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Dynamic Trading with Realization Utility

MIN DAI, CONG QIN, and NENG WANG*

ABSTRACT

An investor receives utility bursts from realizing gains and losses at the individual stock level and dynamically allocates his mental budget between risky and risk-free assets at the trading account level. Using savings, he reduces his stockholdings and is more willing to realize losses. Using leverage, he increases his stockholdings beyond his mental budget and is more reluctant to realize losses. While leverage strengthens the disposition effect, introducing leverage constraints mitigates it. Our model predicts that investors with stocks in deep losses sell them either immediately or after stocks rebound a little.

ONE OF THE MOST ROBUST findings about individual investors' trading behaviors is the disposition effect (Shefrin and Statman (1985)), whereby an investor has greater propensity to sell a stock that has gone up in value since purchase than one that has gone down.[1] Prospect theory (Kahneman and Tversky (1979), Tversky and Kahneman (1992)), which defines preferences over gains and losses, is often perceived as a natural explanation for the

*Min Dai is with the School of Accounting and Finance in the Faculty of Business and the Department of Applied Mathematics in the Faculty of Computer and Mathematical Sciences at The Hong Kong Polytechnic University. Cong Qin is with School of Finance, Shanghai University of Finance and Economics. Neng Wang is with Cheung Kong Graduate a School of Business. The authors thank Wei Xiong (the Editor); two anonymous referees; Li An; Nick Barberis; Patrick Bolton; Yi-Chun Chen; Kent Daniel; Steve Dimmock; Darrell Duffie; Bing Han; Xuedong He (discussant); David Hirshleifer; Harrison Hong; Chris Hsee; Lawrence Jin; Dong Lou; Michaela Pagel; Cameron Peng; Thomas J. Sargent; Paul Tetlock; Zuoquan Xu; Liyan Yang; and Jianfeng Yu; and seminar participants at Australasian Finance and Banking Conference, Chinese Finance Annual Meeting, China International Conference in Finance, Columbia University for helpful comments. Min Dai acknowledges support from the Hong Kong Research Grants Council (Grant Nos. 15217123, 15213422, and T32-615/24-R), the Hong Kong Polytechnic University (Grant Nos. P0042708, P0042456, and P0039114), and the National Natural Science Foundation of China (Grant No. 12071333). Cong Qin acknowledges support from the Fundamental Research Funds for the Central Universities and the National Natural Science Foundation of China (Grant No. 12571516). Neng Wang acknowledges support from CKGSB Research Institute and the RMI at NUS. We have read *The Journal of Finance* disclosure policy and have no conflicts of interest to disclose.

Correspondence: Neng Wang, Cheung Kong Graduate School of Business, Oriental Plaza, Tower E3, 3F, One East Chang An Avenue, Beijing 100738, China; e-mail: newang@gmail.com.

[1] For evidence on disposition effects, see Odean (1998) for retail investors' stock trading, Heath, Huddart, and Lang (1999) for executive stock option exercise, and Genesove and Mayer (2001) for the housing market.

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disposition effect. Barberis and Xiong (2009) (henceforth BX (2009)) show that the realization-utility implementation of prospect theory, where the investor receives a utility burst from realized gains and losses, more reliably predicts a disposition effect.

Barberis and Xiong (2012) and Ingersoll and Jin (2013) (henceforth BX (2012) and IJ (2013), respectively) develop tractable intertemporal realization-utility models that significantly advance our understanding of realization-utility investors' trading behaviors. We build on the powerful frameworks developed in BX (2012) and IJ (2013) but remove a key assumption in their papers: the investor has to allocate his entire stock-sale proceeds to the new stock that he buys. In practice, the investor does not restrict his trading strategies this way. We show that once we lift this trading restriction, the model's predictions change significantly.

To understand why it is crucial to remove this assumption, consider the following commonly shared gambling experience. A gambler allocates budget Π_0 and leaves his credit cards at home before going to a casino. He starts his first bet by choosing a fraction of Π_0. If he wins G dollars ($G > 0$ implies a win and $G < 0$ a loss), he receives a utility burst that depends on G and then closes this betting episode. His budget for subsequent gambles on this trip then becomes $\Pi_1 = \Pi_0 + G$ and the size of his next bet depends on Π_1. This betting process continues until the end of the trip.[2] This narrative suggests that in addition to the series of mental accounts for each individual bet, the gambler has a mental account for his entire trip with a stochastically evolving budget: $\{\Pi_n; n = 0, 1, 2, \ldots\}$.

Adapting the gambling narrative above to our trading model, an investor has a mental account with initial budget of Π_0 for his *intertemporal* realization-utility optimization purpose. (This account is mentally separate from his other accounts, e.g., those for consumption-smoothing and retirement purposes.) The investor begins his first investment episode when he buys a stock with a fraction of Π_0, and he saves the unused fraction of the budget in the risk-free asset. When selling this stock at time t, he receives a utility burst from the realized gains/losses (G_t) and ends this investment episode. He then obtains a new budget of Π_t by combining his stock-sale proceeds with his savings, and he starts his next investment episode by buying a new stock with a new allocation (out of his new budget Π_t), saving the unused part of the budget.

In sum, the investor has two layers of mental accounts that are interconnected but serve different purposes. For each investment episode, he uses a stock-level mental account to evaluate his utility burst based on the realized gains/losses. In addition, he has a dynamic mental account that aggregates

[2] In an experimental setting in which MBA student subjects play with real money, Thaler and Johnson (1990) study how prior outcomes affect risk choices. One of their findings is that a prior gain can stimulate risk-seeking—the "house money" effect. As Thaler (1999) writes, "If a series of gambles are bracketed together then the outcome of one gamble can affect the choices made later." Read, Loewenstein, and Rabin (1999) introduce "choice bracketing" to group individual choices together into sets and discuss narrow versus broad bracketing. See Kahneman and Lovallo (1993), among others, for related research.

all of the individual stock investment episodes via a stochastically evolving budget $\{\Pi_t; t \geq 0\}$ at the trading account level. The latter mental account is broader than the stock-level mental account and is analogous to the gambler's mental account for his casino trip. Our two-layered mental accounts are consistent with An et al. (2024), who show that investors have multiple mental accounts, for example, at the individual stock and brokerage account levels, and that mental accounts interact.

The interaction between the two layers of mental accounts suggests that both the *extensive margin* (when to begin the next investment episode and which stock to buy), analyzed in the literature, and the *intensive margin* (the size of the next stock position), are important to the investor. At each time t, the investor can choose to save a fraction of his budget Π_t or use leverage when buying a new stock. By saving, the investor spreads his stock trades out over time, reducing downside risk and mitigating the disposition effect. Alternatively, by using leverage the investor increases his trading size beyond his budget.

The other key new feature of our model is that stock prices are discontinuous with downward jumps, as commonly observed in markets. Incorporating jumps not only makes stock price processes more realistic, but more importantly also generates new predictions on trading strategies, for example, for stocks in deep losses, due to the interaction between the two new features of our model—jumps and the dynamic mental account.

Our model features three state variables: risk-free wealth W_t, risky wealth X_t, and reference level B_t, which we use to calculate realized gains and losses, at time t. Using the homogeneity property, we solve the model by working with (i) scaled risky wealth $x_t = X_t/B_t$ as in BX (2012) and IJ (2013), and (ii) scaled risk-free wealth $w_t = W_t/B_t$, which is new in our model. Note that the investor's risky wealth and risk-free savings are not completely fungible due to his two-layered mental accounts.

Turning to our model's key predictions, the first set of predictions pertains to the investor's *intensive margin*, that is, the allocation of his budget between the new stock that he chooses and the risk-free asset, and the effect of this intensive margin on his trading behavior.

Saving and Leverage (Intensive Margin) and Implications. When trading stocks at t, the investor allocates a fraction of his budget Π_t to the risk-free asset and the remaining part of his budget to a stock by targeting a constant ratio between his risk-free wealth and risky wealth, w^* (after netting out transaction costs). By saving ($w^* > 0$), the investor makes smaller trades and spreads his trades out over time, reduces his transaction costs (in dollars), and becomes less sensitive to losses in his stock position. Because only a fraction of his budget is at risk, he realizes both gains and losses more frequently. In contrast, by using leverage ($w^* < 0$), the investor can increase his trade size beyond his budget. Because of the increased risk exposure and larger transaction costs (in dollars) associated with leverage, the investor is more reluctant to realize losses, increasing the disposition effect. In contrast, the option to save and use leverage has a relatively smaller effect on the investor's gain realizations than on his loss realizations.

Our model's predictions on gain and loss realizations are consistent with findings in Barber et al. (2019) and Heimer and Imas (2022). According to Heimer and Imas (2022), "Access to leverage increased the disposition effect; … this increase was driven by a greater propensity to hold losses; gains were realized at the same rate" (p. 1646). Barber et al. (2019) find that "margin investors have a stronger disposition effect than cash investors" (p. 4).

Optimal Target: w^*. How does an investor determine the optimal mix between the risk-free asset and the stock that he chooses (w^*) when beginning a new investment episode? Consider the effect of stock return volatility. For volatile stocks, the investor saves a fraction of his budget Π in the risk-free asset ($w^* > 0$) to reduce his trading account's (dollar) risk exposure and trades more frequently. In contrast, for stocks with low volatility, the investor uses leverage to increase his trade size beyond his budget ($w^* < 0$) and trades less frequently. As a result, an investor using leverage is subject to a stronger disposition effect, ceteris paribus (Heimer and Imas (2022)).

Our model predicts that investors prefer stocks with either high or low volatility over stocks with intermediate volatility. This is because the value of saving in the risk-free asset is high for stocks with high volatility and the value of leverage is high for stocks with low volatility, but the value of saving and leverage is low for stocks with intermediate volatility. In sum, the flexibility of saving in the risk-free asset or using leverage can be quite valuable for investors in a high- or low-volatility environment.[3]

Leverage Constraints. Thus far, we have analyzed the effect of leverage for the case in which leverage constraints do not bind. But what if they do bind? By forcing the investor to realize losses sooner than he would prefer (so that creditors break even), binding leverage constraints make the investor realize losses sooner, mitigating his disposition effect. Anticipating this contingency, the investor lowers his leverage ex ante, which in turn makes him more risk-tolerant. Our model's prediction is in line with Heimer and Imas (2022), who find that leverage constraints can improve financial decision making by mitigating the disposition effect.

We next characterize our model solution via two cases that differ in the number of solution regions, resulting in distinct time-series predictions.

Model Solution: Three-Region Case (Case A) and Four-Region Case (Case B). For both Cases A and B, there is a gain-realization region, where $x \geq \overline{x}^*$, and a normal holding region, where $x \in (\underline{x}^*, \overline{x}^*)$. When x exceeds the endogenous gain-realization boundary $\overline{x}^*$, the investor voluntarily realizes gains. If gains ($1 < x < \overline{x}^*$) or losses ($\underline{x}^* < x < 1$) are small or moderate, where $\underline{x}^*$ is the endogenous loss-realization boundary, the investor does not trade. These two predictions

[3] Bian et al. (2024) show that stocks bought in margin accounts tend to have lower systematic volatility and total volatility than stocks bought in cash accounts, consistent with our model's prediction.

are consistent with prior literature. Our model's new predictions come from the region(s) to the left of $\underline{x}^*$.

Case A In the three-region case, the third region—$x \in (0, \underline{x}^*)$—is the loss-realization region, where the investor voluntarily realizes losses for all $x \in (0, \underline{x}^*)$.

Case B In the four-region case, there are two regions to the left of $\underline{x}^*$: the loss-realization region $x \in (\underline{x}^{**}, \underline{x}^*)$, where $\underline{x}^{**}$ and $\underline{x}^*$ are the lower and upper boundaries of this region, respectively, and the deep-loss holding region $x \in (0, \underline{x}^{**})$, which is absent in Case A.

A key difference between the two cases is whether the investor wants to sell his stock at a deep loss. A stock position is in a deep loss if its value X_t is significantly below its reference level B_t, that is, if $x_t = X_t/B_t$ is close to zero.[4] We show that whether the investor voluntarily sells his stock at a deep loss (corresponding to our three-region case) critically depends on whether he has set aside sufficient savings (w^*) in his trading account. Below we further discuss how this (intensive) savings margin, when combined with downside stock price jumps, leads to different time-series predictions.

Selling a Stock at a Deep Loss: Case A. Why would a realization-utility investor want to sell a stock at a deep loss in our model? Such a result could occur if the investor has set aside sufficient savings in the risk-free asset. Consider the following example. Andrew with a mental budget of \$100 in his account allocates \$10, or 10% of his budget, and later loses 90% on this stock. Realizing this stock's deep loss would leave him with a budget of $\$91 = \$10 \times (1 - 90\%) + \$90$ in his mental account, implying a 9% loss of his mental budget of \$100 in his trading account. With a mental budget of \$91, the present value of utility bursts from future trading activities is larger than the immediate utility cost of realizing this stock's deep loss, and thus Andrew is willing to voluntarily sell his stock at a deep loss.

With sufficient savings, voluntarily selling a stock at a deep loss is a unique prediction of our model. This prediction is broadly consistent with the portfolio-driven disposition effect (An et al. (2024)): the disposition effect is large when the portfolio is at a loss but nearly disappears when the portfolio is at a gain. To voluntarily sell stocks at a deep loss, it is necessary for the model to have (downward) stock price jumps and (sufficiently) large savings in the risk-free asset, both of which are new features of our model. Without jumps, the deep-loss region other than the right boundary $\underline{x}^*$ cannot be reached on the optimal path. Without large savings, it is always optimal for the investor to hold onto his stock positions in the deep-loss region.

What if the investor's savings (w^*) are small? This is our Case B.

[4] By "deep-loss realization," we mean that it is possible for an investor to sell a stock at any loss, including a loss arbitrarily close to 100%. As we show below, in other realization-utility models, for example, IJ (2013), it is not possible for an investor to sell a stock at a loss arbitrarily close to 100%. This is because there is a fourth "deep-loss holding region" to the left of the loss-realization region in these models.

Selling a Stock after it Rebounds: Case B. Consider the following example. With a mental budget of \$100, Brian allocates \$80 and later loses 90% on his stock. Realizing the stock's deep loss is too painful, because doing so would leave him with a budget of only $\$28 = \$80 \times (1 - 90\%) + \$20$, which corresponds to a huge (72%) loss of his \$100 trading budget. However, if the stock rebounds, cutting his stock's loss from 90% to 75%, Brian's budget would be $\$40 = \$28 + \$80 \times 15\%$, reducing the loss of his mental budget by just enough ($\$12 = \$40 - \$28$ in our example) that his utility cost of realizing a loss is dominated by the benefit of doing so, which is to reset his reference level for future trading.

This prediction is consistent with our observation that retail investors often sell their losing stocks after these stocks rebound a bit, which cannot be generated by diffusion models with standard reference-point dynamics, that is, those with constant growth rates.[5]

In terms of solution regions, the example above suggests that (i) there are two disconnected holding regions, namely, normal and deep-loss holding regions, and (ii) between the two holding regions is a loss-realization region. In sum, when the investor's savings are not sufficiently large, the solution features four regions: a gain-realization region, a loss-realization region, and the two disconnected holding regions. In addition, stock price jumps are important because they make the deep-loss holding region reachable on the optimal path.

Related Literature. BX (2009) analyze two implementations of prospect theory, one based on realized gains/losses and the other based on annual gains/losses. They find that the former more consistently predicts the disposition effect.[6]

BX (2012) show that an investor with piecewise linear realization utility (and loss aversion greater than one) realizes gains when a stock he owns goes up by a certain percentage but never voluntarily realizes losses, as the utility cost from realizing losses is too high compared with the benefit of resetting the reference for gain realizations in the future. As a result, in BX (2012) there are two regions, a gain-realization region where $x \geq \overline{x}^*$ and a holding region where $x \in (0, \overline{x}^*)$. We show that the BX (2012) result of no voluntary loss realization for investors with piecewise linear utility continues to hold in our model when investors can save a fraction of their budget or use leverage. In addition, the piecewise-linear-utility investor may use leverage but never saves: $w^* \leq 0$.

IJ (2013) incorporate S-shaped utility into the realization-utility framework proposed by BX (2009, 2012). A key prediction of IJ (2013) is that the investor is willing to voluntarily realize losses in order to reset his reference level for

[5] Models of BX (2012) and IJ (2013), when extended to a specification in which the reference point is an average of past stock prices (with exponentially decaying weights), can also generate this prediction. Alternatively, a belief-based model with the law of small numbers (Rabin and Vayanos (2010)) can also generate this prediction. We thank a referee for providing these alternative explanations.

[6] Kyle, Ou-Yang, and Xiong (2006) analyze one-time liquidation problems for a decision maker with prospect theory preferences but with no reinvestment options. Li and Yang (2013) develop a general-equilibrium model to examine the asset pricing and trading volume implications of prospect theory via the disposition effect.

future gain realizations. This is because, unlike in BX (2012), the investor in IJ (2013) becomes less sensitive to losses as his losses increase. Specifically, IJ (2013) analyze the investor's gain- and loss-realization strategies by characterizing the two cutoff thresholds that define the normal holding region along the optimal path.

He and Yang (2019) (henceforth HY (2019)) extend IJ (2013) to allow for a general S-shaped realization utility, a terminal expected utility, and an adaptive reference point. HY (2019) show that in general the solution for diffusion realization-utility models features *four* regions, with a deep-loss holding region being the fourth. To the best of our knowledge, HY (2019) are the first to discuss all four regions and to point out that the deep-loss holding region is not reached on the optimal path in diffusion models.[7] Note that diffusion models with S-shaped utility may feature only two regions for some parameter values because realizing losses is too painful as in BX (2012).[8]

We summarize the differences between our model and the three closely related papers discussed above in Table I. Different from the other papers, Column (1) shows that the investor in our model either saves or uses leverage. This is because the investor in our model has two mental accounts: a narrower mental account at the stock level for utility burst calculations as in the other papers, and a broader mental account for his intertemporal trading. It is this broader mental account that allows him to separate his stock position from the budget of his intertemporal mental account.

Column (2) shows that, also different from the other models, our model allows us to introduce a leverage constraint. We show that the leverage constraint mitigates the disposition effect by forcing the investor to realize losses. Column (3) focuses on jumps, another key feature of our model. We show that only with jumps can all of the regions (up to four) be visited on the optimal path in our model.

Column (4) of Table I shows that our model solution features two mutually exclusive cases: one with three regions (Case A) and one with four regions (Case B), while the solutions in the three other models feature only one case. Column (5) highlights another key difference across the models: all of the regions are reachable on the optimal path due to jumps in our model while in IJ (2013) and HY (2019) the fourth (deep-loss) region is not reachable.

Column (6) shows that Case A in our model uniquely predicts that voluntarily realizing deep losses (including those close to 100%) can be optimal. The mechanism generating this result again goes back to the investor's dynamic

[7] While implicitly containing four regions, IJ (2013) do not mention the deep-loss holding region. Because this fourth region is not on the optimal path, it is sufficient to use the three other regions to fully characterize the solution on the optimal path as in IJ (2013).

[8] For example, when the S-shaped utility is sufficiently close to the piecewise linear utility as in BX (2012), the solution has only two regions. In this special two-region case, the upper boundary for the loss-realization region $\underline{x}^*$ equals zero. See IJ (2013) for further discussion, for example, their Proposition 1.

Table I

Comparing Our Model with BX (2012), IJ (2013), and HY (2019)

This table compares the (infinite horizon with no liquidity shocks) models in the four papers

	Saving or Leverage (1)	Leverage Constraint (2)	Jump Shocks (3)	Number of Regions (4)	All Regions Reachable? (5)	Realize Deep Losses? (6)	Realize Losses after Rebound? (7)
BX (2012)	No	N/A	No	2	Yes	No	No
IJ (2013)	No	N/A	No	4[a]	No	No	No
HY (2019)	No	N/A	No	4[a]	No	No	No
Our Model							
Case *A*	Yes	Yes	Yes	**3**	**Yes**	**Yes**	No
Case *B*	Yes	Yes	Yes	4*	**Yes**	No	**Yes**

[a] Models that feature four-region solutions can produce two-region solutions for certain parameter values.

mental account. With sufficient savings, the investor has incentives to sell his stock even at a deep loss to reset the reference level for his future trading.[9]

Column (7) of Table I shows that Case *B* in our model predicts the following pattern: while an investor is unwilling to sell a stock at a deep loss, he voluntarily realizes losses after the stock price rebounds a bit. This is because this rebound cuts his losses by just enough that his benefit of realizing losses to reset his reference level for future trading is larger than the cost of realizing these losses.

The key difference across the four models can be summarized as follows. While generating very similar gain-realization strategies (i.e., to lock in small gains), the four models predict very different loss-realization strategies. In BX (2012), investors do not voluntarily realize losses. In IJ (2013) and HY (2019), investors are willing to realize moderate losses but not deep losses. In our model, investors voluntarily realize both moderate and deep losses.

The remainder of the paper proceeds as follows. Section I introduces the baseline model and Section II analytically characterizes its solution. Section III explores the model's implications of saving and leverage. Section IV extends the baseline framework by incorporating jump risks. Section V analyzes the special case of piecewise-linear realization utility. Section VI concludes. In Appendices A to F, we provide proofs of the main results and additional results.

I. Model

An investor has an account with budget $\Pi_0 > 0$ at $t = 0$ that is used solely for the purpose of intertemporal realization-utility optimization. He mentally separates this account from his other accounts, for example, those for consumption-smoothing and retirement purposes. The investor receives a utility burst only from realized gains and losses of the stock sale. While the investor has a stock-level mental account for each utility burst calculation, he also has a broader mental account for his intertemporal realization utility at the trading account level. In our two-layered mental account model, the investor is not required to spend his entire budget Π_t when he trades. Instead, he typically holds both a stock and the risk-free asset. In contrast, earlier realization-utility models, for example, BX (2012), IJ (2013), and HY (2019), require the investor to make a binary choice in that he either holds the stock that he chooses or invests in the risk-free asset at any time.[10] In sum, an investor in our model has not only an extensive margin (whether to invest in a stock) as in prior models but also an intensive margin (the size of his stock position). Finally, his mental account is self-financing.

[9] In contrast, in the three papers discussed above, the investor does not voluntarily realize deep losses because his entire trading account holds a single stock. Realizing deep losses would effectively wipe out his mental budget, which would yield a high utility cost but a negligible benefit for future gain realizations.

[10] These papers naturally focus on the economically interesting case in which the investor always holds a stock on the optimal path. See these papers for additional discussions on conditions under which the investor voluntarily holds a stock.

A. Trading Opportunity: Multiple Stocks and Risk-Free Asset

The risk-free asset pays interest at the constant rate of $r > 0$. There are multiple (N) risky assets (stocks) indexed by $n \in \{1, 2, \ldots, N\}$. The cum-dividend price process for a unit of asset n, $P_{n,t}$, follows a geometric Brownian motion (GBM) process,

$$dP_{n,t} = \mu P_{n,t} dt + \sigma P_{n,t} d\mathcal{Z}_{n,t}\,, \quad t > 0, \tag{1}$$

where $\mathcal{Z}_{n,t}$ is a standard Brownian motion. Following the realization-utility literature, we assume that at each moment $t > 0$, the investor can hold at most one of the N stocks due to mental accounting.[11] When purchasing and selling a stock, the investor pays proportional transaction costs. Let θ_p and θ_s denote the proportional purchase and sale cost parameters, respectively. Finally, we set the expected return (drift) μ and volatility $\sigma > 0$ to be the same for all stocks as in the literature.

Let τ_i denote the investor's i^{th} stock trading time. Importantly, in our model, in addition to deciding the extensive margin—which stock to buy at what time τ_i—the investor also chooses the *intensive margin*—the trading size X_{τ_i+}. During an investment episode between two consecutive trading moments, (τ_i, τ_{i+1}), the investor's allocation to the stock in dollars (risky wealth), X_t, follows the same GBM process as $P_{n,t}$,

$$dX_t = \mu X_t dt + \sigma X_t d\mathcal{Z}_{n,t}\,, \quad t \in (\tau_i, \tau_{i+1}), \tag{2}$$

and the investor's risk-free wealth, W_t, evolves according to

$$dW_t = rW_t dt\,, \quad t \in (\tau_i, \tau_{i+1}). \tag{3}$$

At any $t > 0$, the investor has an option to sell his stockholdings X_t and obtain

$$\Pi_t = W_t + (1 - \theta_s)X_t, \tag{4}$$

where $\theta_s X_t$ is the cost of selling the stock.[12] We refer to Π_t as the budget at t for this mental trading account. Upon selling the stock he owns at τ_i, the investor allocates his budget Π_{τ_i} between a new stock and the risk-free asset. When he does not trade, the budget given in (4) measures his mark-to-market wealth in this mental account under the counterfactual that he sells the stock that he owns at time t. The investor's mental budget Π_t proves helpful when we

[11] Because the investor can hold at most one stock at any point in time, the correlation matrix of these N stock returns plays no role in our model, as in this literature. Extending our model to a setting with multiple risky assets represents a promising direction for future research. However, such an extension is technically quite challenging because it involves solving a multidimensional stochastic control problem (as each risky asset has its own reference point). We leave this extension for future research.

[12] For tractability, as in the realization-utility literature, we assume that the investor sells his entire stock position to end his current investment episode before acquiring a new stock.

discuss the economics of the investor's trading behaviors, even though it is not a state variable.

When buying a new stock with a (dollar) position of X_{τ_i+} at τ_i+, the investor pays from his mental account and thus his posttrading risk-free wealth W_{τ_i+} is given by

$$W_{\tau_i+} = \Pi_{\tau_i} - (1+\theta_p)X_{\tau_i+}, \tag{5}$$

where $\theta_p X_{\tau_i+}$ is the purchase cost. In prior realization-utility models, the investor is required to use his entire budget Π_{τ_i} when acquiring a new stock. This implies $X_{\tau_i+} = (1-\theta_s)X_{\tau_i}/(1+\theta_p)$ at trading time τ_i. Therefore, once τ_i is selected, the trading size is determined because there are no savings: $W_t = 0$ at all t.

Leverage Constraint: In addition to saving a fraction of his budget in the risk-free asset, the investor can borrow using the stock that he buys as collateral, provided that

$$X_t \geq -W_t/\kappa \ , \tag{6}$$

where the condition $0 < \kappa < 1-\theta_s$ ensures that creditors bear no credit risk and hence are willing to lend at the risk-free rate.[13] When the leverage constraint (6) binds, the investor is forced to realize losses. If $\kappa = 0$, the investor is not allowed to borrow.

B. Realization Utility

We model the investor's preferences using the realization utility proposed by BX (2012) and IJ (2013). An investor views the investment process as a series of separate episodes whereby his utility payoff comes in bursts when realizing gains or losses from stock sales. To calculate utility from realized gains (losses), we need a reference level. As in BX (2012), we assume that the reference level $B_t > 0$ grows exponentially at the risk-free rate r between two consecutive trading moments, τ_i and τ_{i+1}:

$$dB_t = rB_t dt \quad \text{for} \ \ t \in (\tau_i, \tau_{i+1}). \tag{7}$$

After adjusting his stockholdings at τ_i+, the reference level B_{τ_i} is reset following

$$B_{\tau_i+} = X_{\tau_i+}. \tag{8}$$

Let G_{τ_i} denote the realized gain (loss) at τ_i after ending the investment episode,

$$G_{\tau_i} = (1-\theta_s)X_{\tau_i} - B_{\tau_i}. \tag{9}$$

[13] This is because $(1-\theta_s)X_t + W_t \geq \kappa X_t + W_t \geq 0$, given $0 < \kappa < 1-\theta_s$ and (6).

Anticipating that the homogeneity property plays an important role in our model solution, we define g_{τ_i} as the realized gain or loss, G_{τ_i}, scaled by the reference level B_{τ_i}:

$$g_{\tau_i} = G_{\tau_i}/B_{\tau_i}. \tag{10}$$

As in IJ (2013), the investor derives the following utility burst when selling the stock and realizing a gain or loss at trading time τ_i:

$$U(G_{\tau_i}, B_{\tau_i}) = B_{\tau_i}^{\beta} u(G_{\tau_i}/B_{\tau_i}) = B_{\tau_i}^{\beta} u(g_{\tau_i}), \tag{11}$$

where $\beta \in (0, 1]$ is a constant and $u(\cdot)$ is a function that depends on the scaled realized gain or loss, g_{τ_i}.[14] We specify $u(\cdot)$ by adopting a reference-level-scaled version of the cumulative prospect theory (CPT) utility of Tversky and Kahneman (1992):[15]

$$u(g) = \begin{cases} g^{\alpha_+} & \text{if } g \geq 0, \\ -\lambda(-g)^{\alpha_-} & \text{if } g < 0, \end{cases} \tag{12}$$

where $\lambda \geq 1$ and $\alpha_\pm \in (0, 1]$ are the constant parameters describing $u(\cdot)$.

In addition, $u(\cdot)$ inherits two other features of CPT: (i) diminishing sensitivity, that is, $u(\cdot)$ is concave $(0 < \alpha_+ \leq 1)$ in the gain $(g \geq 0)$ region but convex $(0 < \alpha_- \leq 1)$ in the loss $(g < 0)$ region, and (ii) loss aversion $(\lambda \geq 1)$. A higher value of λ indicates a stronger loss aversion. Note that the point $g = 0$ is a kink, at which $u(\cdot)$ is not differentiable. Finally, as in IJ (2013), we require

$$\beta \leq \min\{\alpha_+, \alpha_-\}, \tag{13}$$

which ensures that $|U(G, B)|$ is decreasing in B for fixed G. When $\beta < \min\{\alpha_+, \alpha_-\}$, the smaller the reference level, the greater the utility impact of a realized gain or loss in absolute value terms $(|G|)$.[16] For the special piecewise linear realization utility case studied in BX (2012), where $\beta = \alpha_+ = \alpha_- = 1$, $U(G, B)$ is independent of the level of B (for a given G) and does not feature the diminishing sensitivity property.

Figure 1 plots the scaled realization utility $u(\cdot)$. Panel A plots the $\alpha_\pm = 0.5$ case used in IJ (2013), and Panel B plots the $\alpha_\pm = 1$ case analyzed in BX (2012); $\lambda = 1.5$ in both panels. In Panel A, $u(\cdot)$ is S-shaped, that is, convex in the loss region and concave in the gain region. In contrast, in the piecewise linear case in Panel B, $u(\cdot)$ is globally concave. In both panels, $u(\cdot)$ is not differentiable at $g = 0$.

[14] The specification in (11) makes growth in our model stationary and tractable, in line with the finance tradition as noted by IJ (2013).

[15] The realization-utility formulation of prospect theory ignores probability weighting, another key feature of CPT. Investors tend to overweight the tail outcomes of a probability distribution. Put differently, investors typically prefer lotteries and insurance compared with predictions of expected utility models.

[16] An illustrative example from IJ (2013): "the gain or loss of \$10 is felt more strongly when the reference level is \$100 than when it is \$500."

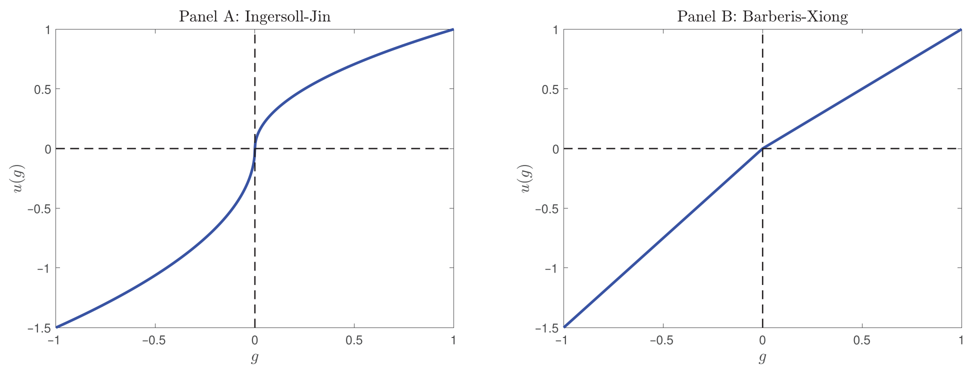

Figure 1. Scaled realization utility $u(\cdot)$. Panel A plots the $\alpha_{\pm} = 0.5$ case in IJ (2013). Panel B plots the piecewise linear $\alpha_{\pm} = 1$ case in BX (2012). In both panels, the loss-aversion parameter is $\lambda = 1.5$ and $u(\cdot)$ is not differentiable at the kink point $x = 0$. In Panel A, $u(\cdot)$ is convex in the loss region and concave in the gain region. In Panel B, $u(\cdot)$ is globally concave. (Color figure can be viewed at wileyonlinelibrary.com)

Liquidity Shocks: Following BX (2012), we assume that the investor faces an exogenous liquidity shock, which arrives stochastically at a constant rate of $\xi > 0$. Upon the arrival of this shock at τ_L, the investor immediately sells his entire stockholdings, realizes a utility burst according to (11), and exits from the asset market. As in BX (2012), we incorporate the liquidity shock for the following reasons: it captures a sudden consumption need that forces the investor to draw on the funds, it makes the investor care about paper gains and losses to some degree, which is reasonable, and it allows us to calibrate the investor's expected trading horizon to $1/\xi$.

Investing in a Stock or Not? As in BX (2012), liquidity shocks force the investor to involuntarily sell the stock that he owns. Anticipating this contingency, the investor may not want to invest in a stock ex ante. Intuitively, if the upside of realizing gains is sufficiently large, the investor will choose to invest in a stock. Before solving for the binary decision at $t = 0$, we first characterize the investor's optimization problem conditional on his choosing to invest in a stock.

C. The Optimization Problem

For a given triple (W_t, X_t, B_t) at time t, the investor chooses a sequence of trading times $\{\tau_i \geq t;\ i = 1, 2, \ldots\}$ and allocation of X_{τ_i+} to a stock at each τ_i+ to solve

$$\max_{\{\tau_i, X_{\tau_i+}\}} \mathbb{E}_t\left[\sum_{i=1}^{\infty} e^{-\delta(\tau_i - t)} U(G_{\tau_i}, B_{\tau_i}) \mathbf{1}_{\{\tau_i < \tau_L\}} + e^{-\delta(\tau_L - t)} U(G_{\tau_L}, B_{\tau_L})\right] \tag{14}$$

subject to the leverage constraint (6) and the dynamics described by (2), (3), (5), (7), and (8). In (14), $\delta > 0$ is the investor's subjective discount rate and $\mathbf{1}_A$

is the indicator function.[17] There are three state variables: risk-free wealth (W), risky wealth (X), and the reference level (B). Let $V(W, X, B)$ denote the value function for the problem defined in (14). We next characterize this problem recursively.

At trading time $\tau \geq t$, the investor realizes a gain or loss, receives a direct utility burst of $U((1-\theta_s)X_\tau - B_\tau, B_\tau)$, and moves forward with a continuation value of

$$\widehat{V}(\Pi_\tau) = \max_{X_{\tau+}} V(W_{\tau+}, X_{\tau+}, X_{\tau+}), \tag{15}$$

where $\Pi_\tau = W_\tau + (1-\theta_s)X_\tau$ is the postrealization budget given in (4). Equation (15) states that at $\tau+$ (before the liquidity shock arrives), the investor chooses $X_{\tau+}$ out of his budget Π_τ to maximize $V(W_{\tau+}, X_{\tau+}, X_{\tau+})$ subject to (5) and the leverage constraint (6). Let $F(W_\tau, X_\tau, B_\tau)$ denote the total (utility) payoff function when ending an episode:

$$F(W_\tau, X_\tau, B_\tau) = U((1-\theta_s)X_\tau - B_\tau, B_\tau) + \widehat{V}(\Pi_\tau). \tag{16}$$

Then, we can express the optimization problem given in (14) as

$$V(W_t, X_t, B_t) = \max_{\tau \geq t} \mathbb{E}_t\Big[e^{-\delta(\tau - t)}F(W_\tau, X_\tau, B_\tau)\mathbf{1}_{\{\tau < \tau_L\}} + e^{-\delta(\tau_L - t)}U(G_{\tau_L}, B_{\tau_L})\Big] \tag{17}$$

subject to (2) to (8). Intuitively, the following Hamilton-Jacobi-Bellman (HJB) equation characterizes the investor's value function in the region in which the investor does not trade:

$$\delta V = \frac{1}{2}\sigma^2 X^2 V_{XX} + \mu X V_X + rWV_W + rBV_B + \xi\,[U(G, B) - V]. \tag{18}$$

At trading time τ, $V(W_\tau, X_\tau, B_\tau) = F(W_\tau, X_\tau, B_\tau)$.

Voluntary Participation: Finally, to ensure that the investor voluntarily invests in stocks, we require $\widehat{V} > 0$, where $\widehat{V}$ is given by (15).

II. Solution

We solve the optimization problem (17) in two steps. First, conditioning on his voluntarily investing in a stock, we simplify the investor's problem by using our model's homogeneity property to characterize the solution.[18] Second, we provide a condition under which the investor voluntarily invests a fraction of his budget in the stock.

Using the Homogeneity Property to Simplify Problem (17): Let w_t and x_t denote W_t and X_t scaled by the contemporaneous reference level B_t for all t,

[17] $\mathbf{1}_A = 1$ if event A occurs and $\mathbf{1}_A = 0$ otherwise.

[18] We solve the optimization problem using the variational inequality in Appendix A.

respectively,

$$w_t = \frac{W_t}{B_t} \quad \text{and} \quad x_t = \frac{X_t}{B_t}. \tag{19}$$

Between two consecutive trading moments (τ_i, τ_{i+1}), the x_t process is a GBM,

$$dx_t = (\mu - r)x_t dt + \sigma x_t d\mathcal{Z}_{n,t}. \tag{20}$$

Since both the reference point B_t and risk-free asset holdings W_t grow at the risk-free rate r, the scaled risk-free asset holding w is constant over (τ_i, τ_{i+1}) and therefore

$$dw_t = 0. \tag{21}$$

The homogeneity property allows us to write $V(W, X, B)$ and $F(W, X, B)$ as $V(W, X, B) = B^\beta v(w, x)$ and $F(W, X, B) = B^\beta f(w, x)$ for any $B > 0$, where $v(w, x)$ and $f(w, x)$ are the scaled value and payoff functions to be characterized next.

Following BX (2012), we define $\delta_e = \delta - \beta r$ and interpret δ_e as the investor's effective discount rate. Using the homogeneity property, we can simplify problem (17) as

$$v(w_t, x_t) = \max_{\tau \geq t} \mathbb{E}_t\left[e^{-\delta_e(\tau - t)} f(w_\tau, x_\tau)\mathbf{1}_{\{\tau < \tau_L\}} + e^{-\delta_e(\tau_L - t)} u((1 - \theta_s)x_{\tau_L} - 1)\right], \tag{22}$$

where $f(w_\tau, x_\tau)$ equals the sum of the utility burst and the continuation value,

$$f(w_\tau, x_\tau) = u((1 - \theta_s)x_\tau - 1) + [(1 - \theta_s)x_\tau + w_\tau]^\beta\, \widehat{v}. \tag{23}$$

The continuation value is homogeneous of degree β in the budget, $\pi_\tau = (1 - \theta_s)x_\tau + w_\tau$, and equals $\pi_\tau^\beta\, \widehat{v}$, where $\widehat{v}$ is the (utility) value with a budget of one dollar and solves

$$\widehat{v} = \max_{w \geq -\kappa} m(w), \tag{24}$$

where $m(w)$ is the (utility) value given scaled risk-free wealth w,

$$m(w) = \left(\frac{1}{w + 1 + \theta_p}\right)^\beta v(w, 1). \tag{25}$$

An investor with a budget of $\Pi_\tau = 1$ at τ, targeting a ratio of $W_{\tau+}/X_{\tau+} = w$ at $\tau+$, must allocate $X_{\tau+} = \frac{1}{w+1+\theta_p}$ to the stock.[19] Since $B_{\tau+} = X_{\tau+}$, we can rewrite (25) as $B_{\tau+}^\beta v(w, 1)$, which is homogeneous in $B_{\tau+}$ with degree β, as expected. The investor chooses the optimal w^* at $\tau+$ to maximize his utility given in (24) and (25). In contrast, in prior literature, (24) and (25) are the same up to a proportionality constant $\left(\frac{1}{1+\theta_p}\right)^\beta$ because $w_t = 0$ for all t by assumption.

In sum, because the investor has a dynamic mental budget at the trading account level, he saves a fraction of his budget in the risk-free asset or uses

[19] This is because summing $W_{\tau+}$, $X_{\tau+}$, and the trading cost $\theta_p X_{\tau+}$ equals the budget $\Pi_\tau = 1$.

leverage by managing w_t subject to the leverage constraint in (6). This is very different from prior literature, where $w_t = 0$ is assumed to hold at all t conditional on the investor's voluntary participation. We show that this new decision margin (w) fundamentally changes the trading strategies and value function. We next characterize the solution.

Characterizing the Solution Using the HJB Equation: At any t, the investor can either trade or passively hold his stock position. The solution thus features two types of domains. If the investor realizes gains or losses, his value function must equal the payoff function $v(w, x) = f(w, x)$. We refer to the set of (w, x) on which the investor trades as the realization domain $\mathcal{R}$, where $v(w, x) = f(w, x)$.

If the investor chooses to hold his stock position, doing so must yield a higher value than realizing gains (losses). We refer to this set of (w, x) as the holding domain $\mathcal{H}$, where $v(w, x) > f(w, x)$. The following HJB equation holds for $v(w, x)$ defined in (22):

$$\delta_e v(w,x) = \frac{1}{2}\sigma^2 x^2 v_{xx}(w,x) + (\mu - r)x v_x(w,x) + \xi\,[u((1-\theta_s)x - 1) - v(w,x)]. \quad (26)$$

Our model solution features two cases as summarized in Table I: Case A, the three-region case, and Case B, the four-region case. The three regions in Case A are a gain-realization region ($x \in (\overline{x}^*, \infty)$), a loss-realization region ($x \in (0, \underline{x}^*)$), and a normal holding region ($x \in (\underline{x}^*, \overline{x}^*)$) that lies between the two realization regions. The investor voluntarily sells his stock even at a deep loss because he has set aside some savings for future trading. This is a unique prediction of our model.

The four regions in Case B are a gain-realization region ($x \in (\overline{x}^*, \infty)$), a normal holding region ($x \in (\underline{x}^*, \overline{x}^*)$), a loss-realization region ($x \in (\underline{x}^{**}, \underline{x}^*)$), and a deep-loss holding region ($x \in (0, \underline{x}^{**})$).[20] We further discuss the implications of these two cases in Section IV, where we analyze the effect of jumps. This is because the economically interesting deep-loss region is not reachable on the optimal path in diffusion models but is reachable in models with jumps.

Note that the leverage constraint (6) plays a prominent role in our analysis. When it binds ($w = -\kappa x$), the investor is forced to realize losses and $v(-\kappa x, x) = f(-\kappa x, x)$.

Finally, an investor is willing to invest in a stock if $\widehat{v} > 0$, where $\widehat{v}$ given in (24) is the (utility) value with a budget of one dollar.

III. Implications of Saving and Leverage (Intensive Margin)

In this section, we analyze how an investor with a dynamic mental trading account can use savings or leverage to manage his trading strategy over time.

First, we choose the parameter values. A period is one year. As in IJ (2013), we set $\alpha_+ = \alpha_- = 0.5$, $\delta = 5\%$, $\beta = 0.3$, $\mu = 9\%$, $\sigma = 30\%$, and $\theta_s = \theta_p = 1\%$. We

[20] In IJ (2013) and HY (2019), the solution features four regions. Voluntary deep-loss realization in their models is impossible. This is because the investor's entire trading account holds only a single stock and selling the stock at a deep loss is too painful.

Table II

Parameter Values for the Baseline Case

The parameters (r, δ, μ, σ) are continuously compounded and annualized

α_+	α_-	λ	β	r	δ	μ	σ	θ_s	θ_p	κ	ξ
0.5	0.5	1.5	0.3	3%	5%	9%	30%	1%	1%	0.79	0

set the loss-aversion parameter at $\lambda = 1.5$ based on the estimate of Andersen et al. (2022). To target the risk premium $(\mu - r)$ at 6%, a commonly used value for the U.S. stock market and housing risk premia,[21] we set the risk-free rate at $r = 3\%$. We turn off the liquidity shock by setting $\xi = 0$. Finally, we set $\kappa = 0.79$ so that the ratio of debt and liquidating net worth never exceeds 80%.[22] As the stock is the collateral with 20% equity subordination, debt is risk-free. Table II summarizes the 12 parameter values.

A. Saving in the Risk-Free Asset to Spread Trades out over Time

Using the parameter values in Table II, we obtain $w^* = 1.76$, which implies that the investor only allocates $1/(1 + \theta_p + w^*) = 36.1\%$ of his dynamic mental budget to the stock each time he trades and saves the remaining $w^*/(1 + \theta_p + w^*) = 63.5\%$ of his budget for future trades (netting out of the 0.4% trading costs). What determines the level of w^*?

Panel A of Figure 2 shows that $m(w)$ increases in w for $w \le w^*$, reaches the maximum value $\hat{v} = 8.53$ at $w^* = 1.76$, and then decreases in w for $w \ge w^*$. The intuition for this single-peaked $m(w)$ is as follows. The investor's total (utility) value includes his realization utility from selling the stock and his continuation value. Choosing a larger w implies a smaller stock allocation $1/(1 + \theta_p + w)$, which leads to (i) a smaller realization utility and (ii) slower but also less risky growth of his budget (Π). The continuation value is concave in Π as $\beta = 0.3 < 1$.[23] As w increases, the realization utility flow decreases, but his continuation value increases due to the risk-reduction effect on his future budget Π. This trade-off pins down w^* (Panel A).

Next, we consider the following counterfactual: How much compensation does an investor require to permanently forgo the option to invest in the risk-free asset? We use a certainty-equivalent wealth-based measure (i.e., the fraction Δ of his budget Π_0) to capture this compensation payment. For an

[21] See Hansen and Singleton (1982) and Mehra and Prescott (1985) for the equity risk premium and Piazzesi and Schneider (2016) for the housing risk premium.

[22] Since the liquidation value of the stock equals $(1 - \theta_s)X$, the investor's net worth upon liquidating the stock is $W + (1 - \theta_s)X$. To target the maximal loan-to-value (LTV) ratio at 80%, we require the ratio $X/[W + (1 - \theta_s)X]$ not to exceed $\phi = 1/(1 - 0.8) = 5$. Rewriting this leverage constraint yields $W \ge -\kappa X$, where $\kappa = 1 - \theta_s - 1/\phi = 0.79$, as $\phi = 5$ and $\theta_p = 1\%$ in our baseline quantitative analysis.

[23] Recall that the continuation value is homogeneous of degree β in the investor's budget (Π) at the end of the investment episode.

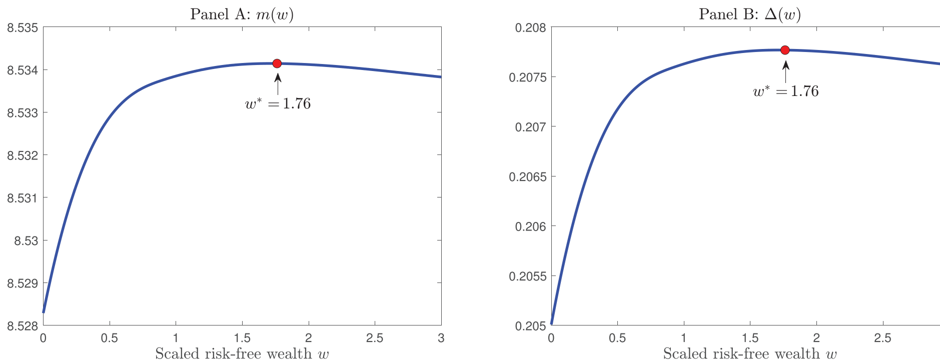

Figure 2. Determining w^* and quantifying the value of saving in the risk-free asset. Panels A and B plot $m(w)$ given in (25) and $\Delta(w)$ in (27), respectively. Both functions are hump-shaped and maximized at $w^* = 1.76$. See Table II for parameter values. (Color figure can be viewed at wileyonlinelibrary.com)

investor to be indifferent between (i) living in our model and (ii) living in the economy analyzed by BX (2012) and IJ (2013) but with a (higher) budget of $(1+\Delta)\Pi_0$, the condition

$$V(W_{0+}, X_{0+}, X_{0+}) = V_N\left(\frac{(1+\Delta)\Pi_0}{1+\theta_p}, \frac{(1+\Delta)\Pi_0}{1+\theta_p}\right)$$

must hold, where $W_{0+} = \Pi_0 - (1+\theta_p)X_{0+}$ is the risk-free wealth after the investor purchases the stock and $V_N(\cdot)$ is the value function with "no options" to invest in the risk-free asset. Using the homogeneity property, we obtain the following expression for $\Delta(w)$:

$$\Delta(w) = \left(\frac{m(w)}{m_N}\right)^{1/\beta} - 1, \tag{27}$$

where $w = W_{0+}/X_{0+}$, $m(\cdot)$ is defined in (25), and $m_N = V_N(1,1)/(1+\theta_p)^\beta$.

Panel B of Figure 2 shows that $\Delta(\cdot)$ depends on w and does indeed have the same monotonicity property as $m(w)$, also peaking at $w^* = 1.76$. Using the parameter values in Table II, we obtain $\Delta(w^*) = 21\%$ at $w^* = 1.76$. That is, the option to save 63.5% of his budget at each trading time and make smaller trades over time is worth about 21% of the investor's mental budget. While the investor can hold only one stock, the option to spread his trades out over time via saving in the risk-free asset is quite valuable.

Savings Makes the Investor Trade More Frequently and Less subject to the Disposition Effect: How do savings influence the investor's trading strategies? Panel A of Figure 3 shows that the investor in our model realizes losses much sooner than in IJ (2013): the lower loss-realization boundary $\underline{x}^*$ is 0.69 in our model, much higher than the 0.55 in IJ (2013). By saving 63.5% of his budget in the risk-free asset, his downside loss (in dollars) in our model is only about

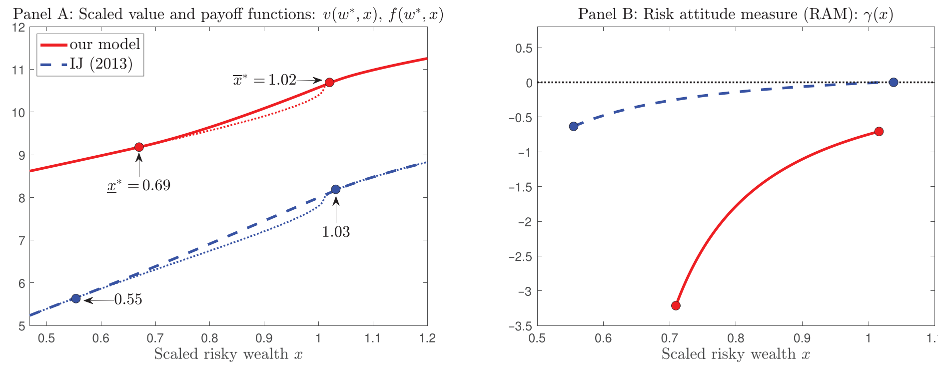

Figure 3. Comparing value functions, payoff functions, and risk attitude measures (RAMs) between our model and the IJ (2013) model. This figure shows that the investor is more willing to realize losses (Panel A) and take risk (Panel B) in our model, in which he has an option to save in the risk-free asset, than in IJ (2013), in which he does not. See Table II for parameter values. (Color figure can be viewed at wileyonlinelibrary.com)

one-third of that in IJ (2013), and thus he is more willing to realize losses to reset the reference level B for future gain realizations.[24]

This intuition is corroborated in Panel B of Figure 3, which plots the risk attitude measure (RAM), a value function curvature measure defined as

$$\text{RAM} = -\frac{XV_{XX}}{V_X} = -\frac{xv_{xx}(w^*, x)}{v_x(w^*, x)} = \gamma(x). \tag{28}$$

Because of the homogeneity property, RAM depends only on x, and hence we write RAM as $\gamma(x)$. Panel B of Figure 3 shows that the investor is endogenously risk-seeking as his value function is convex and RAM is negative ($\gamma(x) < 0$) in the holding region. In addition, the investor is more risk-seeking in our model than in IJ (2013).

In sum, the investor with a broader dynamic mental account saves a sizeable fraction of his budget in the risk-free asset and makes smaller stock trades. In addition, he realizes losses more frequently and is less subject to the disposition effect.

When does the investor want to use leverage rather than save in the risk-free asset? We turn to this question next.

B. Using Leverage to Increase Trading Size

In this subsection, we decrease volatility σ from 30% (used in the previous subsection) to 20%, ceteris paribus. The optimal allocation in the risk-free asset is then $w^* = -0.36 < 0$, which is very different from the value of $w^* = 1.76$

[24] The investor in our model realizes gains slightly sooner: the upper boundary $\overline{x}^*$ is 1.02 in our model, which is slightly lower than the 1.03 in the IJ (2013) model.

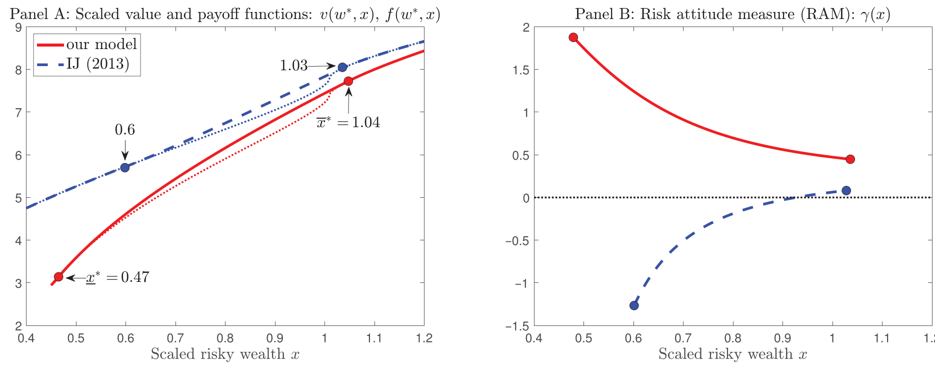

Figure 4. Effect of optimal leverage on value function $v(w^*, x)$ and payoff function $f(w^*, x)$. The optimal w is $w^* = -0.36$. Leverage makes the investor more reluctant to realize losses and more risk-averse. The volatility parameter is set at $\sigma = 20\%$. See Table II for other parameter values. (Color figure can be viewed at wileyonlinelibrary.com)

when $\sigma = 30\%$. Notice that the investor switches from *saving* 63.5% of his trading budget to *borrowing* 55.4%. With a mental budget of $\Pi_0 = 100$, he invests $X_0 = \Pi_0/(1 + \theta_p + w^*) = 154$ (since $w^* = -0.36 < 0$) in the stock he chooses if $\sigma = 20\%$. This trading size is 4.3 times the value of $X_0 = 36$ when $\sigma = 30\%$ (since $w^* = 1.76$). This exercise shows the very large effect of return volatility σ on the investor's trading strategy. The investor uses leverage when volatility is low because the stock yields a higher Sharpe ratio.

Because leverage significantly increases the investor's exposure to the stock, he becomes more reluctant to realize losses: his loss-realization threshold $\underline{x}^*$ decreases from 0.6 in the IJ (2013) model to 0.47 in our model.[25] The gain-realization threshold $\overline{x}^*$ increases slightly from 1.03 in the IJ (2013) model to 1.04 because of leverage. As a result, the holding region $(\underline{x}^*, \overline{x}^*)$ widens (see Panel A of Figure 4). Consistent with our model's prediction on how leverage impacts gain- and loss-realization strategies, Heimer and Imas (2022) find that access to leverage increases the disposition effect by significantly deferring loss realizations but having little effect on gain realizations.

Panel B of Figure 4 corroborates our main result on the leverage effect. The investor is risk-averse ($v(w^*, x)$ is concave). Moreover, the RAM $\gamma(x)$ decreases with x, which means that the investor becomes less risk-averse as paper losses decrease (x increases). Finally, leverage makes the investor more risk-averse. The RAM measure in the IJ (2013) model is lower than in our leverage model for all levels of x. Again, leverage makes the trading size large and the investor endogenously more averse to risk. Finally, we note that quantitatively, the

[25] Note that the investor still voluntarily realizes losses. The leverage constraint (6) is equivalent to $-\kappa \leq W_t/X_t = (W_t/B_t)/(X_t/B_t) = w_t/x_t$ in terms of the scaled variables. That is, the leverage constraint (6) is slack if and only if $\underline{x}^* > -w^*/\kappa$. The loss-realization threshold $\underline{x}^* = 0.47$ is slightly larger than the involuntary liquidation threshold—$-w^*/\kappa = 0.36/0.79 \approx 0.46$—and leave the leverage constraint (6) does not bind.

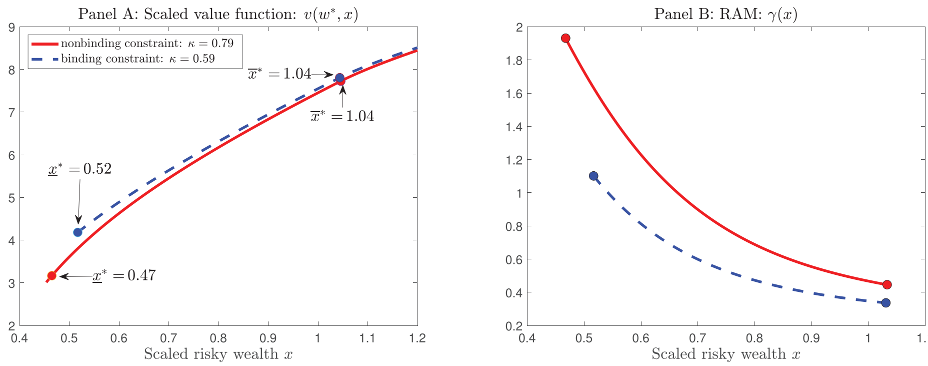

Figure 5. Effect of leverage constraints. Leverage constraints can reduce the disposition effect by making the investor realize losses sooner. When $\kappa = 0.79$, the optimal loss-realization threshold is $\underline{x}^* = 0.47$ and the leverage constraint (6) does not bind. As we tighten the constraint (6) by decreasing κ to 0.59, the optimal loss-realization threshold $\underline{x}^*$ increases to 0.52 and the leverage constraint (6) binds. The volatility parameter is set to $\sigma = 20\%$. See Table II for other parameter values. (Color figure can be viewed at wileyonlinelibrary.com)

option to use leverage is quite valuable, worth about $\Delta = 31\%$ of the investor's trading budget.

In our preceding analysis, the leverage constraint (6) does not bind (see footnote). This is because $\kappa = 0.79$, which is sufficiently large that the investor's aversion to losses makes him choose a prudent level of w^*, so it is in his own interest to voluntarily realize losses. But what if the leverage constraint is sufficiently tight such that it binds?

Leverage Constraints: Figure 5 shows that the investor realizes losses sooner when facing a tighter leverage constraint by comparing the cases of $\kappa = 0.79$ and $\kappa = 0.59$.

First, recall that when $\kappa = 0.79$, the investor uses leverage ($w^* = -0.36$) and voluntarily realizes losses when x reaches $\underline{x}^* = 0.47$. In this case, the leverage constraint (6) $x \geq 0.46$ does not bind. Second, as we tighten the constraint (6) by decreasing κ to 0.59, it becomes more likely to bind and force the investor to realize losses. Anticipating this contingency, the investor prudently reduces his leverage by setting $w^* = -0.31$. The net effect of a tighter leverage constraint (6) and lower leverage is a higher loss-realization threshold: $\underline{x}^* = 0.52$, where the constraint (6) binds. Our model's prediction is consistent with Heimer and Imas (2022), who find that introducing leverage constraints mitigates the disposition effect by making the investor realize losses sooner. Because of a lower leverage ratio, the investor is less averse to realizing losses, as we can see by comparing the two lines for RAMs in Panel B of Figure 5.

In contrast, the gain-realization threshold $\overline{x}^*$ remains unchanged at 1.04. This is consistent with our prior result that the investor's gain-realization strategy is not sensitive to changes in his trading opportunity. This is because

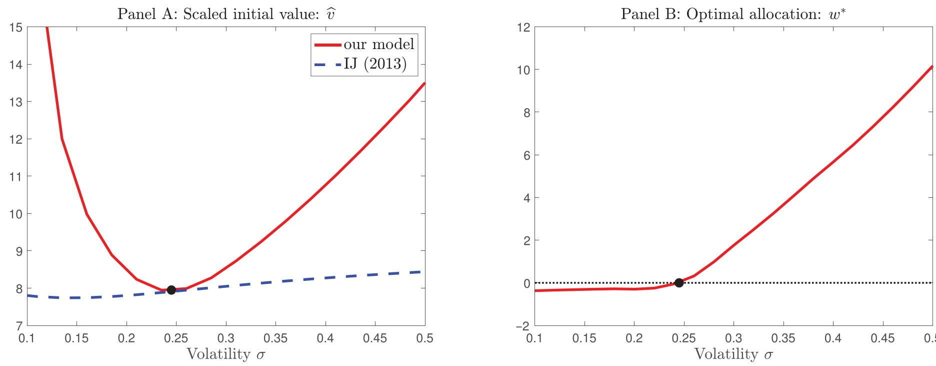

Figure 6. Scaled value $\widehat{v}$ and optimal w^* as functions of volatility σ. Panel A shows that $\widehat{v}$ decreases with σ when $\sigma < 25\%$ but increases with σ when $\sigma > 25\%$. This is because the investor (i) uses leverage when $\sigma < 25\%$ and the value of leverage decreases with σ and (ii) saves in the risk-free asset when $\sigma > 25\%$ and the value of saving increases with σ (Panel B). Finally, the value of saving and leverage is zero when $\sigma = 25\%$. See Table II for parameter values other than σ. (Color figure can be viewed at wileyonlinelibrary.com)

the investor's preference is concave in gains (in the gain region), and hence the benefit of locking in gains is sufficiently large.

In sum, our model predicts that (i) investors using leverage have stronger disposition effects and (ii) introducing leverage constraints can reduce investors' disposition effects by disciplining their behavioral biases. These predictions are consistent with the findings in Barber et al. (2019) and Heimer and Imas (2022).

Thus far, we have analyzed how the investor's dynamic mental account allows him to use savings and leverage to manage his trading strategies by separating his trading size from his trading budget. We next study how the investor's saving/leverage and trading strategies vary with changes in stock return volatility σ. We show that the *interaction* between stock return volatility and the investor's endogenous response via dynamically managing his broader mental account generates new predictions on the investor's trading and risk-management policies.

C. Does the Investor Prefer More or Less Volatile Stocks?

To answer the question posed in this subsection's title, we calculate the (utility) value $\widehat{v}$ defined in (24) and (25) for a range of σ and plot this relation in Panel A of Figure 6.

We see that in our model (red solid line) the (scaled) value $\widehat{v}$ at the moment of trading first decreases with σ, reaches a minimal value of 7.95 when $\sigma = 25\%$, and then increases with σ for $\sigma \geq 25\%$. The minimal value of $\widehat{v} = 7.95$ is represented in the figure as a solid black dot. At this point, our model solution agrees with the solution in IJ (2013), where the investor has no option to save or to use leverage when $\sigma = 25\%$. This is because the investor optimally chooses $w^* = 0$ when $\sigma = 25\%$. In this case, it is optimal for the investor to neither save nor use leverage—the optimal dynamic mental account calls for an all-in stock strategy.

For $\sigma \leq 25\%$, the investor in our model uses leverage to increase his (dollar) exposure to the stock ($w^* < 0$ to the left of the solid black dot in Panel B of Figure 6). Because of his larger risk exposure, the investor is endogenously more risk-averse as he may be forced to realize losses. This is why he prefers less risky stocks after using leverage, and hence why his value function $\widehat{v}$ *decreases* with σ in this range.

In contrast, for $\sigma \geq 25\%$, the investor saves a fraction of his budget for his future trading opportunities ($w^* > 0$ to the right of the solid black dot in Panel B), thus decreasing his risk exposure. Because of his reduced exposure to the stock, the investor is more willing to take on a more risky stock, which explains why $\widehat{v}$ increases with σ in the high-σ range.

An important takeaway from Figure 6 is that investors who use leverage prefer stocks with low volatility, while investors who save to spread their trades out over time prefer riskier stocks. This prediction is consistent with Bian et al. (2024), who find that stocks bought in margin accounts tend to have lower systematic volatility and total volatility than stocks bought in cash accounts. The trade-off between risk-taking at the trading account level (the size of w^*) and the stock level (higher or lower σ) yields an optimal level of risk exposure for the investor. This balancing act reflects the interaction between the two layers of mental accounts in our model: when an investor has both a stock-level mental account for each utility burst calculation and a broader intertemporal mental budget for his trading account, he uses savings or leverage to target a volatility level for his trading account. The endogenous response of the investor's risk-taking given the volatility of stocks that he trades is at the core of our model's mechanism.

IV. A Jump-Diffusion Model

In this section, we analyze the effects of jumps in stock prices on investors' trading. To ease exposition, we turn off liquidity shocks by setting $\xi = 0$.[26] We highlight two key predictions. The first prediction is that the investor voluntarily sells a stock at a deep loss when he has sufficiently high savings in his trading account. Due to high savings, even when the stock that he owns is at a deep loss, his trading account is at only a moderate or even small loss. What happens however if the investor's savings are not sufficiently high?

In this case, the investor is unwilling to sell a stock at a deep loss. However, he may be willing to sell the stock after its price rebounds a bit. This is because the value of his broader mental account—the sum of his savings and the value of his stockholdings—increases just enough with the rebound to

[26] If liquidity shocks are present, we would need to discuss the participation conditions as in our baseline model (see Appendix A.3). Our key results continue to hold in settings with liquidity shocks.

make him willing to realize a loss in exchange for future gain realizations. This sell-after-rebound pattern is the second prediction of our model.

In sum, these two predictions generated by the two cases of our model solution arise from the interaction between the two-layered mental accounting structure of our model.

Jump-Diffusion Model: We model the price process for stock n, where $n \in \{1, 2, \ldots, N\}$, by incorporating jumps into the GBM process given in (1) as

$$\frac{dP_{n,t}}{P_{n,t-}} = \mu dt + \sigma d\mathcal{Z}_{n,t} - (1-Y)d\mathcal{J}_{n,t}\,, \quad P_0 > 0, \tag{29}$$

where $\mathcal{J}_n$ is a pure jump with arrival rate ρ and the random variable $Y \in [0, 1]$ is drawn from the cumulative distribution function (cdf) $\Omega(Y)$. Let $\tau^{\mathcal{J}}$ denote the jump arrival time. If a jump occurs at t ($d\mathcal{J}_{n,t} = 1$), then the stock price falls from $P_{n,t-}$ to $P_{n,t} = YP_{n,t-}$. The other assumptions are the same as in our diffusion model (Section I).[27]

Solution: As in our diffusion model, using the homogeneity property, we work with the scaled state variables $w = W/B$ and $x = X/B$, and the scaled value functions $v(w, x)$ and $f(w, x)$. When $dw_t = 0$, a jump term appears in the x_t process:

$$dx_t/x_{t-} = (\mu - r)dt + \sigma d\mathcal{Z}_{n,t} - (1-Y)d\mathcal{J}_{n,t}. \tag{30}$$

The solution has two domains. In the holding domain $\mathcal{H}$ where $v(w, x) > f(w, x)$, the investor holds onto his stock position and $v(w, x)$ solves the HJB equation

$$\delta_e v(w, x) = \frac{\sigma^2 x^2}{2} v_{xx}(w, x) + (\mu - r)x v_x(w, x) + \rho\left(\mathbb{E}[v(w, Yx)] - v(w, x)\right), \tag{31}$$

where $\delta_e = \delta - \beta r$ is the investor's effective discount rate. In the realization domain $\mathcal{R}$ where $v(w, x) = f(w, x)$, the scaled payoff function $f(w, x)$ is given by (23).

Solution-wise, there are two cases: the three-region case and the four-region case. Both cases feature a gain-realization region, a normal holding region, and a loss-realization region.[28] The key difference between the two cases boils down to whether it is optimal for the investor to voluntarily realize deep stock losses when x is close to zero. If yes, then the solution features three regions. Otherwise, there is a fourth region, namely, the deep-loss holding region. Before analyzing the two cases, we first choose parameter values.

[27] To ease exposition, we focus on the case in which the investor chooses to save rather than use leverage.

[28] It is possible that this loss-realization region $\mathcal{R}_-$ does not exist, for example, when $\alpha_+ = \alpha_- = \beta = 1$, as in BX (2012). We consider this two-region case as a special subcase of the four-region case.

Parameter Choices: We specify the cdf $\Omega(Y)$ for $Y \in [0, 1]$ using a widely used power law as in Barro (2006) and the rare-disaster literature: $\Omega(Y) = Y^{\psi}$, where $\psi > 0$ is the power-law parameter. We set ψ to 6.3 as in Barro and Jin (2011), which implies an expected decrease in stock prices of 14% for each jump: $\mathbb{E}(1 - Y) = \frac{1}{\psi+1} = 14\%$. We set the jump arrival rate to $\rho = 0.73$ per annum, which corresponds to one jump arrival every 1.4 years on average (Huang and Huang (2012)). We set loss aversion to $\lambda = 2.25$ (Tversky and Kahneman (1992)) and target risk premium at 6%, which yields $\mu = 19\%$.[29] To highlight the very different economic predictions for the two cases, we consider two levels of volatility: $\sigma = 30\%$ and $\sigma = 24\%$. All other parameter values are as given in Table II.

A. Case A: Three-Region Solution ($\sigma = 30\%$)

When $\sigma = 30\%$, the solution features three regions. First, the investor realizes gains when x_t exceeds the gain-realization threshold $\overline{x}^* = 1.03$, similar to our diffusion models. Second, the investor holds onto his stock position in the $x \in (0.38, 1.03)$ region. This holding region is wide because of the investor's strong aversion to realizing losses. These two regions are standard as in diffusion models (see, e.g., IJ (2013) and Section I).

In the third region, where $x \in (0, 0.38)$, the investor voluntarily realizes losses. Importantly, even when stock losses are close to 100%, realizing losses is optimal. This is because 19.2% of the investor's budget is allocated to the risk-free asset when he trades ($w^* = 0.24$). With 19.2% set aside in savings, the value of resetting the reference level for future gain realizations is then larger than the utility costs of realizing deep losses. Specifically, the value function at $x = 0$ is positive: $v(w^*, 0) = f(w^*, 0) = 2.3 > 0$.

Selling a Stock at a Deep Loss: With sufficient savings, voluntarily selling a stock in deep losses is a unique prediction of our model. To generate voluntary deep-loss realizations, it is necessary for the model to have both (downward) jumps in asset prices and an option for the investor to save in the risk-free asset. The intuition is as follows. First, without jumps, the deep-loss region cannot be reached on the optimal path. Second, without savings in his trading account, it is always optimal for the investor to hold onto his stock position in the deep-loss region as realizing losses is too painful and there is almost no future. In contrast, with savings in his trading account, a deep loss at an individual stock level does not imply a deep loss in his trading account. An investor with a dynamically evolving mental budget for his trading account is thus willing to sell a stock at a deep loss in exchange for future gain realizations at the trading account level. This prediction is broadly consistent with An et al. (2024), who find that the disposition effect is large when the portfolio is at a loss but nearly disappears when the portfolio is at a gain.[30]

[29] This follows from $r = 3\%$ and $\mu - \rho(1 - \mathbb{E}[Y]) - r = 6\%$.

[30] Hartzmark (2015) documents the rank effect for the disposition effect: individuals are more likely to sell the extreme winning and extreme losing positions in their portfolio.

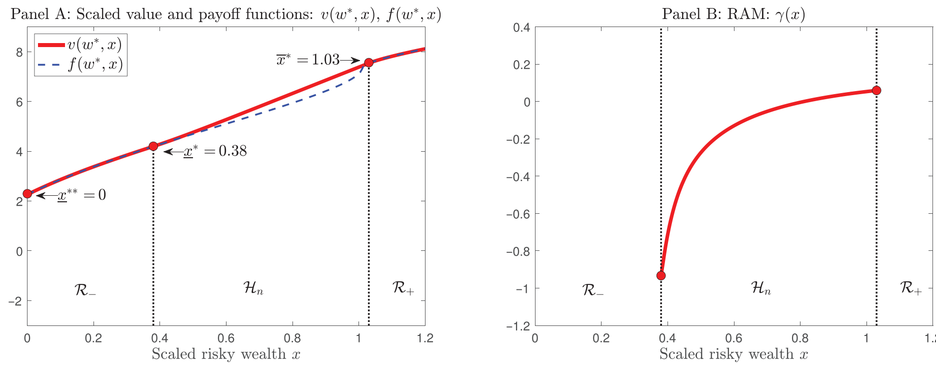

Figure 7. Three-region solution for a jump-diffusion model. Unlike diffusion models, the investor may voluntarily realize deep losses (where x is close to zero) on the optimal path. The investor saves 19.2% of his budget in the risk-free asset: $w^* = 0.24$. Realizing deep losses resets the reference level for future gain realizations, which outweighs high utility costs from realizing deep losses. Panel A plots $v(w^*, x)$ and $f(w^*, x)$. Panel B plots the RAM: $\gamma(x)$. (Color figure can be viewed at wileyonlinelibrary.com)

As in IJ (2013), our model also predicts that the probability of selling a stock increases as its paper gains or losses increase, generating a V-shaped selling propensity pattern as shown by Ben-David and Hirshleifer (2012) and An (2016).[31]

We next analyze Case B, which features four regions. Although models in the literature also feature four regions, the time-series predictions of our Case B are quite different due to jumps and the interaction between the two mental accounts.

B. Case B: Four-Region Solution ($\sigma = 24\%$)

By decreasing σ to 24%, we obtain a four-region solution (Figure 8), which has one more region than for Case A where $\sigma = 30\%$ (Figure 7). The gain-realization region where $x \geq 1.03$ and the normal holding region where $x \in (0.34, 1.03)$ are qualitatively similar in Cases A and B. The other parts of the solution for the two cases are quite different. To the left of $\underline{x}^* = 0.34$ is the loss-realization region, $x \in (0.04, 0.34)$, where the investor voluntarily realizes losses. Finally, in the far-left region where $x \in (0, 0.04)$, the stock is at a deep loss and the investor passively holds onto his stock. In this region, realizing losses is too painful, as he has only 1.9% of his budget in savings ($w^* = 0.02$).[32] This holding prediction in the deep-loss region (where $x \in (0, 0.04)$) is the opposite of the stock-selling prediction of Case A in the same region. Finally, the investor only realizes losses in Case B when $x \in (0.04, 0.34)$, suggesting that

[31] As in IJ (2013), to generate this V-shaped selling propensity, we need heterogeneous preferences. Liu et al. (2022) provide survey results in support of preference heterogeneity.

[32] Since it is optimal to hold the stock in this region, $v(w^*, 0) = 0$ and $f(w^*, 0) < 0$ (see Figure 8).

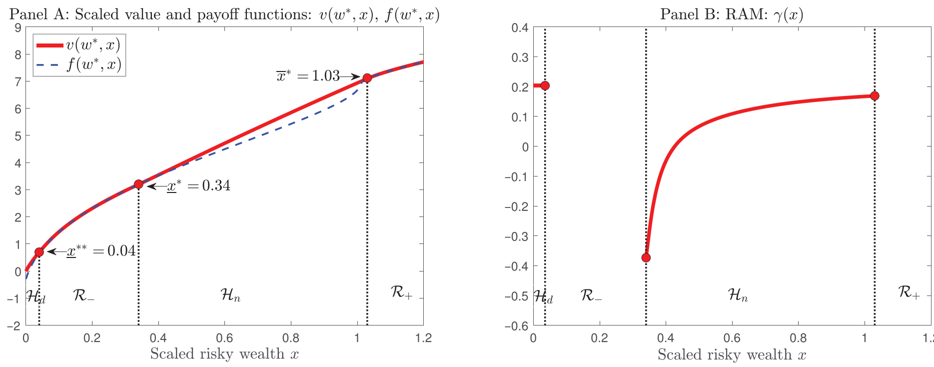

Figure 8. Four-region solution for a jump-diffusion model. The investor saves a tiny fraction (1.9%) of his budget in the risk-free asset: $w^* = 0.02$. In addition to the gain-realization and loss-realization regions as well as the normal holding region that lies between these two realization regions, there is a fourth, deep-loss holding, region. Unlike diffusion models, the investor may find himself in the deep-loss holding region on the optimal path. Other than $\sigma = 24\%$, all other parameters are the same as in Figure 8. Panel A plots $v(w^*, x)$ and $f(w^*, x)$. Panel B plots the RAM: $\gamma(x)$. (Color figure can be viewed at wileyonlinelibrary.com)

his loss-realization propensity is nonmonotonic in x. We next discuss the new time-series prediction of Case B.

Selling a Stock after it Rebounds: Consider an investor whose stock in his trading account is at a very deep loss: $x = 0.03$. While holding the stock as $x < \underline{x}^{**} = 0.04$ (see Figure 8), the stock rebounds, cutting his losses a bit and bringing his x up to 0.04, in which case the investor immediately realizes his stock loss as he has entered the loss-realization region: $x \in [0.04, 0.34]$. Realizing this loss allows him to reset his reference level and start anew from $x = 1$ in the normal holding region. The process then repeats.

This path captures the following prediction that would not have been possible in diffusion models. The investor, while unwilling to sell his stock at a deep loss, voluntarily sells the stock after it rebounds a bit. Intuitively, a rebound cuts his losses by just enough to enter into the loss-realization region. This testable prediction is consistent with our observation that retail investors often sell their losing stocks after these stocks rebound a little (although there is no direct evidence). We next compare our Case B with diffusion models.

Comparing Case B (Featuring Four-Region Solution) with Diffusion Models: Recall that solutions of diffusion models with S-shaped realization utilities, for example, IJ (2013), HY (2019), and our model in Section I, also feature four regions. What then are the differences between our jump-diffusion model and these other diffusion models?

In diffusion models, the deep-loss holding and the loss-realization regions (other than the upper loss-realization threshold of this region) are never reached on the optimal path. This is because diffusion processes are continuous and the x_t process never falls below the upper (right) boundary $\underline{x}^*$ for the loss-realization region due to optimality. In contrast, in our jump-diffusion model, because x_t in our model can jump downward at any time t, all four regions are on the optimal path. Indeed, any value of x lower than the gain-realization threshold $\overline{x}^*$ can be on the optimal path in our jump-diffusion model.

Comparing Cases A and B: We conclude this section by highlighting a key difference between Cases A and B, namely, whether the investor is willing to sell his stock at a deep loss. The key distinction is whether the investor has set aside sufficient savings in the investor's dynamic trading account. Recall that $w^* = 0.24$ in Case A, where $\sigma = 30\%$ (Figure 7), and $w^* = 0.02$ in Case B, where $\sigma = 24\%$ (Figure 8). As stock volatility σ decreases, the demand for savings (w^*) decreases, which means the incentive to realize a loss so as to reset the reference level for future gain realizations decreases. In sum, in our model with two layers of mental accounts, the investor's demand for savings plays a crucial role in generating new time-series predictions.

V. Piecewise Linear Realization Utility

In this section, we allow for piecewise linear realization utility.[33] We obtain the following two main results. First, the investor uses leverage to increase his exposure to the stock that he chooses. Second, he realizes losses only when he is forced to do so by a binding leverage constraint (6). That is, the result in BX (2012) that an investor with piecewise linear utility never voluntarily realizes losses continues to hold in our model. The following proposition summarizes these two results.

PROPOSITION 1: *In the absence of liquidity shocks, for piecewise linear realization utility $U(G, B)$ ($\alpha_{\pm} = \beta = 1$), the investor uses leverage and never saves: $w^* \leq 0$. In addition, he does not realize losses until the leverage constraint (6) binds, that is, when $x = -w^*/\kappa$.*

To ease comparison, we keep the parameter values the same as in our baseline model whenever feasible. We set $\delta = 35\%$ to satisfy the transversality condition (see Internet Appendix Section II).[34] We show that the investor highly values leverage and adopts very different gain/loss realization policies compared with the BX (2012) model.

Figure 9 compares our model solution with the BX (2012) model solution. First, recall that the value function in BX (2012) is convex and thus the RAM is negative: $\gamma(x) = -1.51$ (the dashed blue lines in Panels A and B).[35] This is because realizing losses is too painful compared with the benefit of resetting the reference level B for gain realizations in the future. Since the investor only realizes gains and never realizes losses, he prefers higher volatility σ, ceteris paribus.

[33] To ease exposition, we turn off liquidity shocks by setting $\xi = 0$.

[34] The Internet Appendix may be found in the online version of this article.

[35] Note that the investor's value equals zero at the origin. This is because $x = 0$ is an absorbing state and no loss realization is optimal.

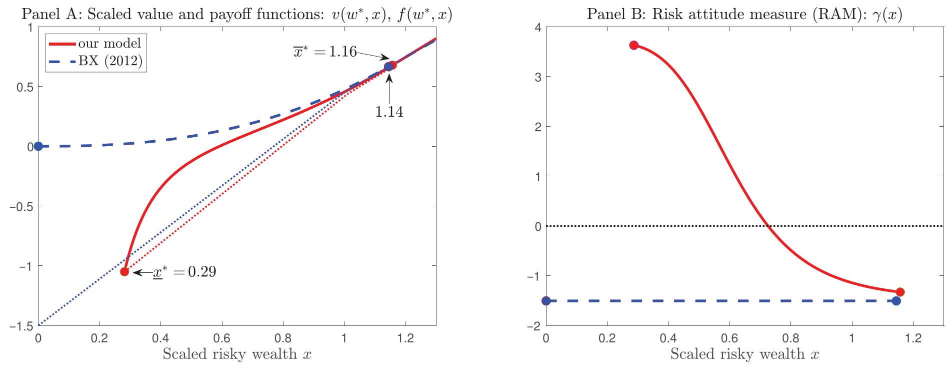

Figure 9. Piecewise linear realization utility: $\alpha_\pm = \beta = 1$. The optimal w is $w^* = -0.23$ and the investor realizes losses only when the leverage constraint (6) binds. The discount rate is set at $\delta = 35\%$ for convergence. For other parameter values, see Table II. (Color figure can be viewed at wileyonlinelibrary.com)

As the investor is risk-seeking in BX (2012) and can borrow in our model, it is optimal for him to use some leverage. However, high leverage makes his holdings volatile, which increases the probability of forced loss realization (as the leverage constraint is more likely to bind sooner). Because of the large utility cost of forced loss realization, the investor chooses a prudent leverage ratio by setting $w^* = -0.23$ and thus only realizes losses if x falls by 71% to $-w^*/\kappa = 0.23/0.79 = 29\%$ (solid red line in Panel A).[36]

In sum, the investor is risk-averse ($\gamma(x) > 0$) in the $x \in (0.29, 0.73)$ region, where he is close to the leverage constraint and the incentive to manage downside risk is strong. However, he is risk-loving ($\gamma(x) > 0$) in the $x \in (0.73, 1.16)$ region, where he is sufficiently far away from the constraint and the risk-taking force in BX (2012) dominates. The solid red lines in the two panels of Figure 9 depict this highly nonmonotonic risk-taking incentive. Note that the option to use leverage turns a convex value function in BX (2012) to a value function that is concave and then convex-shaped. Finally, the value of using leverage is quantitatively substantial, worth $\Delta = 23\%$ of the investor's budget.

VI. Conclusion

Building on BX (2009, 2012) and IJ (2013), we develop a jump-diffusion model in which an investor receives utility bursts from realizing stock gains and losses. In addition to a series of stock-level mental accounts for each utility burst, the investor also has a mental budget that brackets all of his investment episodes together to evaluate his intertemporal realization utility. A key implication of our two-layered mental account model is that the investor does not have to use his entire mental budget when trading stocks. Instead,

[36] An investor with an initial budget of $\Pi_0 = 100$ allocates $X_0 = \Pi_0/(1 + \theta_p + w^*) = 128$ to the stock and finances $28/128 = 22\%$ of his stockholdings with leverage.

he can save a fraction of his trading budget and/or use leverage to separate his trading size from his trading budget.

By saving a fraction of his budget ($w^* > 0$), the investor makes smaller trades and spreads his trades out over time, lowers transaction costs (in dollars), and is less subject to the disposition effect. By using leverage ($w^* < 0$), the investor increases his trading size beyond his budget. Because of increased risk exposure and larger transaction costs, the investor is more reluctant to realize losses, strengthening the disposition effect (Barber et al. (2019), Heimer and Imas (2022)). Introducing leverage constraints mitigates this effect by making the investor realize losses sooner (Heimer and Imas (2022)).

Our model generates new predictions for an investor selling a stock at a deep loss. With enough savings in his mental trading account, he voluntarily sells the stock at a deep loss. This prediction is consistent with the portfolio-driven disposition effect (An et al. (2024)). If the savings in his trading account is low, the investor is unwilling to sell the stock at a deep loss, but he will realize a dampened loss after the stock rebounds a bit. This sell-after-rebound prediction is consistent with our observation that retail investors often sell their losing stocks after these stocks rebound a bit.

Quantitatively, we find that having access to the risk-free asset is worth over 20% of the investor's total wealth in our calibrated diffusion models. Incorporating jumps further enhances the quantitative importance of our model mechanism.

In our model, the investor holds or trades a single stock each period. In reality, investors hold and trade multiple stocks. In future work, we aim to shed light on how a realization-utility investor trades in this richer setting.

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Appendix A: Variational Inequality

We first use a variational inequality to characterize the value function $V(W, X, B)$ defined by (14) or equivalently (17). We then discuss transversality conditions. For proofs of existence and uniqueness of the variational inequality as well as the associated verification theorem, we refer readers to Section III of the Internet Appendix.[37] The variational-inequality method recently has been applied to various problems in dynamic corporate finance. For example, using this approach, Décamps, Mariotti, Rochet, and Villeneuve (2011), Bolton, Chen, and Wang (2011, 2013), Hugonnier, Malamud, and Morellec (2015), and Bolton, Wang, and Yang (2019) analyze corporate policies and valuation under costly external equity financing, and Dai, Giroud, Jiang, and Wang (2024) study internal capital markets for a financially constrained conglomerate subject to independently and identically distributed shocks. Dai, Jiang, and

[37] The optimization problem (14) is an impulse-control problem (Øksendal and Sulem (2002), Altarovici, Reppen, and Soner (2017)) and the solution to the variational inequality (A.1) should be interpreted in a weak sense, that is, as a viscosity solution (Crandall, Ishii, and Lions (1992)).

Wang (2025) extend this method to a real-option (stopping-time) game setting to study the first- and second-mover advantages.

A.1. Characterization by Variational Inequality

In the region in which $W > -\kappa X$, $X \geq 0$, and $B > 0$, the leverage constraint (6) does not bind and the value function $V(W,X,B)$ defined in (14) satisfies the following variational inequality (see, e.g., Pham (2009) on the standard optimal-stopping theory):

$$\max\Big\{\mathcal{L}V(W,X,B),\ F(W,X,B) - V(W,X,B)\Big\} = 0\ , \tag{A.1}$$

where

$$\mathcal{L}V = \frac{1}{2}\sigma^2 X^2 V_{XX} + \mu X V_X + rWV_W + rBV_B - \delta V + \xi\,[U(G,\,B) - V], \tag{A.2}$$

and $G = (1-\theta_s)X - B$ is the realized gain (if positive) or loss (if negative).

The intuition for (A.1) is as follows. At each time t, the investor can either keep the holding in his trading account unchanged or sell the stock he owns to realize a gain or loss. If it is optimal not to change his stockholding, the first term in (A.1) is larger than the second term, which implies that the standard Hamilton-Jacobi-Bellman (HJB) equation in the waiting/holding region, $\mathcal{L}V(W,X,B) = 0$, holds. In contrast, if it is optimal to sell the stock to realize a gain or loss, we must have $V(W,X,B) = F(W,X,B)$. This is the case in which the second term in (A.1) is larger than the first term. As either keeping holdings unchanged or trading must be optimal at any given t, the variational inequality (A.1) holds at all times.

Finally, when the leverage constraint (6) binds, the investor has no choice but to realize a loss in order to satisfy the leverage constraint. Therefore,

$$V(W,X,B) = F(W,X,B), \quad \text{when } X = -W/\kappa > 0. \tag{A.3}$$

Homogeneity: We can simplify the optimization problem as follows. First, in the domain $\mathcal{S} = \{x > 0,\ w > -\kappa x\}$ where the leverage constraint (6) does not bind, the variational inequality (A.1) for $V(W,X,B)$ is equivalent to the following simplified variational inequality for the scaled value function $v(w,x) = B^{-\beta}V(W,X,B)$:

$$\max\big\{\mathscr{L}v(w,x),\ f(w,x) - v(w,x)\big\} = 0, \tag{A.4}$$

where $f(w,x)$ is the scaled payoff function given in (23), $\mathscr{L}v$ is given by

$$\mathscr{L}v = \frac{1}{2}\sigma^2 x^2 v_{xx} + (\mu - r)xv_x - \delta_e v + \xi\,[u((1-\theta_s)x - 1) - v], \tag{A.5}$$

and $\delta_e = \delta - \beta r$. Finally, if the leverage constraint (6) binds, that is, when $x = -w/\kappa > 0$, (A.3) is simplified to $v(w,x) = f(w,x)$.

A.2. Transversality Conditions

We next provide transversality conditions, which ensure that the value functions are finite. For brevity, we only report results for our diffusion model with liquidity shocks (see Section I). We provide transversality conditions for our jump-diffusion model in Internet Appendix Section II.

It is convenient to introduce the following notation:

$$K = \frac{1-\theta_s}{1-\theta_s-\kappa}\,. \tag{A.6}$$

We propose the following transversality conditions for our diffusion model:

(i) If $\beta = 1$, then

$$\delta + \xi > \mu + \max\{0, (\mu - r)(K-1)\}\,, \tag{A.7}$$

where K is defined in (A.6).

(ii) If $\beta < 1$ and $\frac{\mu - r}{(1-\beta)\sigma^2} \in (0, K)$, then

$$\delta + \xi > \beta r + \max\left\{0,\ \frac{\sigma^2}{2}\alpha_+(\alpha_+ - 1) + (\mu - r)\alpha_+,\ \frac{\beta(\mu - r)^2}{2(1-\beta)\sigma^2}\right\}. \tag{A.8}$$

(iii) If $\beta < 1$ and $\frac{\mu - r}{(1-\beta)\sigma^2} \notin (0, K)$, then

$$\delta + \xi > \beta r + \max\left\{0, \frac{\sigma^2}{2}\alpha_+(\alpha_+ - 1) + (\mu - r)\alpha_+, \frac{\sigma^2}{2}\beta(\beta - 1)K^2 + (\mu - r)\beta K\right\}. \tag{A.9}$$

We next use a two-step procedure to verify these transversality conditions. First, we construct a sufficiently smooth supersolution of the variational inequality (A.4) satisfying (A.7) to (A.9). Second, we prove that the value function is bounded from above by the supersolution. Internet Appendix Section II provides calculation details.

We now turn to the voluntary participation condition introduced in Section I. Specifically, we characterize the set of drift and volatility parameters (μ, σ) in which the investor is willing to invest a fraction of his mental budget in stocks and the remaining in the risk-free asset. (Leverage is possible provided that the leverage constraint is satisfied.)

A.3. Participation Condition: Drift and Volatility Effects

Recall that the participation condition is $\widehat{v} > 0$. In Figure A.1, we characterize the participation condition $\widehat{v} > 0$ by focusing on the drift μ and volatility σ parameters (see Table II for all other parameter values). The solid red line divides the admissible domain for (μ, σ) into two parts. Above the line is the set of (μ, σ) pairs such that the investor voluntarily invests in stocks. Below this line he does not invest in stocks, implying that his realization utility equals

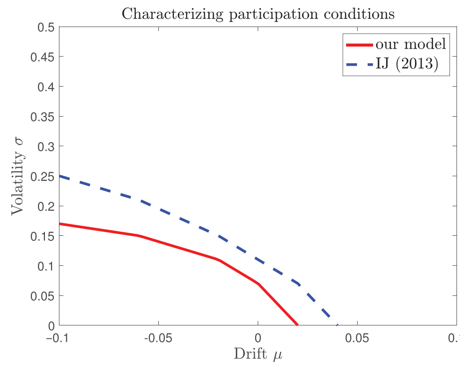

Figure A.1. Voluntary participation conditions for diffusion models: $\widehat{v} > 0$. The solid red line divides the (μ, σ) plane for our model. Above this solid line, the investor voluntarily holds a combination of the risk-free asset and the stock he chooses, so that $\widehat{v} > 0$ is satisfied. The dashed blue line divides the (μ, σ) plane for the corresponding realization-utility models where at each t the investor has a binary choice of either only holding the stock or investing solely in the risk-free asset. Above this dashed line, the participation condition $\widehat{v} > 0$ is satisfied. Parameter values: $\delta = 0.05$, $\xi = 0.1$, $\kappa = 0.79$, $r = 3\%$, $\theta_p = \theta_s = 1\%$, $\alpha_+ = 0.5$, $\alpha_- = 0.5$, $\lambda = 1.5$, and $\beta = 0.3$. (Color figure can be viewed at wileyonlinelibrary.com)

zero. When investing in a stock that he chooses, he allocates a fraction of his budget to the stock and saves his remaining (mental) trading budget. It is possible that he allocates more than 100% of his mental budget to the stock he chooses, using leverage subject to the leverage constraint (6).

To highlight how our model relaxes the investor's voluntary participation constraint $\widehat{v} > 0$ (increasing the set of admissible parameter values), we also characterize the participation condition for realization-utility models, for example, BX (2012), IJ (2013), and HY (2019), where the investor has a *binary* choice between allocating his entire budget to a stock that he chooses or allocating his entire budget to the risk-free asset (a combination of the two assets is not allowed). The participation condition in this case is satisfied if and only if the parameters (μ, σ) are in the region above the dashed blue line.

The solid red line, which defines the participation condition for our model, is lower than the dashed blue line for realization-utility models in the literature, which corresponds to a binary choice between two options. This result is consistent with our intuition that the participation condition $\widehat{v} > 0$ for the investor in our model is easier to satisfy because of the additional investment flexibility along the intensive margin (how much to invest in a stock that he chooses).

Appendix B: Solutions and Transition Dynamics

In this appendix, we first derive closed-form solutions for our baseline model with no liquidity shocks. We then analyze the model-implied transition dynamics. Finally, for the generalized jump-diffusion model, we propose a numerical procedure based on the penalty method proposed in Dai and Zhong (2010).

B.1. Closed-Form Solution for Diffusion Models without Liquidity Shocks

While in general the model solution features four regions (a gain-realization region, a loss-realization region, a normal holding region, and a deep-loss holding region), only the first three regions are reached on the optimal path in our diffusion model as in IJ (2013) and HY (2019). This is because (i) as soon as the investor realizes losses or gains, he resets x to one and returns to the normal holding region, and (ii) stock prices follow diffusion processes, which are continuous.

Since we focus on the optimal path, it is convenient to use the heuristic real-option approach as in IJ (2013), who correctly identify the three regions on the optimal path.[38] Smooth-pasting conditions also correctly characterize the optimal realization strategies in IJ (2013), as pointed out by HY (2019).

In the (normal) holding region, we conjecture and later verify the following closed-form expression for $v(w,x)$:

$$v(w,x) = C_1(w)x^{\eta_1} + C_2(w)x^{\eta_2}, \tag{B.1}$$

where $\eta_1 > 0$ and $\eta_2 < 0$ are the two roots of the fundamental quadratic equation,[39]

$$h(\eta) = \frac{\sigma^2}{2}\eta(\eta-1) + (\mu - r)\eta - \delta_e. \tag{B.2}$$

Next, we determine $C_1(w)$ and $C_2(w)$ as functions of w.[40]

We show that the normal holding region is characterized by two endogenous threshold functions, $\underline{x}(w)$ and $\overline{x}(w)$, satisfying $\max\{-w/\kappa, 0\} \le \underline{x}(w) < 1 \le \overline{x}(w)$, where $-w/\kappa$ is the involuntary liquidation threshold implied by the leverage constraint (6) when $w < 0$ (the leverage case). That is, the investor optimally keeps his scaled savings constant at w in the $x \in (\underline{x}(w), \overline{x}(w))$ region and trades only when $x = \underline{x}(w)$ or $x = \overline{x}(w)$. We refer to $\overline{x}(w)$ and $\underline{x}(w)$ as the optimal gain- and loss-realization boundary, respectively.

Using (23) and (B.1), we obtain the following expression for the scaled value function $f(w,x)$ in both the gain- and loss-realization regions:

$$f(w,x) = u\big((1-\theta_s)x - 1\big) + \frac{C_1(w^*) + C_2(w^*)}{[w^* + (1+\theta_p)]^\beta}\big[w + (1-\theta_s)x\big]^\beta, \tag{B.3}$$

[38] By the heuristic real-option approach, we refer to the following commonly used solution method in the real-options literature. First, conjecture that the value function satisfies an HJB equation in the holding (waiting) region. Second, specify the payoff functions in the realization (exercise) region. Finally, impose the value-matching and smooth-pasting conditions. We find that, while intuitive and easy to use compared to the variational inequality, the heuristic real-option approach may give incorrect solution in a jump-diffusion model. Researchers should exercise caution when applying the heuristic approach to solve real-option problems.

[39] Different from the classical real-options literature, for example, McDonald and Siegel (1986) and Dixit and Pindyck (1994), the positive root η_1 may be less than one, which means that the value function may not be globally convex in X. See Figure 3 for an example,

[40] Recall that w_t is constant over time.

where w^* is the optimal postrealization scaled savings given by

$$w^* = \arg\max_{\widehat{w} \geq -\kappa} \frac{C_1(\widehat{w}) + C_2(\widehat{w})}{[\widehat{w} + (1+\theta_p)]^\beta}. \tag{B.4}$$

There are two scenarios for the solution. First, if the investor is willing to voluntarily realize losses, that is, when $\underline{x}(w) > \max\{-w/\kappa, 0\}$ holds, then the two realization boundaries, $\overline{x}(w)$ and $\underline{x}(w)$, satisfy the following value-matching and smooth-pasting conditions:

$$C_1(w)[\overline{x}(w)]^{\eta_1} + C_2(w)[\overline{x}(w)]^{\eta_2} = f(w, \overline{x}(w)), \tag{B.5}$$

$$C_1(w)[\underline{x}(w)]^{\eta_1} + C_2(w)[\underline{x}(w)]^{\eta_2} = f(w, \underline{x}(w)), \tag{B.6}$$

$$C_1(w)\eta_1[\overline{x}(w)]^{\eta_1-1} + C_2(w)\eta_2[\overline{x}(w)]^{\eta_2-1} = f_x(w, \overline{x}(w)), \tag{B.7}$$

$$C_1(w)\eta_1[\underline{x}(w)]^{\eta_1-1} + C_2(w)\eta_2[\underline{x}(w)]^{\eta_2-1} = f_x(w, \underline{x}(w)), \tag{B.8}$$

where $f(w,x)$ is given in (B.3). Second, if the investor realizes losses only when the leverage constraint (6) binds, we replace the smooth-pasting condition (B.8) with $\underline{x}(w) = \max\{0, -w^*/\kappa\}$ implied by (6).

We next analyze the case $\kappa > 0$. We then turn to the no-leverage case where $\kappa = 0$.

The Case $\kappa > 0$: First, we solve for the five numbers w^*, $C_1(w^*)$, $C_2(w^*)$, $\overline{x}(w^*)$, and $\underline{x}(w^*)$ using the five-equation system (B.4) to (B.8). Second, we substitute the values of w^*, $C_1(w^*)$, and $C_2(w^*)$ into (B.3) to obtain $f(w,x)$. Finally, we solve for the four functions $C_1(w)$, $C_2(w)$, $\overline{x}(w)$, and $\underline{x}(w)$ by substituting $f(w,x)$ into the four-equation system (B.5) to (B.8).

Step 1: Show $w^* > -\kappa$: The optimal new (scaled) allocation to the risk-free asset w^* when the investor trades must be in the interior region of w: $w^* > -\kappa$. Otherwise, immediately after trading, the investor has to incur trading costs again, which is suboptimal. Therefore, the first-order condition for (B.4) implies[41]

$$[w^* + 1 + \theta_p][C_1'(w^*) + C_2'(w^*)] - \beta[C_1(w^*) + C_2(w^*)] = 0. \tag{B.9}$$

We solve for w^* and the functions $C_1(w)$, $C_2(w)$, $\underline{x}(w)$, and $\overline{x}(w)$ in the following two steps.

Step 2: Solve for five numbers: w^*, $C_1(w^*)$, $C_2(w^*)$, $\overline{x}(w^*)$, and $\underline{x}(w^*)$: Recall that $w_s = w^*$ is absorbing in that for all $s \geq t$, where $t = \{u : \inf_u w_u = w^*\}$. Conditional on $w = w^*$, the solution boils down to a one-dimensional problem. There are two possible scenarios, as we show next.

[41] We verify that the second-order condition holds at w^*, that is, for $m''(w^*) < 0$.

- Scenario (i), the leverage constraint does not bind: $\underline{x}(w^*) > \max\{0, -w^*/\kappa\}$. In this case, the value-matching and smooth-pasting conditions hold at both $x = \underline{x}(w^*)$ and $x = \overline{x}(w^*)$. We therefore obtain a candidate solution for these five numbers by solving a system of five equations: (B.5) to (B.9).
- Scenario (ii), the leverage constraint binds: $\underline{x}(w^*) = \max\{0, -w^*/\kappa\}$. We then obtain a candidate solution for $\{w^*, C_1(w^*), C_2(w^*), \overline{x}(w^*)\}$ by solving a system of four equations: (B.5) to (B.7), and (B.9), as $\underline{x}(w^*) = \max\{0, -w^*/\kappa\}$.

Step 3: Solve for the four functions $\{C_1(w), C_2(w), \underline{x}(w), \overline{x}(w)\}$, where $w \neq w^*$:

- For scenario (i) above, using the value-matching and smooth-pasting conditions at $x = \underline{x}(w)$ and $x = \overline{x}(w)$, we derive a candidate solution for $\{C_1(w), C_2(w), \underline{x}(w), \overline{x}(w)\}$ by solving a system of four equations: (B.5) to (B.8). Note that we can use the explicit expression for the payoff function $f(w, x)$ obtained from our analysis in Step 2 (for $w = w^*$).
- For scenario (ii) above, we obtain the candidate solution for $\{C_1(w), C_2(w), \overline{x}(w)\}$ by solving a system of three equations: (B.5) to (B.7). This is because $\underline{x}(w) = \max\{-w/\kappa,\ 0\}$ holds as the fourth equation.

Finally, comparing the candidate solutions from the two scenarios and choosing the one that gives the larger value of $v(w^*, 1)$, we obtain the optimal solution.

We have now described the procedure for obtaining the optimal solution for the case $\kappa > 0$, where the investor can borrow. We next consider the no-borrowing case: $\kappa = 0$.

The Case $\kappa = 0$: In this case, since the investor cannot borrow, there is no forced loss realization, which implies $\mathcal{S} = \{x > 0,\ w \geq 0\}$. Since it may be optimal to invest all of his budget in the stock he chooses, there are two possible cases for $w = w^*$: (i) $w^* > 0$, where the first-order condition (B.9) holds, and (ii) $w^* = 0$. For both cases, we have two scenarios depending on whether the investor is willing to voluntarily realize losses: (i) $\underline{x}(w^*) > 0$ and (ii) $\underline{x}(w^*) = 0$. Using the same argument as in the $\kappa > 0$ case, we can obtain the optimal solution for w^* and the four functions $\{C_1(w), C_2(w), \underline{x}(w), \overline{x}(w)\}$.

For the optimal trading strategy and the value function, we provide a verification theorem in Proposition IA.3 and a proof in the Internet Appendix. We next describe the transition dynamics of (x, w) to the target $(1, w^*)$.

B.2. Transition Dynamics to the Target Position $(x, w) = (1, w^*)$

We analyze the transition dynamics for an investor starting with a position given by (W, X, B) or equivalently in the scaled variables $(x, w) = (X/B, W/B)$. Let $\tau(x, w)$ denote the first time the investor realizes gains or losses starting

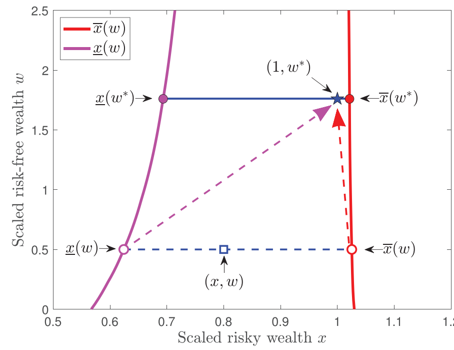

Figure B.1. Transition dynamics of (x_t, w_t). (Color figure can be viewed at wileyonlinelibrary.com)

from (x, w).[42] Immediately after $\tau(x, w)$, the investor adjusts his position to $(x_{\tau+}, w_{\tau+}) = (1, w^*)$. For all $t \geq \tau(x, w)$, $w_t = w^*$ and x_t follows (20) before the investor trades again.

Figure B.1 illustrates the transition dynamics.[43] First, starting from the position (x, w) (blue square), the investor's scaled risky wealth x_t moves stochastically along the horizontal dashed blue line (because $dw_t = 0$) in response to shocks until x_t reaches either the open pink circle or the open red circle. After immediately realizing losses or gains at $\tau(x, w)$, the investor readjusts his allocation to the blue starred position $(1, w^*)$. Note the discrete change of his scaled risk-free wealth from w to w^* at $\tau(x, w)$.

After $\tau(x, w)$, the investor stays on the horizontal dashed blue line as $w_t = w^*$ for all t. The scaled risky wealth x evolves stochastically until either the loss-realization boundary $\underline{x}(w^*)$ or the gain-realization boundary $\bar{x}(w^*)$ is reached. After reaching either boundary line, the investor again readjusts his position to $(1, w^*)$.

We next propose a numerical scheme for our jump-diffusion model.

B.3. Numerical Scheme: Penalty Method

For the generalized jump-diffusion model, we do not have closed-from solutions. We obtain the solution by working with the following variational inequality:

$$\max\{\mathscr{L}^{\mathcal{J}} v(w, x),\ f(w, x) - v(w, x)\} = 0 \quad \text{for } x \geq 0,\ w \geq 0, \tag{B.10}$$

[42] Mathematically, given the optimal double-barrier policy, $\bar{x}(w)$ and $\underline{x}(w)$, because $dw_t = 0$, we have $\tau(x, w) = \inf\{t \geq 0 \mid x_0 = x,\ x_t \notin (\underline{x}(w), \bar{x}(w))\}$.

[43] We start the dynamics in the normal holding region. Parameter values are from our quantitative analysis in Table II in Section III.

where $f(w, x)$ is given by

$$f(w,x) = u\big((1-\theta_s)x - 1\big) + \max_{\widehat{w}\geq 0} \frac{v(\widehat{w},1)}{(\widehat{w}+1+\theta_p)^{\beta}} \,[(1-\theta_s)x + w]^{\beta}, \tag{B.11}$$

and the operator $\mathscr{L}^{\mathcal{J}}$ is

$$\mathscr{L}^{\mathcal{J}}v(w,x) = \frac{\sigma^2 x^2}{2} v_{xx}(w,x) + (\mu - r)xv_x(w,x) - \delta_e v(w,x) + \rho\big(\mathbb{E}[v(w,Yx)] - v(w,x)\big). \tag{B.12}$$

We use the following iteration algorithm developed by Dai and Zhong (2010).

Step 1. For $k \geq 0$, calculate the payoff function $f^{(k)}(w, x)$ in (B.11) taking the k$^{\text{th}}$ solution $v^{(k)}(w, x)$ as given (we start the iteration with $v^{(0)}(w, x) = 1$).

Step 2. Update $v^{(k+1)}(w, x)$ by solving the following equation with a penalty term:

$$\begin{aligned} 0 = &\frac{1}{2}\sigma^2 x^2 v_{xx}^{(k+1)}(w,x) + (\mu - r)xv_x^{(k+1)}(w,x) - (\delta_e + \rho)v(w,x) \\ &+\rho\,\mathbb{E}[v^{(k)}(w,Yx)] + P \times \mathbf{1}_{\{f^{(k)}(w,x) - v^{(k)}(w,x) > 0\}}\Big(f^{(k)}(w,x) - v^{(k+1)}(w,x)\Big), \end{aligned}$$

where P is a large penalty constant, for example, $P = 10^6$.

Step 3. Stop iteration if the relative error is less than a given tolerance level ε, for example, $\varepsilon = 10^{-9}$:

$$\frac{||v^{(k+1)} - v^{(k)}||}{\max\{1, ||v^{(k)}||\}} < \varepsilon.$$

Otherwise, go to Step 1 and continue the iterative process.

Appendix C: Deep-Loss Region

In this appendix, we explain why it is possible for the investor in our model to voluntarily sell a stock at a deep loss while it is not possible in IJ (2013) and HY (2019). We illustrate this difference between our model and their models by focusing on the payoff functions near $x = 0$.

Predictions of IJ (2013) and HY (2019): We first show that if the investor has to make a binary choice between a stock and the risk-free asset as in IJ (2013) and HY (2019), it is not possible for the investor to sell a stock at a deep loss.[44] Recall that the scaled payoff function in IJ (2013) is (in our notation)

$$f_N(x) = u((1-\theta_s)x - 1) + \left(\frac{1-\theta_s}{1+\theta_p}x\right)^{\beta} v_N(1),$$

where $v_N(1) = V_N(1, 1)$. In the deep-loss region, that is, when x approaches zero,

$$\lim_{x\to 0} f_N(x) = \lim_{x\to 0} u((1-\theta_s)x - 1) + \left(\frac{1-\theta_s}{1+\theta_p}x\right)^{\beta} v_N(1) = -\lambda < 0.$$

Therefore, voluntarily realizing a deep loss, which yields negative utility, must be suboptimal in IJ (2013) and HY (2019). This is because the investor can always achieve zero realization utility by never realizing a loss. In sum, being passive is optimal if the stock that he owns is at a deep loss because the entire budget is invested in a single stock.

Also note that this no-deep-loss-realization result holds when the solution features only two regions. In that case, realizing losses is so painful that the investor does not want to realize a loss of any size as in BX (2012). This two-region-solution case is a special case of our four-region solution analyzed above.

Predictions of Our Model: We now return to our model and explain why there are two Cases, A and B, that generate quite different economic predictions when an investor has two-layered mental accounts and does not have to invest his entire mental budget in a single stock.

- Case A. The solution features three regions, and realizing a deep loss is optimal. This is a new key result of our model, summarized in the first row for "our model" in Table II.
- Case B. The solution features four regions, and realizing a deep loss is not optimal as in IJ (2013) and HY (2019). This case generates sell-after-rebound. The second row for "our model" in Table II describes this case.

We next address the question of which case gives the solution under what conditions, and provide intuition. Recall that in our model, the investor's scaled payoff function is

$$f(w^*, x) = u((1-\theta_s)x - 1) + \left(\frac{w^* + (1-\theta_s)x}{w^* + 1 + \theta_p}\right)^{\beta} v(w^*, 1).$$

[44] HY (2019) provide a proof of this result.

It is helpful to first consider the limit of $f(w^*, x)$ for the case $w^* > 0$:

$$\lim_{x\to 0} f(w^*, x) = \lim_{x\to 0}\left[u((1-\theta_s)x - 1) + \left(\frac{w^* + (1-\theta_s)x}{w^* + 1 + \theta_p}\right)^{\beta} v(w^*, 1)\right]$$

$$= -\lambda + \left(\frac{w^*}{w^* + 1 + \theta_p}\right)^{\beta} v(w^*, 1). \tag{C.1}$$

Equation (C.1) implies the following two possibilities:

(i) For w^* large enough that $f(w^*, 0) > 0$ (e.g., when compared with λ), it is optimal for the investor to voluntarily realize a deep loss, as doing so yields a higher value $v(w^*, 0)$ than being permanently passive, which yields zero utility. This is our Case A where the solution features three regions. Being able to save a fraction of his trading budget for future potential gain realizations in our model is the key force driving this result.

(ii) For w^* sufficiently close to zero, the scaled payoff is negative: $f(w^*, x) < 0$. Because the investor can always achieve zero realization utility by never realizing a loss, realizing a deep loss cannot be optimal (as $f(w^*, x) < 0$). This is why being passive is optimal in the deep-loss region.[45] This is our Case B where the solution features four regions as in IJ (2013) and HY (2019).

In sum, if $w^* > 0$, the solution features either three regions (Case A) or four regions (Case B) depending on how large w^* is. Finally, if it is optimal to use leverage ($w^* < 0$), the investor has no option but to realize a loss when the scaled risky wealth x approaches the involuntary realization boundary, $-w^*/\kappa > 0$, implied by the leverage constraint (otherwise, lenders cannot break even and the investor cannot borrow ex ante).

Appendix D: Comparative Statics

In this appendix, we conduct comparative statics for the optimal policies: the steady-state value w^* and trading strategies $(\underline{x}^*, \overline{x}^*)$. We focus on the effects of liquidity shocks, the investment opportunity, and the investor's preferences.

D.1. Effects of Liquidity Shocks: ξ

In this subsection, we consider the effect of liquidity shocks on the investor's risk-taking and value function curvature. We show that liquidity shocks provide discipline on the investor's option to sit out and encourage risk-taking. However, the effects of liquidity shocks on the value function curvature also critically depend on whether the leverage constraint (6) binds.

[45] In IJ (2013) and HY (2019), $f(w^*, 0) = -\lambda < 0 \le v(w^*, 0)$ holds automatically.

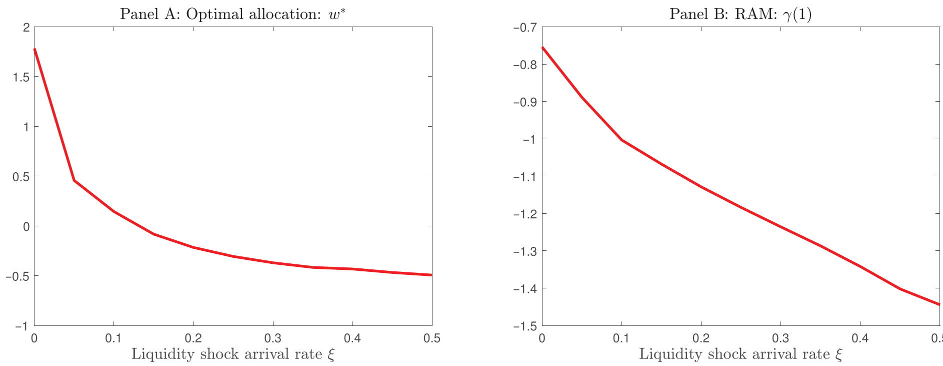

Figure D.1. Effects of liquidity shocks on w^* and value function curvature: RAM $\gamma(1)$ when leverage constraint (6) does not bind. Panels A and B plot the optimal allocation to the risk-free asset w^* and RAM $\gamma(1)$, respectively, as we vary the liquidity shock arrival rate ξ. As ξ increases, the investor increases his stockholdings (w^* decreases) and becomes more willing to take risk. See Table II for parameter values. (Color figure can be viewed at wileyonlinelibrary.com)

When the Leverage Constraint (6) Does Not Bind: In Figure D.1, we plot w^* and the value function curvature measure, risk attitude measure (RAM) $\gamma(1)$, as we vary the liquidity shock arrival rate ξ, for an S-shaped utility function with $\alpha_+ = \alpha_- = 0.5$, $\lambda = 1.5$, and $\beta = 0.3$. Panel A of Figure D.1 shows that as the liquidity shock becomes more frequent (ξ increases), the investor increases his stockholdings by decreasing w^*. This is because liquidity shocks effectively shorten the investment horizon and make savings (in the risk-free asset), which yield no utility bursts, more costly. The effect of liquidity shocks on w^* is substantial: increasing the liquidity shock arrival rate from $\xi = 0$ to $\xi = 0.5$ leads the investor's stockholdings to increase 5.4 times from 36% of his total trading budget to 196% (as w^* decreases from 1.76 to -0.5). Note that w^* turns negative for $\xi \geq 0.15$, which means that the investor uses leverage to amplify his stockholdings when the arrival rate is sufficiently high ($\xi \geq 0.15$). We verify that the leverage constraint (6) (for Figure D.1) does not bind by checking $\underline{x}^* > \max\{0, -w^*/\kappa\}$.

Panel B of Figure D.1 shows that the investor's value function curvature, measured by RAM $\gamma(1)$, is negative and its absolute value $\gamma(1)$ increases as ξ increases. This means that the investor is risk-seeking and becomes more willing to take risk as ξ increases. This is because liquidity shocks provide discipline on the investor's option to sit out and increase the convexity of the investor's value function: $\gamma(1)$.

When the Leverage Constraint (6) Binds: In Figure D.2, we plot w^* and RAM $\gamma(1)$ for the case in which the investor has a piecewise linear realization utility as in BX (2012), that is, we set $\alpha_+ = \alpha_- = \beta = 1$, loss-aversion parameter to $\lambda = 1.5$, and $\delta = 0.35$, and we keep all other parameters the same as in Table II. Panel A of Figure D.2 confirms that the main result continues to

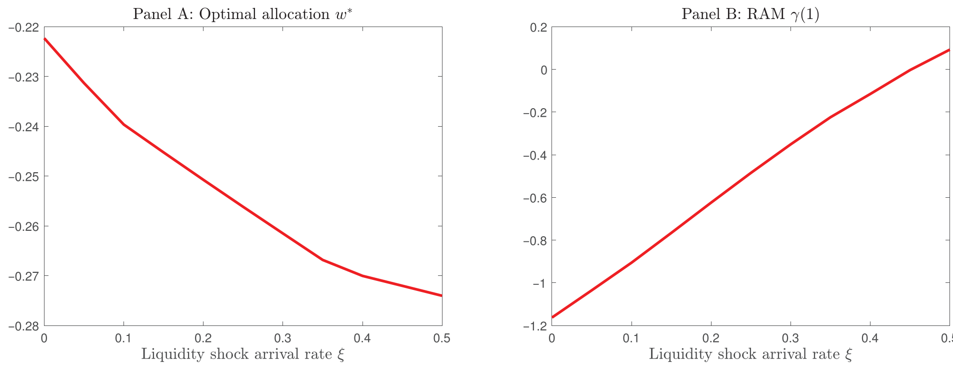

Figure D.2. Effects of liquidity shocks on w^* and value function curvature: RAM $\gamma(1)$ when leverage constraint (6) binds. Panels A and B plot the optimal allocation to the risk-free asset w^* and RAM $\gamma(1)$, respectively, as we vary the liquidity shock arrival rate ξ. As ξ increases, the investor increases his stockholdings (w^* decreases) and becomes less risk-taking due to binding of leverage constraint. Parameter values: $\alpha_{\pm} = \beta = 1$, $\delta = 0.35$, and other parameter values are listed in Table II. (Color figure can be viewed at wileyonlinelibrary.com)

hold, that is, the investor's stock allocation increases with the arrival rate ξ. This is because liquidity shocks provide discipline on the investor's option to sit out and encourage him to take risk.

However, the investor's value function curvature measure, RAM $\gamma(1)$, behaves quite differently from the case analyzed in Figure D.1. Rather than decreasing with ξ, RAM $\gamma(1)$ increases with ξ because the leverage constraint (6) binds for all ξ (recall our analysis in Section V for the case in which the investor has a piecewise linear realization utility). The intuition is as follows. When the leverage constraint binds, the investor is forced to realize losses to meet his debt payments, which generates a large utility cost (a negative utility burst). Anticipating this scenario, the investor is endogenously averse to stock return volatility when he is close to the binding leverage constraint. This example illustrates that leverage constraints have a significant effect on the investor's risk-taking and can mitigate his disposition effect, as we discuss above.

D.2. Investment Opportunity: (r, μ)

Panels A1 and A2 of Figure D.3 plot the ratio between the investor's risk-free savings and stockholdings, w^*, as we vary the risk-free rate r and the expected stock return μ, respectively. The investor increases his stock allocations by decreasing w^* as μ increases or r decreases.

The quantitative effects of changing the investment opportunity within an economically plausible range are large. What is the effect of decreasing r from 3% to 0? The ratio w^* decreases significantly from 1.76 to −0.14 (Panel A1), which means that the investor's stock allocation increases 3.2 times from 36% of his budget to a leveraged position with 115% of his budget. Similarly, he increases his stock allocation by decreasing w^* from 1.76 to −0.15, which

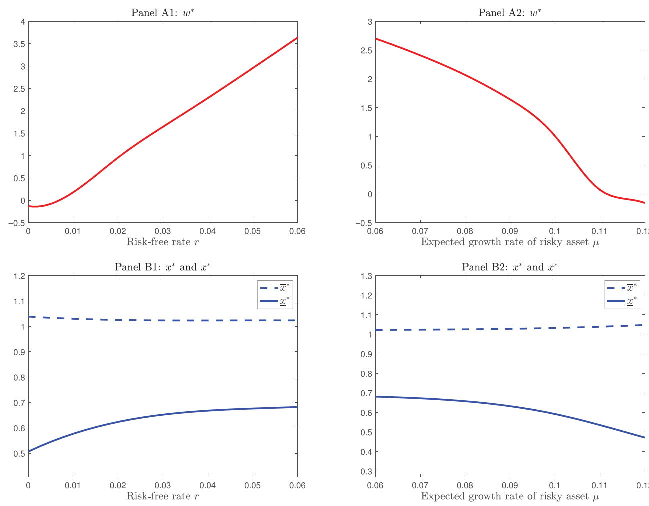

Figure D.3. Comparative statics: effects of changing investment opportunity: (r, μ). Panels A1 and A2 plot the optimal w^* as we vary r and μ, respectively. Panels B1 and B2 plot the optimal gain-realization threshold $\overline{x}^*$ (dashed blue lines) and loss-realization threshold $\underline{x}^*$ (solid blue lines), as we vary r and μ, respectively. See Table II for all other parameter values. (Color figure can be viewed at wileyonlinelibrary.com)

implies that his stock allocation increases 3.2 times from 36% of his budget to a leveraged position amounting to 116% of his budget (Panel A2).

Panels B1 and B2 of Figure D.3 plot the gain-realization threshold $\overline{x}^*$ (dashed blue lines) and the loss-realization threshold $\underline{x}^*$ (solid blue lines) as we vary r and μ, respectively. Recall that as we increase μ or decrease r, the investor increases his dollar exposure to the stock (decreasing w^*), which in turn makes him more reluctant to realize losses and gains. As a result, the holding region $(\underline{x}^*, \overline{x}^*)$ widens as μ increases (Panel B1) or r decreases (Panel B2).

Finally, the quantitative effects of changing r or μ on loss realization (lower boundary) are much more significant than on gain realization (upper boundary).

D.3. Realization Utility: $(\lambda, \alpha_-, \alpha_+)$

In Figure D.4, we conduct comparative static analysis for three key preference parameters: loss aversion λ, loss sensitivity α_-, and gain sensitivity α_+. We plot the optimal allocation, w^*, as functions of λ, α_-, and α_+ in Panels A1, A2, and A3, respectively. Similarly, we plot the gain-realization threshold $\overline{x}^*$ (dashed blue lines) and the loss-realization threshold $\underline{x}^*$ (solid blue lines) as functions of λ, α_-, and α_+ in Panels B1, B2, and B3, respectively.

To ease comparison with IJ (2013), we first analyze the lower row of Figure D.4. The key takeaways are as follows. First, as the investor becomes more loss-averse (higher λ), more sensitive to losses (lower α_-), or less sensitive to gains (higher α_+), the loss-realization threshold $\underline{x}^*$ decreases and the

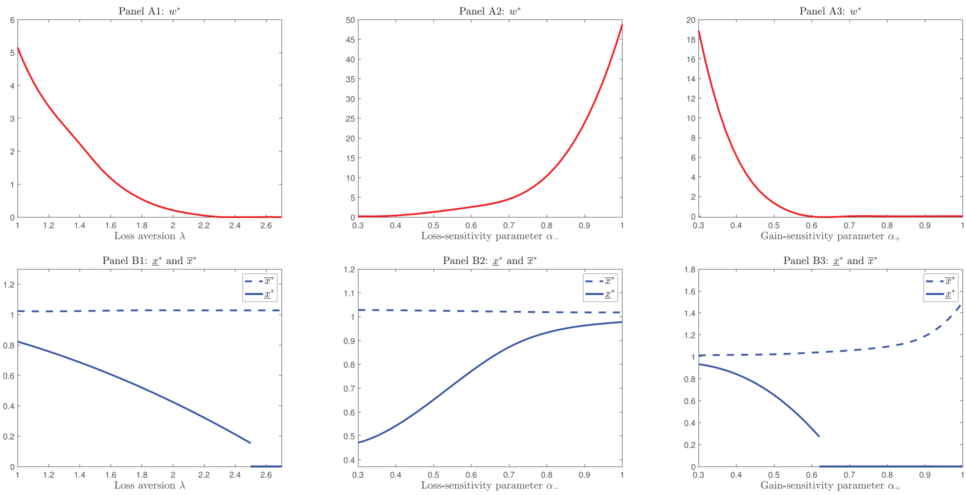

Figure D.4. Comparative statics: effects of changing realization utility: $(\lambda, \alpha_-, \alpha_+)$. Panels A1, A2, and A3 plot the optimal w^* as we vary λ, α_-, and α_+, respectively. Panels B1, B2, and B3 plot the optimal gain-realization threshold $\overline{x}^*$ (dashed blue lines) and the optimal loss-realization threshold $\underline{x}^*$ (solid blue lines) as we vary λ, α_-, and α_+, respectively. All parameter values other than the one being studied are reported in Table II. (Color figure can be viewed at wileyonlinelibrary.com)

gain-realization threshold $\overline{x}^*$ increases. The effects on $\underline{x}^*$ and $\overline{x}^*$ reinforce each other, widening the holding region. A more loss-averse or more loss-sensitive investor is less willing to realize losses. A less gain-sensitive (higher α_+) investor waits longer to realize gains, reducing the option value of resetting the reference level and lowering the loss-realization threshold $\underline{x}^*$.

Second, comparing the loss-realization threshold $\underline{x}^*$ (dashed lines) with the gain-realization threshold $\overline{x}^*$ (solid lines), we clearly see that the quantitative effects of changing these preference parameters on $\underline{x}^*$ are much more significant than on $\overline{x}^*$. For loss aversion satisfying $\lambda \geq 2.5$ or gain sensitivity satisfying $\alpha_+ \geq 0.62$, the investor never realizes losses. These results are similar to those in IJ (2013).

The upper row shows that w^* decreases with loss aversion λ (Panel A1), increases with α_- (Panel A2), and decreases with α_+ (Panel A3), reinforcing the results discussed above for the lower row. Recall that as we decrease loss aversion (reducing λ), decrease loss sensitivity (increasing α_-), and increase gain sensitivity (decreasing α_+), the investor trades more frequently (narrower holding region $(\underline{x}^*, \overline{x}^*)$). As a result, the benefit of placing more frequent and smaller stock trades leads w^* to increase.

Quantitatively, the effects of changing these preference parameters within economically relevant ranges on w^* are large. For example, the investor's stockholdings will increase from 64% to 100% of his total mental trading budget (following a decrease of w^* from 1.76 to zero) if any one of the following three changes takes place: (i) loss aversion λ increases from 1.5 to 2.5, (ii) the parameter α_- decreases from 0.5 to 0.3, and (iii) the parameter α_+ increases from 0.5 to 0.62.

In sum, the option to save a fraction of his mental budget or use leverage significantly alters the investor's dynamic trading strategies and has large quantitative effects.

We next analyze the effect of liquidity shocks on w^* and the value function curvature.

Appendix E: Proof of Proposition 1 (Piecewise Linear Realization Utility)

We prove this proposition in two steps. First, we show that the investor never realizes losses as in BX (2012) so that $\underline{x}(w^*) = \max\{-w^*/\kappa,\ 0\}$. Second, we show that the investor has no incentives to save, $w^* \leq 0$, despite having the option to do so.

Step 1: Prove $\underline{x}(w^*) = \max\{-w^*/\kappa,\ 0\}$: First, for piecewise linear $u(\cdot)$, where $\alpha_\pm = \beta = 1$, $f(w^*, x)$ is given by

$$f(w^*, x) = \begin{cases} (1-\theta_s)x - 1 + \frac{v(w^*,1)}{w^*+1+\theta_p}\big(w^* + (1-\theta_s)x\big)\,, \\ \qquad\qquad\qquad\qquad \text{if } x \in [1/(1-\theta_s), \infty)\,, \\ -\lambda\big(1-(1-\theta_s)x\big) + \frac{v(w^*,1)}{w^*+1+\theta_p}\big(w^* + (1-\theta_s)x\big)\,, \\ \qquad\qquad\qquad\qquad \text{if } x \in [0, 1/(1-\theta_s)). \end{cases}$$

Note that $f(w^*, x)$ is linear in x with slope $\left(\frac{v(w^*,1)}{w^*+1+\theta_p} + \lambda\right)(1-\theta_s)$ in the region where $x \in [1/(1-\theta_s), \infty)$ and $f(w^*, x)$ is linear in x with slope $\left(\frac{v(w^*,1)}{w^*+1+\theta_p} + 1\right)(1-\theta_s)$ in the region where $x \in [0, 1/(1-\theta_s))$. Because $\lambda \geq 1$, the slope in the loss region where $x \in [0, 1/(1-\theta_s))$ is larger than that in the gain region where $x \in [1/(1-\theta_s), \infty)$. Also, $f(w^*, x)$ is increasing and globally concave in x.

We prove $\underline{x}(w^*) = \max\{-w^*/\kappa,\ 0\}$ by contradiction. Suppose that the opposite holds, with $\max\{-w^*/\kappa,\ 0\} < \underline{x}(w^*) < \overline{x}(w^*)$. Then the optimal trading thresholds are interior, implying that the smooth-pasting conditions apply at both $\underline{x}(w^*)$ and $\overline{x}(w^*)$. Using these conditions together with the concavity of $f(w^*, x)$ in x, we obtain

$$0 < v_x(w^*, \overline{x}(w^*)) = f_x(w^*, \overline{x}(w^*)) \leq f_x(w^*, \underline{x}(w^*)) = v_x(w^*, \underline{x}(w^*)).$$

Using the smooth-pasting conditions at the two boundaries and the result that $v > f$ in the holding region, we know that v_x must attain a local maximum at a point denoted by $\check{x}$ between the two boundaries: $\check{x} \in (\underline{x}(w^*),\ \overline{x}(w^*))$. That is, $v_{xx}(w^*, \check{x}) = 0 \geq v_{xxx}(w^*, \check{x})$. As $\mathscr{L}v(w^*, x) = 0$ in the holding region, differentiating $\mathscr{L}v(w^*, x) = 0$ with respect to x, we obtain the following contradiction at $(w, x) = (w^*, \breve{x})$:[46]

$$0 = \frac{\sigma^2 x^2}{2} v_{xxx} + (\mu - r + \sigma^2) x v_{xx} - (\delta - \mu) v_x \leq -(\delta - \mu) v_x < 0, \qquad \text{(E.1)}$$

where the first inequality uses $v_x(w^*, \breve{x})$ being a local maximum and the last equality follows from the monotonicity condition $v_x > 0$ and the transversality condition $\delta > \mu$. Therefore, $\underline{x}(w^*) = \max\{-w^*/\kappa,\ 0\}$ and the investor never voluntarily realizes losses.

Step 2: Prove $w^* \leq 0$: To prove this result, note that it is equivalent to proving that the investor always chooses zero savings ($w^* = 0$) even if he can save in the risk-free asset.

Recall that the general solution of $v(w, x)$ in the holding region is

$$v(w, x) = C_1(w) x^{\eta_1} + C_2(w) x^{\eta_2}, \qquad \text{(E.2)}$$

where $C_1(\cdot)$ and $C_2(\cdot)$ are two functions of w to be determined, and $\eta_1 > 0$ and $\eta_2 < 0$ are the two roots of (B.2). We prove $w^* \leq 0$ by contradiction.

Suppose that $w^* > 0$. Using $\underline{x}(w^*) = \max\{-w^*/\kappa,\ 0\} = 0$ obtained from Step 1 and $\eta_2 < 0$, we obtain $C_2(w) = 0$. Using the first-order condition for w^*, we obtain

$$C_1'(w^*)(w^* + 1 + \theta_p) = C_1(w^*). \qquad \text{(E.3)}$$

The value-matching and smooth-pasting conditions at $x = \overline{x}(w^*)$ imply

$$C_1'(w^*)\big(\overline{x}(w^*)\big)^{\eta_1} = \frac{C_1(w^*)}{w^* + 1 + \theta_p}. \qquad \text{(E.4)}$$

Combining (E.3) and (E.4), we obtain

$$\overline{x}(w^*) = 1, \qquad \text{(E.5)}$$

which implies that the investor sells immediately after buying a new stock. This is clearly suboptimal as there are no gains and the investor continuously pays transaction costs. Therefore, the investor chooses not to save ($w^* = 0$) even if he can save.

In sum, an investor with piecewise linear realization utility (and loss aversion) has no incentives to save but may use leverage (so that $w^* \leq 0$) when he has access to the risk-free asset.

Appendix F: Stationary Distributions of x in Jump-Diffusion Models

In this appendix, we characterize the stationary distribution of x and the duration of an investment episode. As there are no closed-form solutions,

[46] In the holding region, the coefficients of the differential equation are constant. The classic regularity theory for elliptic equations implies that the solution is infinitely smooth in the holding region (Evans (2010)).

we use the finite difference method to numerically solve the corresponding Kolmogorov forward and backward equations.

Given the trading policy, characterized by the optimal w^* and the four regions $\mathcal{H}_d = (0, \underline{x}^{**})$, $\mathcal{R}_- = [\underline{x}^{**}, \underline{x}^*]$, $\mathcal{H}_n = (\underline{x}^*, \overline{x}^*)$, and $\mathcal{R}_+ = [\overline{x}^*, \infty)$, where $0 \le \underline{x}^{**} \le \underline{x}^* < 1 < \overline{x}^*$, the density function $\varphi(\cdot)$ for the stationary distribution of x for our jump-diffusion model satisfies the Kolmogorov forward equation

$$\mathcal{K}^*\varphi(x) = 0, \tag{F.1}$$

in the regions where $x \in \mathcal{H}\backslash\{1\} = \mathcal{H}_d \cup \mathcal{H}_n\backslash\{1\}$ and $\mathcal{K}^*$ is the operator defined by

$$\mathcal{K}^*\varphi(x) = \frac{d^2}{dx^2}\left(\frac{1}{2}\sigma^2 x^2 \varphi(x)\right) - \frac{d}{dx}\big((\mu - r)x\,\varphi(x)\big) + \rho\left(\mathbb{E}\left[\frac{\varphi(x/Y)}{Y}\right] - \varphi(x)\right). \tag{F.2}$$

Note that (F.1) holds for the two holding regions excluding the $x = 1$ point.[47] The last term in (F.2) captures the effect of jumps on φ. We next generalize the duration analysis for diffusion models in IJ (2013) to allow for jumps.

Duration of Investment Episodes: Let τ denote the calendar time that the investor realizes the next trading gain or loss. Let $D(x_t)$ denote the expectation of $(\tau - t)$ conditional on the value of x_t at current time t, that is, $D(x) = \mathbb{E}_t[(\tau - t)|x_t = x]$. The following result holds in the two holding regions where $x \in \mathcal{H} = \mathcal{H}_d \cup \mathcal{H}_n$:

$$\mathcal{K}D(x) = -1, \tag{F.3}$$

where $\mathcal{K}$ is the operator, which is adjoint to $\mathcal{K}^*$ defined in (F.2), defined by

$$\mathcal{K}D(x) = \frac{1}{2}\sigma^2 x^2 D_{xx}(x) + (\mu - r)xD_x(x) + \rho\,\big(\mathbb{E}[D(Yx)] - D(x)\big). \tag{F.4}$$

By definition, $D(x) = 0$ if the investor immediately realizes gains or losses: $x \in \mathcal{R}_- \cup \mathcal{R}_+$.

Let Φ_G denote the fraction of time that the stock has unrealized gains (on paper). Using the stationary distribution $\varphi(x)$, we obtain $\Phi_G = \int_1^{\overline{x}^*} \varphi(x)dx$. The fraction of time that the asset has unrealized losses is therefore given by $\Phi_L = 1 - \Phi_G$. The stock has paper losses either in the deep-loss holding region $\mathcal{H}_d$ or when $\underline{x}_1^* < x < 1$ in the normal holding region $\mathcal{H}_d$. Let $\Phi_{\mathcal{H}_d}$ denote the fraction of time that the investor is in the deep-loss holding region $\mathcal{H}_d$: $\Phi_{\mathcal{H}_d} = \int_0^{\underline{x}^{**}} \varphi(x)dx$. Then, $\Phi_L - \Phi_{\mathcal{H}_d}$ is the fraction of time that the investor incurs losses in the normal holding region $\mathcal{H}_n$ with $x \in (\underline{x}^*, 1)$.

[47] The $x = 1$ point in $\mathcal{H}_n$ does not satisfy (F.1) as $\varphi(x)$ is not differentiable there. But $\varphi(x)$ is continuous at $x = 1$: $\lim_{x\to 1^-} \varphi(x) = \lim_{x\to 1^+} \varphi(x)$. Other conditions for $\varphi(x)$ are: (i) the fraction of time spent in the gain- and loss-realization regions is zero: $\varphi(x) = 0$ for $x \in \mathcal{R}_- \cup \mathcal{R}_+$, as the investor immediately resets to $x = 1 \in \mathcal{H}_n$, and (ii) the density function $\varphi(x)$ integrates to one: $\int_0^{\overline{x}^*} \varphi(x)dx = 1$.

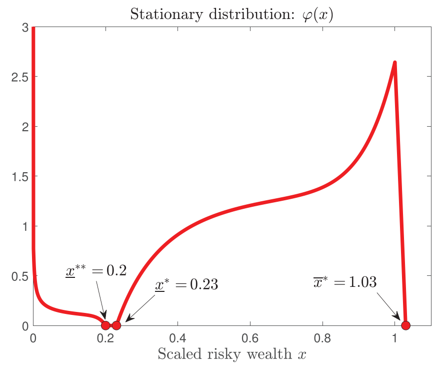

Figure F.1. Stationary density function $\varphi(x)$ for the jump-diffusion case. The long-run probability for $x \in \mathcal{H}_d$ is 3% (the area under the left part of $\varphi(x)$). Parameter values: $\mu = 19\%$, $\sigma = 20\%$, $\rho = 0.73$, $\psi = 6.3$, $r = 3\%$, $\theta_p = \theta_s = 1\%$, $\alpha_+ = 0.5$, $\alpha_- = 0.5$, $\lambda = 2.5$, $\beta = 0.3$, and $\delta = 5\%$. (Color figure can be viewed at wileyonlinelibrary.com)

In Figure F.1, we plot the stationary density function $\varphi(x)$ for x_t for parameter values $\mu = 19\%$, $\sigma = 20\%$, $\rho = 0.73$, $\psi = 6.3$, $r = 3\%$, $\theta_p = \theta_s = 1\%$, $\alpha_+ = 0.5$, $\alpha_- = 0.5$, $\lambda = 2.5$, $\beta = 0.3$, and $\delta = 5\%$, in which case the optimal $w^* = 0$, $\overline{x}^* = 1.03$, $\underline{x}^* = 0.23$, and $\underline{x}^{**} = 0.2$ are such that $\mathcal{H}_d = (0, 0.2)$, $\mathcal{R}_- = [0.2, 0.23]$, $\mathcal{H}_n = (0.23, 1.03)$, and $\mathcal{R}_+ = [1.03, \infty)$.

The area under the left part of the stationary density function $\varphi(x)$ equals $\Phi_{\mathcal{H}_d} = 3\%$, which means that in the long run the investor spends about 3% of his time in the deep-loss holding region $\mathcal{H}_d$. The density function $\varphi(x)$ has a single peak at $x = 0$ in this deep-loss region.

The area under the right part of the density function $\varphi(x)$ equals $1 - \Phi_{\mathcal{H}_d} = 97\%$, which means that in the long run the investor spends about 3% of his time in the deep-loss $\mathcal{H}_d$ region and the remaining 97% of his time in the normal holding region $\mathcal{H}_n$. Of the 97% of time spent in $\mathcal{H}_n$, the investor spends about 4% of his time in the paper gain region $x \in (1, \overline{x}^*) = (1, 1.03)$, as $\Phi_G = 4\%$, and the other 93% of his time in the normal paper loss region $x \in (\underline{x}^*, 1) = (0.23, 1)$. In the normal holding region $\mathcal{H}_n$, $\varphi(x)$ is single-peaked at $x = 1$. This is because the investor can only enter into $\mathcal{H}_n$ from either $\mathcal{R}_+$ (after realizing gains) or $\mathcal{R}_-$ (after realizing losses). For this reason, $\varphi(x)$ is not differentiable at $x = 1$. The expected duration for an investment episode is about 183 days for our baseline jump-diffusion model: $D(1) = 0.5$.

Finally, we compare the two peaks for the density function $\varphi(x)$: one at $x = 0$ and the other at $x = 1$. For the peak at $x = 0$, the density $\varphi(x)$ goes to ∞ because $x = 0$ is an absorbing state. For the peak at $x = 1$, the density $\varphi(x)$ does not go to ∞. Recall that $x = 1$ is the beginning of each investment episode, in which the investor sells the stock that he owns, realizes a gain or loss, and then resets his reference level. The differences between the densities at these two peaks, $\varphi(0)$ and $\varphi(1)$, can be seen in Figure F.1.

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Supporting Information

Additional Supporting Information may be found in the online version of this article at the publisher's website:

Appendix S1: Internet Appendix.
Replication Code.