Gamma_x(\underline{x}, \overline{x}) \le \Gamma_x(\widehat{x}_F, \overline{x}) \le \Gamma_x(\widehat{x}_F, \overline{y}), \tag{IA.K.102}$$

where the first inequality use $L''(x) < 0$ for $x < x_F$ and $\underline{x} \le \widehat{x}_F < x_F$, the second inequality follows from (IA.K.65) and $\underline{x} \le \widehat{x}_F < \widetilde{x}/D$, and the last inequality uses (IA.K.101), $\widehat{x}_F < \widetilde{x}/D$, and $\overline{x} \ge \overline{y} > \widetilde{x}/D$. By using (IA.K.65), $L''(x) < 0$ for any $x \in (0, x_F)$, (IA.K.102), and $\Gamma(\widehat{x}_F, \overline{y}) = F(\widehat{x}_F) = L(\widehat{x}_F) - K_L$, we can show that

$$\Gamma(x, \overline{y}) > L(x) - K_L, \quad x \in (\widehat{x}_F, x_F]. \tag{IA.K.103}$$

Using (IA.K.48) and (IA.K.65), we obtain $\Gamma(x, \overline{y}) > D\Pi(x) - K_L = L(x) - K_L$ for any $x \in (x_F, \overline{y})$. Combining this result with (IA.K.103), we obtain

$$V_*(x) = \Theta(x; \widehat{x}_F, \overline{y}) = \Gamma(x, \overline{y}) > L(x) - K_L, \quad x \in (\widehat{x}_F, \overline{y}). \tag{IA.K.104}$$

Additionally, $\mathcal{A}V_*(x) = 0$ holds for any $x \in (\widehat{x}_F, \overline{y})$.

For any $x \ge \overline{y} > \widetilde{x}/D \ge x_F$, $V_*(x) = D\Pi(x) - K_L = L(x) - K_L$ holds and

$$\mathcal{A}\left(D\Pi(x) - K_L\right) = rK_L - Dx \le rK_L - \widetilde{x} = rK_L\left(1 - \frac{\beta}{\beta - 1}\frac{r - \mu}{r}\right) < 0\,, \tag{IA.K.105}$$

where the first inequality follows from $x \ge \overline{y} > \widetilde{x}/D$. □

Proof of Proposition IA.3: Denote the optimal value in the period $t \in (\tau_L, \tau_F)$ as

$$H(x) = \max_{\tau_F \ge t} \mathbb{E}_t^x\left[\int_t^{\tau_F} e^{-r(s-t)} X_s ds + \int_{\tau_F}^{\infty} e^{-r(s-t)} 2DX_s ds - K_F e^{-r(\tau_F - t)}\right]. \tag{IA.K.106}$$

Then we have

$$H(x) = \Pi(x) + \max_{\tau_F \ge t} \mathbb{E}_t^x\left[e^{-r(\tau_F - t)}\Big((2D-1)\Pi(X_{\tau_F}) - K_F\Big)\right]. \tag{IA.K.107}$$

and

$$W^{\text{co}}(x) = \max_{\tau_L \ge t} \mathbb{E}_t^x\left[e^{-r(\tau_L - t)}\Big(H(X_{\tau_L}) - K_L\Big)\right]. \tag{IA.K.108}$$

The case $D \le 1/2$**.** Since $D - 1/2 \le 0$ and $K_F > 0$, then the optimal stopping to (IA.K.107) is $\tau_F^* = \infty$ and $H(x) = \Pi(x)$. It follows that the optimal stopping to (IA.K.108) is $\tau_L^{\text{co}} = \inf\{s \ge t : X_s \ge \widetilde{x}\}$, where $\widetilde{x} = \frac{\beta}{\beta-1}(r - \mu)K_L$.

The case $D > 1/2$**.** Then the optimal stopping to (IA.K.107) is $\tau_F^* = \inf\{s \ge t : X_s \ge \widetilde{x}_F\}$ and $H(x)$ is given by (IA.G.5)-(IA.G.6), where $\widetilde{x}_F = \frac{\beta}{\beta-1}(r - \mu)\frac{K_F}{2D-1}$. In order to solve

(IA.K.108), we need to solve following variational inequality:

$$\max\left\{\frac{\sigma^2 x^2}{2}\frac{d^2 W^{\rm co}(x)}{dx^2}+\mu x\frac{dW^{\rm co}(x)}{dx}-rW^{\rm co}(x),(H(x)-K_L)-W^{\rm co}(x)\right\}=0. \tag{IA.K.109}$$

- **The case $D > 1$ and $K_L < \frac{K_F}{2D-1}$.** We first show that

$$[H(x_M)-K_L]\left(\frac{x}{x_M}\right)^{\beta} > H(x)-K_L, \quad x < x_M. \tag{IA.K.110}$$

Since $K_L < \frac{K_F}{2D-1}$, we have $x_M < \widetilde{x}_F$. Denote $f(x)=[H(x_M)-K_L]\left(\frac{x}{x_M}\right)^{\beta}-[H(x)-K_L]$. Using (IA.G.5), for any $x \le \widetilde{x}_F$, we have

$$\begin{aligned} f(x) &= K_L-\Pi(x)+\left[\frac{H(x_M)-K_L}{x_M^{\beta}}-\frac{(2D-1)\Pi(\widetilde{x}_F)-K_F}{\widetilde{x}_F^{\beta}}\right]x^{\beta} \\ &= K_L-\Pi(x)+\frac{K_L}{\beta-1}\left(\frac{x}{x_M}\right)^{\beta}. \end{aligned} \tag{IA.K.111}$$

We can verify that $f(x_M)=f'(x_M)=0$, $f(0)=K_L>0$ and $f''(x)>0$. Thus, we have $f'(x)<0$ for any $x<x_M$. It follows that $f(x)>f(x_M)=0$ for any $x<x_M$, i.e. (IA.K.115) holds.

Next, we show that

$$\mathcal{A}(H(x)-K_L) \le 0 \tag{IA.K.112}$$

for $x > x_M$. For $x \in [x_M, \widetilde{x}_F]$, we have

$$\mathcal{A}(H(x)-K_L) = -x+rK_L \le -x_M+rK_L = rK_L\Big(1-\frac{\beta}{\beta-1}\frac{r-\mu}{r}\Big) < 0, \tag{IA.K.113}$$

where the inequality uses $\frac{\beta}{\beta-1}\frac{r-\mu}{r}>1$. For $x > \widetilde{x}_F$, we have

$$\begin{aligned} &\mathcal{A}(H(x)-K_L) = -2Dx+r(K_L+K_F) \le -2D\widetilde{x}_F+r(K_L+K_F) \\ &\le -\frac{\beta}{\beta-1}(r-\mu)\frac{2D}{2D-1}K_F+r\frac{2D}{2D-1}K_F = r\frac{2D}{2D-1}K_F\Big(1-\frac{\beta}{\beta-1}\frac{r-\mu}{r}\Big) < 0, \end{aligned} \tag{IA.K.114}$$

where the second inequality uses $K_L \le \frac{K_F}{2D-1}$, the last inequality uses $\frac{\beta}{\beta-1}\frac{r-\mu}{r}>1$. Hence, (IA.K.112) holds.

Combining (IA.K.115) and (IA.K.112), we can see that $W^{\text{CO}}(x)$ given by (IA.G.3)-(IA.G.4) satisfies (IA.K.109).

- **The case $D > 1/2$ and $K_L \geq \frac{K_F}{2D-1}$.** We first show that

$$[H(x_D) - K_L] \left(\frac{x}{x_D} \right)^{\beta} > H(x) - K_L, \quad x < x_D. \tag{IA.K.115}$$

Since $K_L \geq \frac{K_F}{2D-1}$, we have $x_D \geq \widetilde{x}_F$. Denote $f(x) = [H(x_D) - K_L] \left(\frac{x}{x_D} \right)^{\beta} - [H(x) - K_L]$. We can verify that $f(x_D) = f'(x_D) = 0$ using $x_D \geq \widetilde{x}_F$. Since $H(x)$ is linear in $x > \widetilde{x}_F$ and $H(x_D) - K_L = \frac{K_L + K_F}{\beta - 1} > 0$, we have $f''(x) > 0$ for $x > \widetilde{x}_F$. It follows that $f'(x) < 0$ and $f(x) > 0$ for $x \in [\widetilde{x}_F, x_D)$.

Using (IA.G.5), for any $x \leq \widetilde{x}_F$, we have

$$f(x) = K_L - \Pi(x) + \left[\frac{H(x_D) - K_L}{x_D^{\beta}} - \frac{(2D-1)\Pi(\widetilde{x}_F) - K_F}{\widetilde{x}_F^{\beta}} \right] x^{\beta}. \tag{IA.K.116}$$

If $\left[\frac{H(x_D) - K_L}{x_D^{\beta}} - \frac{(2D-1)\Pi(\widetilde{x}_F) - K_F}{\widetilde{x}_F^{\beta}} \right] \leq 0$, then $f'(x) < 0$ for any $x \leq \widetilde{x}_F$. It follows that $f(x) > f(\widetilde{x}_F) \geq 0$ for any $x \in [0, \widetilde{x}_F]$. If $\left[\frac{H(x_D) - K_L}{x_D^{\beta}} - \frac{(2D-1)\Pi(\widetilde{x}_F) - K_F}{\widetilde{x}_F^{\beta}} \right] > 0$, then $f''(x) > 0$ for any $x < \widetilde{x}_F$. Since $f'(\widetilde{x}_F) < 0$ and $f(\widetilde{x}_F) > 0$, we have $f'(x) < 0$ and $f(x) \geq f(\widetilde{x}_F) > 0$ for any $x \in [0, \widetilde{x}_F]$. In sum, we have $f(x) > 0$ for $x < x_D$, i.e. (IA.K.115) holds.

Since $x_D \geq \widetilde{x}_F$, we conclude from (IA.K.114) that (IA.K.112) holds for any $x > x_D$. Combining (IA.K.115), and $H(x_D) - K_L = 2D\Pi(x_D) - \big(K_L + K_F\big)$, we can see that $W^{\text{CO}}(x)$ given by (IA.G.7)-(IA.G.8) satisfies (IA.K.109).

In sum, we have solved (IA.K.109), and proved that the first term in (IA.K.109) equals to zero when $x < \widetilde{x}_L$, and the second term in (IA.K.109) equals to zero when $x > \widetilde{x}_L$, where $\widetilde{x}_L = \min\{x_M, x_D\}$ equals to x_M if $K_L < \frac{K_F}{2D-1}$ and equals to x_D if $K_L \geq \frac{K_F}{2D-1}$. Finally, we solve (IA.K.108). Recall $W^{\text{CO}}(x)$ is given by (IA.G.3)-(IA.G.4) for the case $D > 1/2$ and $K_L < \frac{K_F}{2D-1}$, and is given by (IA.G.7)-(IA.G.8) for the case $D > 1/2$ and $K_L \geq \frac{K_F}{2D-1}$. Applying Itô's Lemma to $e^{-rs}W^{\text{CO}}(X_s)$ for $s \in [t, \tau]$ and any stopping time τ, and taking expectations at time t with $X_t = x$, we obtain the following expression for $W^{\text{CO}}(x)$:

$$\begin{aligned} W^{\text{CO}}(x) &= \mathbb{E}_t^x[e^{-r(\tau - t)}W^{\text{CO}}(X_\tau)] - \mathbb{E}_t^x \left[\int_t^{\tau} e^{-r(s-t)} \mathcal{A}W^{\text{CO}}(X_s) ds \right] \\ &\geq \mathbb{E}_t^x[e^{-r(\tau - t)}(H(X_\tau) - K_L)], \end{aligned} \tag{IA.K.117}$$

where $\mathcal{A}$ is the infinitesimal generator given in (IA.K.1), the inequality follows from that $W^{\mathrm{co}}(x) \geq H(x) - K_L$ and $\mathcal{A}W^{\mathrm{co}}(x) \leq 0$ (see the variational inequality(IA.K.109)). When $\tau = \tau_L^{\mathrm{co}} = \inf\{s \geq t : X_s \geq \widetilde{x}_L\}$, (IA.K.117) becomes equality because $X_{\tau_L^{\mathrm{co}}} \geq \widetilde{x}_L$, $W^{\mathrm{co}}(x) = H(x) - K_L$ for $x \geq \widetilde{x}_L$, and $X_s < \widetilde{x}_L$ for $s < \tau_L^{\mathrm{co}}$, and $\mathcal{A}W^{\mathrm{co}}(x) = 0$ for $x < \widetilde{x}_L$. □

Proof of Theorem IA.1: We prove Theorem IA.1 for three mutually exclusive regions as follows: 1.) for the $(0, \widehat{x}_L)$ region, the proof of Theorem IA.1 is essentially the same as that of Theorem IA.2 for Case C; 2.) for the $[\widehat{x}_L, \widehat{x}_F]$ region, firms optimally compete to become Leader because $L(x) - K_L \geq F(x)$; and 3.) for the $x > \widehat{x}_F$ region, we prove that the inequalities (A.3) and (A.4) hold.

To ease our exposition of the proof for the $x > \widehat{x}_F$ region, recall that $\varphi_i^* = (\mathcal{E}^*, \lambda^*)$, $\mathcal{E}^* = [\widehat{x}_L, \widehat{x}_F]$, $\lambda^*(x) = 0$ for any $x < \widehat{x}_L$ and $x \in (\widehat{x}_F, \infty) \setminus \mathcal{R}^E$, and $\lambda^*(x) > 0$ given by (19) for any x in the probabilistic entry region $\mathcal{R}^E$ defined in (IA.B.2).

Also note that our proof below covers two subcases: 1.) for Subcase B_1, the mixed-strategy entry region is given by $\mathcal{R}^E = (\widehat{x}_F, \underline{x}] \cup [\overline{x}, \infty)$, where $\underline{x}$ and $\overline{x}$ are given in Lemma 1; 2.) for Subcase B_2, the mixed-strategy entry region is given by $\mathcal{R}^E = [\overline{x}, \infty)$, where $\overline{x}$ is the unique solution for (IA.J.15) in the $y > \frac{1}{D}\widetilde{x}$ region.

Now we are ready to prove that the inequality (A.3) hold for any $x > \widehat{x}_F$ in the following two steps.

Step 1: Prove $V_*(x) \geq J_a(x; \varphi_a, \varphi_b^*)$ in the $x > \widehat{x}_F$ region where $(\varphi_a, \varphi_b^*) \in \Phi$.

Let τ_a and τ_b denote firm a's and b's stochastic entry time associated with φ_a and φ_b^*, respectively, and let $\tau := \min\{\tau_a, \tau_b\}$. Because $\mathcal{E}^* = [\widehat{x}_L, \widehat{x}_F]$ and $X_t = x > \widehat{x}_F$, we have $X_s > \widehat{x}_F$ for any $s \in [t, \tau_b)$. For both Subcase B_1 and Subcase B_2, $V_*(x)$ is twice continuously differentiable except at finite points and is globally continuously differentiable. Applying Itô's Lemma to $e^{-rs}V_*(X_s)$ for $s \in [t, \tau]$, we obtain (IA.K.4). Since $V_*(x)$ satisfies (18) in the $x > \widehat{x}_F$ region, we obtain $V_*(x) \geq L(x) - K_L$ for all $x \geq \widehat{x}_F$. Using this result, (IA.K.4), and $X_s > \widehat{x}_F$ for all $s \in [t, \tau)$, we obtain (IA.K.5). Note that

$$\begin{aligned}
& J_a(x; \varphi_a, \varphi_b^*) \\
= \;& \mathbb{E}_t^x\left[e^{-r(\tau-t)}\left(\mathbf{1}_{\tau_a<\tau_b}(L(X_\tau) - K_L) + \mathbf{1}_{\tau_a>\tau_b}F(X_\tau) + \mathbf{1}_{\tau_a=\tau_b}\frac{L(X_\tau) - K_L + F(X_\tau)}{2}\right)\right] \\
= \;& \mathbb{E}_t^x[e^{-r(\tau-t)}(L(X_\tau) - K_L)] - \mathbb{E}_t^x\left[\mathbf{1}_{\tau_a>\tau_b}e^{-r(\tau-t)}(L(X_\tau) - K_L - F(X_\tau))\right], \qquad \text{(IA.K.118)}
\end{aligned}$$

where the second equality follows from $L(\widehat{x}_F) - K_L = F(\widehat{x}_F)$ (see Proposition 1) and the result that with probability 1, $\tau_a = \tau_b$ only when $X_\tau = \widehat{x}_F$. Using (IA.K.118) and (IA.K.5), we obtain

$$\begin{aligned}&J_a(x;\varphi_a,\varphi_b^*)\\ \leq &V_*(x)+\mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}\mathcal{A}V_*(X_s)ds-\mathbf{1}_{\tau_a>\tau_b}e^{-r(\tau-t)}(L(X_\tau)-K_L-F(X_\tau))\right].\end{aligned} \quad \text{(IA.K.119)}$$

Let $\widehat{\tau}:=\inf\{s\geq t: X_s\in[\widehat{x}_L,\widehat{x}_F]\}$ and $G(s):=1-(1-\mathbf{1}_{s\geq\widehat{\tau}})e^{-\int_t^s\lambda^*(X_u)du}$. We obtain:

$$\begin{aligned}\mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}\mathcal{A}V_*(X_s)ds\right]&=\mathbb{E}_t^x\left[\int_t^{\widehat{\tau}}\int_t^{\tau_a\wedge u}e^{-r(s-t)}\mathcal{A}V_*(X_s)dsdG(u)\right]\\ &=\mathbb{E}_t^x\left[\int_t^{\tau_a\wedge\widehat{\tau}}\int_s^{\widehat{\tau}}dG(u)e^{-r(s-t)}\mathcal{A}V_*(X_s)ds\right]\\ &=\mathbb{E}_t^x\left[\int_t^{\tau_a\wedge\widehat{\tau}}e^{-\int_t^s(r+\lambda^*(X_u))du}\mathcal{A}V_*(X_s)ds\right],\end{aligned} \quad \text{(IA.K.120)}$$

using Tonelli's Theorem, $\mathcal{A}V_*(x)\leq 0$, $G(\widehat{\tau})=1$, and integration by parts.

Because $\mathcal{A}V_*(x)=\mathbf{1}_{x\in\mathcal{R}^E}\left[(rK_L-x)\mathbf{1}_{x<x_F}+(rK_L-Dx)\mathbf{1}_{x>x_F}\right]$ is locally bounded in the $x\geq\widehat{x}_F$ region, we can rewrite (IA.K.120) as follows:

$$\begin{aligned}&\mathbb{E}_t^x\left[\int_{s\in[t,\tau_a\wedge\widehat{\tau})}e^{-\int_t^s(r+\lambda^*(X_u))du}\mathcal{A}V_*(X_s)ds\right]\\ =&\mathbb{E}_t^x\left[\int_{s\in[t,\tau_a\wedge\widehat{\tau})}e^{-\int_t^s(r+\lambda^*(X_u))du}\lambda^*(X_s)[L(X_s)-K_L-F(X_s)]ds\right]\\ =&\mathbb{E}_t^x\left[\mathbf{1}_{\tau_a\wedge\widehat{\tau}>\tau_b}e^{-r(\tau_b-t)}[L(X_{\tau_b})-K_L-F(X_{\tau_b})]\right]\\ =&\mathbb{E}_t^x\left[\mathbf{1}_{\tau_a>\tau_b}e^{-r(\tau_b-t)}[L(X_{\tau_b})-K_L-F(X_{\tau_b})]\right],\end{aligned} \quad \text{(IA.K.121)}$$

where the first equality uses $\mathcal{A}V_*(x)=\lambda^*(x)[L(x)-K_L-F(x)]$ and the last equality follows from $\tau_b\leq\widehat{\tau}$, $X_{\widehat{\tau}}=\widehat{x}_F$, and $L(\widehat{x}_F)-K_L=F(\widehat{x}_F)$.

Combining (IA.K.119), (IA.K.120), and (IA.K.121), we obtain $V_*(x)\geq J_a(x;\varphi_a,\varphi_b^*)$.

Step 2: Prove $V_*(x)=J_a(x;\varphi_a^*,\varphi_b^*)$ in the $x>\widehat{x}_F$ region.

Let τ_a^* and τ_b^* be firm a's and b's stochastic entry time, respectively, associated with strategy $(\varphi_a^*,\varphi_b^*)$, and let $\tau^*:=\min\{\tau_a^*,\tau_b^*\}$. Because $\lambda^*(x)=0$ for any $x\in(\widehat{x}_F,\infty)\setminus\mathcal{R}^E$, we have $X_{\tau^*}\in\mathcal{R}^E\cup\{\widehat{x}_F\}$, which implies $V_*(X_{\tau^*})=L(X_{\tau^*})-K_L$. Therefore, (IA.K.119) holds with equality if $\tau_a=\tau_a^*$, $\tau_b=\tau_b^*$, and $\tau=\tau^*$. We thus have shown $V_*(x)=J_a(x;\varphi_a^*,\varphi_b^*)$.

In sum, we have proven (A.3) via Steps 1 and 2. By symmetry, we can also prove (A.4). □

Proof of Existence of Equilibrium in Theorem IA.2: Let $\tau_a^*=\tau_b^*=\widehat{\tau}:=\inf\{s\geq t: X_s\geq\widehat{x}_L\}$. We prove that (τ_a^*,τ_b^*) is the equilibrium strategy pair in three steps.

First, because $L(x) - K_L \geq F(x)$ holds in $x \geq \widehat{x}_L$ region, it is optimal for firms to compete to enter as Leader in this region. Leader is selected randomly via the rent-equalization principle of Dixit and Pindyck (1994) and Grenadier (1996), which implies $V_i(x) = (L(x) - K_L + F(x))/2$.

Second, we analyze the solution in the $x \in (0, \widehat{x}_L)$ region. As both firms wait in the $(0, \widehat{x}_L)$ region and compete to enter only when $\{X_s; s \geq 0\}$ exceeds $\widehat{x}_L$, firm i's value is given by

$$V_i(x) = \mathbb{E}_t^x \left[e^{-r(\widehat{\tau}-t)} \frac{L(X_{\widehat{\tau}}) - K_L + F(X_{\widehat{\tau}})}{2} \right] = \mathbb{E}_t^x \left[e^{-r(\widehat{\tau}-t)} F(X_{\widehat{\tau}}) \right] = F(x), \qquad \text{(IA.K.122)}$$

where the first equality is due to definition (14), the second equality follows from $L(\widehat{x}_L) - K_L = F(\widehat{x}_L)$ (see Proposition 1) and $X_{\widehat{\tau}} = \widehat{x}_L$, and the last equality follows from the property that Follower's present value is a martingale in its waiting region.

Third, we show that firms have no incentives to deviate from the strategy pair (τ_a^*, τ_b^*). Suppose firm a chooses its entry time τ_a, deviating from τ_a^*, and firm b chooses $\tau_b = \tau_b^*$. Let $\tau := \min\{\tau_a, \tau_b^*\}$. Using the definition of $J_i(x)$ given in (14), we obtain

$$J_a(x) \leq \mathbb{E}_t^x \left[e^{-r(\tau-t)} F(X_\tau) \right] = F(x) = V_a(x), \qquad \text{(IA.K.123)}$$

where the inequality in (IA.K.123) follows from 1.) $\tau_b = \widehat{\tau}$ and 2.) the property that $X(s) \leq \widehat{x}_L$ and $L(X_s) - K_L \leq F(X_s)$ hold for any $s \in [t, \tau]$ (see Proposition 1), the first equality follows from the property that $\{F(X_s); s \geq 0\}$ is a martingale in the pre-entry region $X_s \leq x_F$ and the second equality follows from (IA.K.122). Therefore, firm a has no incentives to deviate from τ_a^*. The same analysis holds for firm b. We thus have proven (τ_a^*, τ_b^*) is the equilibrium strategy pair. □

Proof of Uniqueness of Equilibrium in Theorem IA.2: We prove the uniqueness of equilibrium strategy in three steps. Let $(\mathcal{E}_a^*, \mathcal{E}_b^*)$ denote any equilibrium strategy.

Step 1: We prove $(0, \widehat{x}_L) \cap \mathcal{E}_i^* = \emptyset$*.* Consider firm a deviates to strategy $\mathcal{E}_a = [\widehat{x}_L, \infty)$. Then, starting from any $x \in (0, \widehat{x}_L)$, we have $X_s \leq \widehat{x}_L$ for any $s \leq \tau_L$, where τ_L is Leader's entry time under strategy $(\mathcal{E}_a, \mathcal{E}_b^*)$. Since $X_{\tau_a} = \widehat{x}_L$ and $L(\widehat{x}_L) - K_L = F(\widehat{x}_L)$, we have $L(X_{\tau_a}) - K_L = F(X_{\tau_a})$. Thus, firm a's value for any $x \in (0, \widehat{x}_L)$ is given by

$$J_a(x; \mathcal{E}_a, \mathcal{E}_b^*) = \mathbb{E}_t^x \left[e^{-r(\tau_L - t)} F(X_{\tau_L}) \right] = F(x), \qquad \text{(IA.K.124)}$$

where the second equality uses $X_s < \widehat{x}_L < x_F$ for any $s < \tau_L$. Combining (IA.K.124) with (IA.D.1), we have $J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) \geq F(x)$ for $x \in (0, \widehat{x}_L)$. If there exists $x \in (0, \widehat{x}_L) \cap \mathcal{E}_a^*$, then we have $J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) = L(x) - K_L$ if $x \notin \mathcal{E}_b^*$ and $J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) = \frac{L(x) - K_L + F(x)}{2}$ if $x \in \mathcal{E}_b^*$. Since

$L(x) - K_L < F(x)$ using Proposition 1, we have $J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) < F(x)$. Then we arrive at a contradiction. Hence, $(0, \widehat{x}_L) \cap \mathcal{E}_a^* = \emptyset$. Similarly, we have $(0, \widehat{x}_L) \cap \mathcal{E}_b^* = \emptyset$.

Step 2: We prove $(\widehat{x}_L, \infty) \cap \mathcal{E}_a^* = (\widehat{x}_L, \infty) \cap \mathcal{E}_b^*$ for $R < 1$ and $(\widehat{x}_L, x_F) \cap \mathcal{E}_a^* = (\widehat{x}_L, x_F) \cap \mathcal{E}_b^*$ for $R = 1$. We first consider the case $R < 1$. For the sake of contradiction, assume there exists $x > \widehat{x}_L$, such that $x \in \mathcal{E}_a^*$ and $x \notin \mathcal{E}_b^*$. Then we have $J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) = L(x) - K_L$ and $J_b(x; \mathcal{E}_a^*, \mathcal{E}_b^*) = F(x)$. If firm b deviates to $\mathcal{E}_b = [\widehat{x}_L, \infty)$, then we have $J_b(x; \mathcal{E}_a^*, \mathcal{E}_b) = \frac{L(x) - K_L + F(x)}{2} > F(x) = J_b(x; \mathcal{E}_a^*, \mathcal{E}_b^*)$, which is a contradiction. Hence, for any $x \in (\widehat{x}_L, \infty) \cap \mathcal{E}_a^*$, we have $x \in (\widehat{x}_L, \infty) \cap \mathcal{E}_b^*$. Similarly, we can show that for any $x \in (\widehat{x}_L, \infty) \cap \mathcal{E}_b^*$, we have $x \in (\widehat{x}_L, \infty) \cap \mathcal{E}_a^*$. Therefore, we have $(\widehat{x}_L, \infty) \cap \mathcal{E}_a^* = (\widehat{x}_L, \infty) \cap \mathcal{E}_b^*$. If $R = 1$, then using $\frac{L(x) - K_L + F(x)}{2} > F(x)$ for any $x \in (\widehat{x}_L, x_F)$, we can similarly show that $(\widehat{x}_L, x_F) \cap \mathcal{E}_a^* = (\widehat{x}_L, x_F) \cap \mathcal{E}_b^*$ for $R < 1$.

Step 3: We prove $\{x \in (\widehat{x}_L, \infty) : x \notin \mathcal{E}_a^* \cup \mathcal{E}_b^*\} = \emptyset$.

For the sake of contradiction, assume $\{x \in (\widehat{x}_L, \infty) : x \notin \mathcal{E}_a^* \cup \mathcal{E}_b^*\} \neq \emptyset$. Consider $x \in \{x \in (\widehat{x}_L, \infty) : x \notin \mathcal{E}_a^* \cup \mathcal{E}_b^*\}$. If firm a deviates to $\mathcal{E}_a = [\widehat{x}_L, \infty)$, then we have $J_a(x; \mathcal{E}_a, \mathcal{E}_b^*) = L(x) - K_L$. It follows that $J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) \geq L(x) - K_L$. Similarly, we have

$$J_i(x; \mathcal{E}_a^*, \mathcal{E}_b^*) \geq L(x) - K_L, \quad x > \widehat{x}_L, x \notin \mathcal{E}_a^* \cup \mathcal{E}_b^*. \tag{IA.K.125}$$

When $R < 1$, combining result proved in Steps 1-2 and using $\mathcal{E}_i^*$ is a closed set, we have $\mathcal{E}_a^* = \mathcal{E}_b^*$. Hence, we have

$$J_i(x; \mathcal{E}_a^*, \mathcal{E}_b^*) = \mathbb{E}_t^x \left[e^{-r(\tau_L - t)} \frac{L(X_{\tau_L}) - K_L + F(X_{\tau_L})}{2} \right], \tag{IA.K.126}$$

where $\tau_L = \inf\{s : X_s \in \mathcal{E}_a^* \cup \mathcal{E}_b^*\}$. When $R = 1$, we have $L(x) - K_L = F(x) = Dx - K_L$ for $x \geq x_F$. Combining it with the result proved in Steps 1-2, and $L(\widehat{x}_L) - K_L = F(\widehat{x}_L)$, we can see (IA.K.126) also holds.

When $R < 1$, we have shown that $\mathcal{E}_a^* = \mathcal{E}_b^*$. Since $\mathcal{E}_i^*$ is a closed set, $\mathbb{R} \setminus \mathcal{E}_i^*$ is open set and thus $\{x \in (\widehat{x}_L, \infty) : x \notin \mathcal{E}_i^*\}$ is a open set, that can be represented by the union of open intervals. Let (x_1, x_2) be one of such intervals, where $x_1, x_2 \in \mathcal{E}_a^* = \mathcal{E}_b^*$. Then for any $x \in (x_1, x_2)$, we have

$$L(x) - K_L \leq J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) = \theta_1(x_1, x_2) x^\beta + \theta_2(x_1, x_2) x^\gamma, \tag{IA.K.127}$$

where the inequality uses (IA.K.125), $\theta_1(a, b)$ and $\theta_2(a, b)$ are given in (A.12) but with $L(\cdot) - K_L$ replaced by $\frac{L(\cdot) - K_L + F(\cdot)}{2}$. Using the right side in (IA.K.127), we have $\lim_{x \to x_2-} J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) = \frac{L(x_2) - K_L + F(x_2)}{2}$. Using the left side in (IA.K.127), we have $\lim_{x \to x_2-} J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) \geq L(x_2) - K_L > \frac{L(x_2) - K_L + F(x_2)}{2}$. Then we arrive at a contradiction. Therefore, $\{x \in (\widehat{x}_L, \infty) : x \notin \mathcal{E}_i\} =$

$\emptyset$. As a result, $(\widehat{x}_L, \infty) \subseteq \mathcal{E}_i^*$, $i = a, b$. Combining it with the result proved in Step 1, and using that $\mathcal{E}_i^*$ is a closed set, we conclude that $\mathcal{E}_i^* = [\widehat{x}_L, \infty)$ for $i = a, b$.

Next, we consider the case $R = 1$. Using (IA.K.126) and Lemma 7, we have

$$J_i(x; \mathcal{E}_a^*, \mathcal{E}_b^*) \leq \frac{L(x) - K_L + F(x)}{2}, \quad x \geq \widetilde{x}. \tag{IA.K.128}$$

Since $\mathcal{E}_i^*$ is a closed set, $\{x \in (\widehat{x}_L, \infty) : x \notin \mathcal{E}_a^* \cup \mathcal{E}_b^*\}$ is an open set and thus can be represented by the union of open interval. Let (x_1, x_2) be one of such interval, where $x_1, x_2 \in \mathcal{E}_a^* \cup \mathcal{E}_b^*$. If $x_2 \in (\widehat{x}_L, x_F)$, then using $(\widehat{x}_L, x_F) \cap \mathcal{E}_a^* = (\widehat{x}_L, x_F) \cap \mathcal{E}_b^*$, we conclude that $x_1, x_2 \in \mathcal{E}_i^*$ for $i = a, b$, and (IA.K.127) also holds. Using $L(x) - K_L > F(x)$ for any $x \in (\widehat{x}_L, x_F)$, we still arrive at a contradiction. Hence, we only need to consider the case $x_2 \geq x_F$.

If $x_1 < x_F$, then for $x \in (x_1, x_F)$, we derive from (IA.K.125) that $J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) \geq L(x) - K_L > \frac{L(x) - K_L + F(x)}{2}$, which contradicts with (IA.K.128) and $\widetilde{x} < x_F$. Hence, we must have $x_1 \geq x_F$. For any $x \in (x_1, x_2)$, we have

$$\begin{aligned} &J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) = \mathbb{E}_t^x[e^{-r(\tau_L - t)}(D\Pi(X_{\tau_L}) - K_F)] \\ =&D\Pi(x) - K_F + \mathbb{E}_t^x\left[\int_t^{\tau_L} e^{-r(s-t)}(rK_F - DX_s)ds\right] < D\Pi(x) - K_F = L(x) - K_L, \end{aligned} \tag{IA.K.129}$$

where the first equality uses (IA.K.126), $X_{\tau_L} \geq x_F$, and $L(x) - K_L = F(x) = D\Pi(x) - K_F$ for $x \geq x_F$, the inequality uses $DX_s - rK_F \geq Dx_F - rK_F = rK_F\left(\frac{\beta}{\beta-1}\frac{r-\mu}{r} - 1\right) > 0$ for $s \in (t, \tau_L)$. Using (IA.K.125) with (IA.K.129), we arrive at a contradiction. Therefore, $\{x \in (\widehat{x}_L, \infty) : x \notin \mathcal{E}_a^* \cup \mathcal{E}_b^*\} = \emptyset$, i.e. $(\widehat{x}_L, \infty) \subseteq \mathcal{E}_a^* \cup \mathcal{E}_b^*$. Combining it with $(\widehat{x}_L, x_F) \cap \mathcal{E}_a^* = (\widehat{x}_L, x_F) \cap \mathcal{E}_b^*$, we have $(\widehat{x}_L, x_F) \subseteq \mathcal{E}_i^*$ for $i = a, b$. In the region $x \geq x_F$, Follower will enter the market immediately if Leader is in the market and $L(x) - K_L = F(x) = D\Pi(x) - K_F$. Then we conclude from $[x_F, \infty) \subseteq (\widehat{x}_L, \infty) \subseteq \mathcal{E}_a^* \cup \mathcal{E}_b^*$ that one firm enters the market immediately and the another firm follows simultaneously in the region $x \geq x_F$. Since both firms receive $D\Pi(x) - K_F$, we do not need to differentiate which one is Leader and which one is Follower. $\square$

Proof of Theorem IA.3: Using (IA.F.4), (IA.F.7), $D_L < D_F$, and $K_L = K_F$, we conclude that $L(x) - K_L < F(x)$ for any $x \geq x_F$. Denote $\psi(y) := y - \left(\frac{\beta}{\beta-1}\frac{1+D_F-D_L}{D_F} - 1\right)\left(D_F\frac{\beta-1}{\beta}\right)^\beta y^\beta$ where $y > 0$. Using (IA.F.3), (IA.F.6) and $K_L = K_F$, we have

$$\begin{aligned} L(x) - K_L - F(x) &= \Pi(x) - K_F\left[1 + \left(\frac{x}{x_F}\right)^\beta\left(\frac{\beta}{\beta-1}\frac{1+D_F-D_L}{D_F} - 1\right)\right] \\ &= K_F\left[\psi\left(\frac{x}{(r-\mu)K_F}\right) - 1\right], \quad x \leq x_F. \end{aligned} \tag{IA.K.130}$$

We can verify that $\psi(y)$ is increasing ($\psi'(y) > 0$) for $y \in (0, y^*)$ and decreasing ($\psi'(y) < 0$) for $y > y^*$, where

$$y^* = \frac{1}{D_F} \frac{\beta}{\beta - 1} \left(\frac{1}{\beta(1 - D_L) + D_F} \right)^{1/(\beta-1)}. \tag{IA.K.131}$$

Using $\beta(1 - D_L) + D_F > 1 - D_L + D_F > 1$, we can see $y^* < \frac{1}{D_F}\frac{\beta}{\beta-1}$. In addition, we can show that $\psi(0) = 0$, $\psi(y^*) = \frac{1}{D_F}\left(\frac{1}{\beta(1-D_L)+D_F}\right)^{1/(\beta-1)}$, and $\psi(\frac{1}{D_F}\frac{\beta}{\beta-1}) = 1 + \frac{\beta}{\beta-1}\frac{D_L - D_F}{D_F} < 1$. Using these properties, we can show that for $D_L > 1 + \frac{D_F - D_F^{1-\beta}}{\beta}$, we have $\psi(y^*) = \frac{1}{D_F}\left(\frac{1}{\beta(1-D_L)+D_F}\right)^{1/(\beta-1)} > 1$, the $\psi(y) = 1$ equation has two positive roots: $\widehat{y}_L$ and $\widehat{y}_F$, satisfying $0 < \widehat{y}_L < y^* < \widehat{y}_F < \frac{1}{D_F}\frac{\beta}{\beta-1}$. Note that $\psi(y)$ is increasing in $y \in [0, y^*]$ and decreasing in $y \geq y^*$. We then infer

$$\psi(y) < 1, \quad y \in (0, \widehat{y}_L) \cup (\widehat{y}_F, +\infty), \tag{IA.K.132}$$

$$\psi(y) > 1, \quad y \in (\widehat{y}_L, \widehat{y}_F). \tag{IA.K.133}$$

Using (IA.K.130), (IA.K.132)-(IA.K.133), we obtain (IA.K.88), where $\widehat{x}_F = \widehat{y}_F(r - \mu)K_F$, $\widehat{x}_L = \widehat{y}_L(r - \mu)K_F$.

When $D_L < 1 + \frac{D_F - D_F^{1-\beta}}{\beta}$, we have $\psi(y^*) = \frac{1}{D_F}\left(\frac{1}{\beta(1-D_L)+D_F}\right)^{1/(\beta-1)} < 1$, which implies that $\psi(y) < 1$ for any $y > 0$. Using (IA.K.130), we conclude that $L(x) - K_L < F(x)$ for $x > 0$.

The proof of equilibrium value function and equilibrium strategy in the case where $D_L < 1 + \frac{D_F - D_F^{1-\beta}}{\beta}$ is the same as that of Theorem 1. The proof of equilibrium value function and equilibrium strategy in the case where $D_L > 1 + \frac{D_F - D_F^{1-\beta}}{\beta}$ is the same as that of Theorem IA.1.

□

Proof of Theorem IA.4: The proof is similar to that in Theorem IA.1. □

Proof of Theorem 6: The proof is similar to that in Theorem IA.1. □

Proof of Theorem IA.5: The proof is similar to that in Theorem 2. □

Proof of Proposition 4: First, we conclude from Lemma 5 that $P_L(x) = V_*(x)$.

Next, we prove that $\mathcal{E}_a^* = [x_L, \infty)$ and $\mathcal{E}_b^* = \emptyset$ form an asymmetric pure-strategy entry equilibrium in two steps.

Step 1: We show that Leader has no incentives to deviate its strategy from $\mathcal{E}_a^* = [x_L, \infty)$ to an alternative strategy $\mathcal{E}_a$.

Let $\tau_a = \inf\{s \geq t : X_s \in \mathcal{E}_a\}$. Using

$$\mathbb{E}_t^x \left[e^{-r(\tau_L^*-t)}(L(X_{\tau_L^*}) - K_L)\right] = V_*(x) = \max_{\tau \geq t} \mathbb{E}_t^x \left[e^{-r(\tau-t)}(L(X_\tau) - K_L)\right], \tag{IA.K.134}$$

we obtain

$$\mathbb{E}_t^x \left[e^{-r(\tau_L^*-t)}(L(X_{\tau_L^*}) - K_L)\right] \geq \mathbb{E}_t^x \left[e^{-r(\tau_a-t)}(L(X_{\tau_a}) - K_L)\right]. \tag{IA.K.135}$$

Inequality (IA.K.135) together with $\mathcal{E}_b^* = \emptyset$ implies that Leader has no incentive to deviate.

Step 2: We show that Follower has no incentives to deviate its strategy from $\mathcal{E}_b^* = \emptyset$ to another strategy $\mathcal{E}_b$.

Using the definition of $J_F(x)$ given in (32), we obtain $J_b(x; \mathcal{E}_a^*, \mathcal{E}_b^*) = J_F(x)$. Let $\tau_a^* := \inf\{s \geq t : X_s \geq \overline{x}\}$, $\tau_b := \inf\{s \geq t : X_s \in \mathcal{E}_b\}$, and $\tau := \min\{\tau_a^*, \tau_b\}$. For any $x \geq \overline{x}$, we conclude from the properties that $F(x) \geq L(x) - K_L$ and $e^{-rs}F(X_s)$ is a supermartingale that

$$J_b(x; \mathcal{E}_a^*, \mathcal{E}_b) \leq \mathbb{E}_t^x[e^{-r(\tau-t)}F(X_\tau)] \leq F(x) = J_F(x).$$

For any $x \in (0, \overline{x})$, applying Itô's Lemma to $e^{-rs}J_F(X_s)$ where $s \in [t, \tau]$, we obtain

$$\begin{aligned} J_F(x) =& \mathbb{E}_t^x[e^{-r(\tau-t)}J_F(X_\tau)] = \mathbb{E}_t^x \left[e^{-r(\tau-t)}\left(F(X_{\tau_a^*})\mathbf{1}_{\tau_a^* \leq \tau_b} + J_F(X_\tau)\mathbf{1}_{\tau_a^* > \tau_b}\right)\right] \\ \geq& \mathbb{E}_t^x \left[e^{-r(\tau-t)}\left(F(X_{\tau_a^*})\mathbf{1}_{\tau_a^* \leq \tau_b} + (L(X_\tau) - K_L)\mathbf{1}_{\tau_a^* > \tau_b}\right)\right] \geq J_b(x; \mathcal{E}_a, \mathcal{E}_b^*), \end{aligned}$$

where the second equality uses $X_{\tau_a^*} \geq \overline{x}$ and $J_F(x) = F(x)$ for all $x \geq \overline{x}$, the first inequality follows from $X_\tau \leq \overline{x}$ and $J_F(x) \geq L(x) - K_L$ for any $x \leq \overline{x}$, and the last inequality uses the result: $F(x) \geq L(x) - K_L$ for any $x > 0$. ☐

Proof of Proposition IA.1: This is implied by Theorem IA.1 and parts (i)-(ii) of Lemma 4. ☐

Proof of Proposition IA.2: Our proof uses similar arguments as those for the proofs of Theorem IA.1 and Propositions 4-2. ☐

Proof of Lemma 5: Using similar arguments as those for the proofs of Lemma 3, we can see the solution for the variational-inequality problem (18) is given by (A.6)-(A.7) when $R > R_{A_1A_2}$, and is given by (IA.J.6)-(IA.J.9) when $1 < R \leq R_{A_1A_2}$. Moreover, $\mathcal{R}^E = [\overline{x}, \infty)$ with $\overline{x}$ is given by (A.8), when $R > R_{A_1A_2}$, and $\mathcal{R}^E = [\widetilde{x}, \underline{x}] \cup [\overline{x}, \infty)$, with $\underline{x}$ and $\overline{x}$, are given in Lemma 1, when $1 < R \leq R_{A_1A_2}$.

Applying Itô's Lemma to $e^{-rs}V_*(X_s)$ for $s \in [t, \tau]$ where $\tau \geq t$ is a stopping time, we obtain

$$V_*(x) = \mathbb{E}_t^x[e^{-r(\tau-t)}V_*(X_\tau)] - \mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}\mathcal{A}V_*(X_s)ds\right] \geq \mathbb{E}_t^x\left[e^{-r(\tau-t)}(L(X_\tau) - K_L)\right], \tag{IA.K.136}$$

where the inequality follows from $\mathcal{A}V_*(x) \leq 0$ and $V_*(x) \geq L(x) - K_L$. Also note that when $\tau = \tau_L^*$, (IA.K.136) holds with equality. This is because $\mathcal{A}V_*(x) = 0$ for $x \notin \mathcal{R}^E$, and $V_*(x) = L(x) - K_L$ for $x \in \mathcal{R}^E$.

Therefore, we have shown

$$V_*(x) = \max_{\tau\geq t}\ \mathbb{E}_t^x\left[e^{-r(\tau-t)}(L(X_\tau) - K_L)\right] = J_L(x)\,. \tag{IA.K.137}$$

□

Proof of Lemma 6: In Internet Appendix IA.B.2, we show that the optimal strategy to problem (31) is $\tau_L^* = \inf\{t : X_t \in \mathcal{R}^E\}$, where $\mathcal{R}^E(K_L) := \{x > 0 : J_L(x) = L(x) - K_L\}$.

Consider $K_L > K_F$ and a sufficiently small $\epsilon > 0$ such that $K_L - \epsilon > K_F$. For $x \in \mathcal{R}^E(K_L)$, we have

$$\begin{aligned}
&\max_{\tau\geq t}\ \mathbb{E}_t^x\left[e^{-r(\tau-t)}(L(X_\tau) - (K_L - \epsilon))\right] \qquad \text{(IA.K.138)}\\
\leq&\max_{\tau\geq t}\ \mathbb{E}_t^x\left[e^{-r(\tau-t)}(L(X_\tau) - K_L)\right] + \max_{\tau\geq t}\ \mathbb{E}_t^x\left[e^{-r(\tau-t)}\epsilon\right]\\
=&L(x) - K_L + \epsilon,
\end{aligned}$$

where the equality follows from that the optimal strategy to problem (31) is entering the market immediately when $x \in \mathcal{R}^E(K_L)$. Hence, for $x \in \mathcal{R}^E(K_L)$, the optimal strategy to problem (IA.K.138) is entering the market immediately and $x \in \mathcal{R}^E(K_L - \epsilon)$. It follows that $\mathcal{R}^E(K_L) \subseteq \mathcal{R}^E(K_L - \epsilon)$.

Since $F(x) > L(x) - K_L$ for any $x > 0$, and $\lambda^*(x; K_L) = \frac{(\mathbf{1}_{x<x_F} + D\mathbf{1}_{x\geq x_F})x - rK_L}{F(x) - (L(x) - K_L)} > 0$ for $x \in \mathcal{R}^E(K_L)$, we have

$$\lambda^*(x; K_L) = \frac{(\mathbf{1}_{x<x_F} + D\mathbf{1}_{x\geq x_F})x - rK_L}{F(x) - (L(x) - K_L)} < \frac{(\mathbf{1}_{x<x_F} + D\mathbf{1}_{x\geq x_F})x - r(K_L - \epsilon)}{F(x) - (L(x) - (K_L - \epsilon))} = \lambda^*(x; K_L - \epsilon), \tag{IA.K.139}$$

for any $x \in \mathcal{R}^E(K_L)$, and small $\epsilon > 0$, where the last equality uses $\mathcal{R}^E(K_L) \subseteq \mathcal{R}^E(K_L - \epsilon)$. Since $\lambda^*(x; K_L) = 0$ for $x \notin \mathcal{R}^E(K_L)$, and $\lambda^*(x; K_L - \epsilon) \geq 0$ for all x, we have $\lambda^*(x; K_L - \epsilon) \geq \lambda^*(x; K_L)$ for all $x > 0$. □

Proof of Lemma 7: We first show that

$$\frac{L(x_M) - K_L + F(x_M)}{2}\left(\frac{x}{x_M}\right)^{\beta} > \frac{L(x) - K_L + F(x)}{2}, \quad x < x_M. \tag{IA.K.140}$$

Since $K_L \le K_F < K_F/D$, we have $x_M < x_F$. Thus, for any $x < x_M$,

$$\frac{L(x) - K_L + F(x)}{x^{\beta}} = \frac{\Pi(x) - K_L}{x^{\beta}} + \frac{(2D-1)\Pi(x_F) - K_F}{x_F^{\beta}}. \tag{IA.K.141}$$

Since $\frac{\Pi(x)-K_L}{x^{\beta}}$ achieves the maximum exactly at $x = x_M$, we derive from (IA.K.141) that $\frac{L(x)-K_L+F(x)}{x^{\beta}}$ achieves the maximum exactly at $x = x_M$, and thus (IA.K.140) holds.

For any $x \in (x_M, x_F)$, we have

$$\mathcal{A}(L(x) - K_L + F(x)) = -x + rK_L \le -x_M + rK_L = rK_L\Big(1 - \frac{\beta}{\beta-1}\frac{r-\mu}{r}\Big) < 0, \tag{IA.K.142}$$

where the inequality uses $\frac{\beta}{\beta-1}\frac{r-\mu}{r} > 1$. For any $x > x_F$, we have

$$\begin{aligned}\mathcal{A}(L(x) - K_L + F(x)) &= -2Dx + r(K_L + K_F) \le -2Dx_F + r(K_L + K_F)\\ &\le -2(r-\mu)\frac{\beta}{\beta-1}K_F + 2rK_F = 2rK_F\Big(1 - \frac{\beta}{\beta-1}\frac{r-\mu}{r}\Big) < 0,\end{aligned} \tag{IA.K.143}$$

where the second inequality uses $K_L \le K_F$, the third inequality uses $\frac{\beta}{\beta-1}\frac{r-\mu}{r} > 1$.

Combining (IA.K.140), (IA.K.142), (IA.K.143), we can see that the optimal stopping to problem (IA.J.17) is $\inf\{s \ge t : X_s \ge x_M\}$, and the optimal value, is given by (IA.J.18)-(IA.J.19). □