Reallocating and Pricing Illiquid Capital: Two productive trees*

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Abstract

We develop a tractable two-sector equilibrium model with capital accumulation and adjustment costs. We study capital reallocation decisions and asset pricing. With two sectors, the consumer balances diversification benefits against reallocation costs and efficiency losses in production. The distribution of capital between the two sectors determines the risk-free rate, risk premium, investment, and Tobin's q at both sectoral and aggregate levels. Our framework highlights the importance of sectoral heterogeneity and capital liquidity for economic growth and asset pricing. An unbalanced distribution of capital increases risk and reduces welfare, but making sectors more balanced through capital reallocation reduces efficiency and growth.

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1 Introduction

Economic fluctuations often begin in a distinct sector, and then propagate throughout the economy. Yet equilibrium models in macro and finance typically assume a representative agent and/or a representative firm, emphasizing the equilibrium response to aggregate shocks and de-emphasizing distribution and propagation. Existing models that incorporate multiple sectors assume either that capital is perfectly liquid and can be reallocated frictionlessly, as in Cox, Ingersoll, and Ross (1985), hereafter CIR, or that capital is completely illiquid and fixed, as in Lucas (1978) and Breeden (1979), and the two-sector version by Cochrane, Longstaff, and Santa Clara (2008), hereafter CLS.[1] When capital is perfectly liquid as in CIR, Tobin's q is one at all times, heterogeneity plays no role and perfect aggregation holds in equilibrium.[2] When capital is completely illiquid as in CLS, investment is zero at all times and prices have to adjust to sustain the no-trade equilibrium. CIR and CLS thus correspond to the two extreme cases of capital mobility.

However, in reality, we live in a production economy and capital reallocation while feasible, is often costly. Motivated by this fundamental observation, we develop a parsimonious and analytically tractable model to capture the impact of capital illiquidity on equilibrium resource allocation between consumption and production, and between different productive sectors, as well as intertemporal equilibrium asset pricing. Building on the neoclassical q theory of investment,[3] we use convex adjustment costs to model capital illiquidity. We show that the distribution of capital effectively is the single state variable determining equilibrium capital reallocation and asset pricing. For example, the distribution of capital between the two sectors determines investment, the value of capital, and the risk premium both at the sectoral and aggregate levels, as well as the equilibrium risk-free rate and aggregate quantities. Unlike in CIR and CLS, both prices and quantities adjust as the economy optimally accumulates and reallocates capital in response to shocks in our model. We first develop a

[1] Martin (2009) extends CLS to multi-tree, i.e. a Lucas orchard setting and obtains analytic asset pricing results. Santos and Veronesi (2006) is a pure exchange economy with two different sources of income, dividends and wages. They show the ratio of labor income to consumption predicts stock returns.

[2] See Jones and Manuelli (2005) for essentially the same modeling in endogenous growth context.

[3] Brainard and Tobin (1968) and Tobin (1969) define the ratio between the firm's market value to the replacement cost of its capital stock, as Q and propose to use this ratio to measure the firm's incentive to invest in capital. This ratio has become known as Tobin's average Q. Hayashi (1982) provides conditions under which average Q is equal to marginal q. Abel and Eberly (1994) develop a unified q theory of investment in neoclassic settings. Lucas and Prescott (1971) and Abel (1983) are important early contributors.

baseline case, with log utility and two *ex ante* identical sectors, to establish analytic findings in a simple benchmark case. Then, we extend the model to a more general non-expected Epstein-Zin utility framework[4] and allow for *ex ante* asymmetry between the two sectors.

When the two sectors are *ex ante* identical, the economy tends toward a balanced capital distribution, where the two sectors are of equal size and the consumer achieves maximum diversification and utility. With two symmetric sectors, the interest rate is higher than with one sector alone, since the consumer must be induced to save despite being well-diversified. At the sectoral level, there is substantial impact of the capital distribution on value, investment, and asset returns. In particular, as one sector becomes small, its investment rate skyrockets - not because its marginal returns increase (we have constant-returns-to-scale), but rather because its cost of capital falls. This occurs because the small sector becomes virtually risk-free (when the shocks to two sectors are uncorrelated), and retaining the small sector and its potential for diversification becomes extremely valuable.[5] We can therefore gauge the importance of general equilibrium on prices and quantities, by scaling from a negligibly small sector that approximates "partial equilibrium" to a single dominating sector that corresponds to standard one sector general equilibrium analyses.

However, the benefits of diversification are counterbalanced by the costs of reallocation. A high cost of reallocating capital acts as a tax on savings, so high adjustment costs deter savings, instead promoting consumption, and dampening growth. Asset prices rise with adjustment costs because the rents to installed capital are higher, but the rates of return to capital investment are lower. These results give a hint of our findings when liquidity varies endogenously, expanding upon these comparative static results.

When we extend beyond log utility or allow for asymmetry, the effect of the capital distribution becomes more pronounced - even for aggregate variables. With higher risk aversion, when the household is less diversified because of an unbalanced distribution of capital, the household responds by adjusting the consumption-savings decision: saving more and consuming less. This raises investment and growth, but also results in higher risk

[4]We choose the agent's utility function to be the one proposed by Epstein and Zin (1989) and Weil (1990) in discrete time, but use the continuous-time formulation developed by Duffie and Epstein (1992a). This utility specification allows us to separate the coefficient of relative risk aversion and elasticity of intertemporal substitution, and still generate a stochastic balanced growth path, which is desirable.

[5]Note that the equilibrium interest rate and aggregate risk premium can be computed using the one-sector economy, where the sector is the dominating one.

premia and expected returns, so asset prices fall. Hence, the distribution of capital affects both economic growth and asset prices.

Similarly, when the sectors are not *ex ante* identical, and the two sectors have different adjustment costs (so that one is more liquid than the other), shocks to the distribution of capital affect the overall liquidity of the economy. For example, when the economy has too much illiquid capital (one might think of too much housing capital), aggregate volatility and the aggregate risk premium are high, while aggregate investment and growth both decline. The interest rate falls because the consumer has a greater incentive to save in the illiquid economy, and saving goes to rebuild the liquid sector, which has a high value (and higher Tobin's q). This unpleasant combination of real outcomes and risk pricing continues until the economy can rebalance by saving and investing in liquid capital. Interestingly, these endogenous changes in liquidity also cause aggregate investment and Tobin's q to move in opposite directions.

Our findings convey both the importance of heterogeneity and also caution about the lessons of modeling heterogeniety. Initially, with symmetry and log utility, we establish a benchmark where the two sectors show interesting internal dynamics but have quantitatively small effects on the aggregate. However, when we depart from log utility or allow for asymmetry, the aggregate effects are substantial. First, even in the symmetric case, higher risk aversion (we consider risk-aversion as high as four), the endogenous response of savings to an undiversified economy causes investment and growth to depend on the distribution of capital. Moreover, this generates a higher value of capital that is reflected in higher asset prices and lower rates of return. With *ex ante* asymmetry between the two sectors, changes in the distribution of capital determine the overall liquidity in the economy. These endogenous compositional changes in liquidity drive investment and growth, as well as asset prices and rates of return.

Related Literature. As we have noted, our paper is most closely related to CIR, CLS and the neoclassical q theory of investment. Additionally, our paper is also linked to investment-based general equilibrium models. Jermann (1998) develops a one-sector DSGE model with habit formation preferences and capital adjustment costs to generate historical risk premium, risk-free rate, and aggregate quantities. Tallarini (2000) introduces the Epstein-Zin utility (with arbitrary risk aversion but unit EIS) into a one-sector DSGE model and finds that risk

aversion has little impact on business cycle properties, and mostly influences the asset pricing and welfare costs (of business cycle) calculations. He does not model capital adjustment costs and hence $q = 1$ in his model at all times. Boldrin, Christiano, and Fisher (2001) introduce habit formation in a two-sector DSGE model to generate price and quantity dynamics. Unlike these papers, we use Epstein-Zin utility to separate risk aversion from the EIS and use the capital adjustment costs to capture illiquidity. Finally, we solve the model analytically, while all the above referenced papers solve the model numerically.

Although our paper does not directly address issues in international finance, the two sectors in our model can be potentially interpreted as foreign and domestic economies. With that interpretation, our paper can also be interpreted as one for international finance. Obstfeld (1994) uses Epstein-Zin utility and shows that reallocation of capital achieves perfect diversification in the absence of adjustment costs in the international context. In terms of the model construction, ours features capital illiquidity and nests Obstfeld (1994) as a special case. Dumas (1992) studies a two-country general equilibrium economy with linear/proportional shipping costs between countries and derives analytic solutions featuring trade and no-trade regions.[6]

Our paper also contributes to the literature on capital reallocation. Ramey and Shapiro (1998) examine the effect of fiscal shocks in a two-sector general equilibrium model with costly reallocation of capital and find significant effects of the capital frictions. Eisfeldt and Rampini (2006) emphasize the costs of reallocation, showing in a quantiative setting that these costs must be time-varying and countercyclical in order to generate the observed pattern of capital reallocation. In our model, the cost of reallocation varies endogenously, generating large shifts in sectoral investment (e.g., as described above for a small sector) and reallocation of capital.

There has been surging interest in linking corporate investments and q to cross-sectional returns. See Cochrane (1991, 1996), Berk, Green, and Naik (1999), and Zhang (2003), for example. These models focus on firm's optimal investment and price securities by using an exogenously specified stochastic discount factor. For a general equilibrium model with cross-sectional stock return predictions, see Gomes, Kogan, and Zhang (2003).

[6]Technically, our model is an example of analytically tractable general equilibrium models featuring heterogeneity. For an example of heterogenous investors with different risk aversion, see Dumas (1989).

2 Baseline Model

Consider an infinite-horizon continuous-time production economy. There are two productive sectors in the economy, sectors 0 and 1. We introduce the model in this section with the case where the two sectors are symmetric. Hence, functional forms and parameter values are not sector specific, but random variables are. We allow for sector asymmetry in the general formulation of Section 5. Let K_n, I_n, and Y_n denote the representative firm's capital stock, investment and output, respectively, in sector n, where $n = 0, 1$. The representative firm in each sector has an "AK" production technology:

$$Y_n(t) = AK_n(t)\,, \tag{1}$$

where $A > 0$ is a constant. Capital accumulation is given by

$$dK_n(t) = \Phi(I_n(t), K_n(t))dt + \sigma K_n(t)dB_n(t)\,, \tag{2}$$

where $\sigma > 0$ is the volatility parameter, and the function $\Phi(I_n, K_n)$ measures the effectiveness of converting investment goods into installed capital. In this section, we assume the correlation between the Brownian motions B_0 and B_1 is zero. Shocks appear in the capital accumulation dynamics (2) as in CIR and the endogenous growth models in macroeconomics (e.g., see the handbook chapter by Jones and Manuelli (2005)). Similarly, Dumas (1992) considers shocks to allocations in a two-country model with linear adjustment (shipping) costs. As in Hayashi (1982) and Jermann (1998), we assume that the adjustment technology in each sector is homogeneous of degree one in I and K, so we can write the installation function as follows:

$$\Phi(I_n, K_n) = \phi(i_n)K_n, \tag{3}$$

where $i_n \equiv I_n/K_n$ is the investment-capital ratio in sector n. We require $\phi'(\,\cdot\,) > 0$ and $\phi''(\,\cdot\,) \leq 0$. Our model nests frictionless "AK" models (such as CIR and Jones and Manuelli (2005)) as special cases.[7] In an earlier paper, Eberly and Wang (2009), we use this specification in a deterministic model to examine the effects of capital reallocation on growth.

A representative consumer has a logarithmic utility given by:

$$\mathbb{E}\left(\int_0^\infty e^{-\alpha t}\alpha \ln C(s)\, ds\right), \tag{4}$$

[7]Kogan (2004) considers a two sector model in which investment in one sector is irreversible (bounded below by zero) and also bounded above. This is a special case of convex adjustment costs, applied to one sector.

where $\alpha > 0$ is the subjective discount rate. We consider the more general recursive utility formulation in Section 5. The consumer is endowed with financial claims on the aggregate output from both sectors in the economy. Markets are complete.

Now consider the market equilibrium. The representative consumer chooses his consumption and a complete set of financial claims to maximize (4). The representative firm in each sector takes the equilibrium stochastic discount factor as given and maximizes firm value. All produced goods are either consumed or invested in one or the other of the two sectors, so the goods-market clearing condition holds:

$$C = Y_0 + Y_1 - I_0 - I_1. \tag{5}$$

In equilibrium, the representative consumer holds his financial claims on aggregate output in both sectors. Using the standard results in complete-markets competitive equilibrium analysis, we obtain the equilibrium allocation by solving a central planner's problem and then decentralize the allocation using the price system. Details are included in the appendix.

3 Model Solution

We first summarize the model solution for the one-sector economy. Then, we solve the allocation in the two-sector economy using the one-sector solution as the natural boundary condition.

3.1 The one-sector economy

The one-sector economy serves as a benchmark and also is the solution to the model in the extreme case where all capital is invested in one sector. In this case, the sectoral capital stock is the aggregate capital K, which is the single state variable in this economy. The equilibrium of the one-sector economy features stochastic growth, where the stochastic growth rates of consumption, investment, and capital, and output are all equal. Moreover, these growth rates are independently and identically distributed. Therefore, after scaling by capital, the consumption-capital ratio $c = C/K$, investment-capital ratio $i = I/K$, and Tobin's q are all constant. For logarithmic utility, the first-order condition (FOC) with respect to consumption gives $c = \alpha q$, where q is the firm value-capital ratio, also referred to as average q

or Tobin's q. The FOC with respect to investment directly links Tobin's q to the investment-capital ratio i: $q = 1/\phi'(i)$. The FOCs for consumption and investment together with goods market clearing condition (investment equals saving, i.e. $A - c = i$) jointly determine the optimal investment-capital ratio as the solution to the equation $(A - i)\phi'(i) = \alpha$. The equilibrium interest rate r is given by $r = \alpha + \phi(i) - \sigma^2$, the sum of the subjective discount rate α and the expected growth rate $\phi(i)$, minus the standard precautionary saving term for logarithmic utility. The expected return of a financial claim on aggregate output is $\mu^m = \alpha + \phi(i)$ implying that the aggregate risk premium is equal to σ^2. The CIR model is a special case with $q = 1$ because capital is perfectly liquid (no adjustment cost, i.e. $\phi'(i) = 1$).

In the appendix, we show that the representative consumer's value function is $J(K) = \ln(pK)$, where

$$p = (A - i) \exp\left[\frac{1}{\alpha}\left(\phi(i) - \frac{\sigma^2}{2}\right)\right], \tag{6}$$

and where i solves $(A - i)\phi'(i) = \alpha$. See the appendix for details. The constant value p will help to determine the boundary conditions for a one-sector economy.

3.2 The two-sector economy

With two sectors, the natural state variables in the model are the capital stocks in the two sectors. By exploiting the homogeneity properties of the model, the effective state variable is the *relative* size of capital stocks in the two sectors, defined by

$$z \equiv \frac{K_1}{K_0 + K_1}, \tag{7}$$

the ratio between sector-1 capital K_1 and the aggregate capital $(K_0 + K_1)$. Since physical capital is non-negative, we have $0 \leq z \leq 1$.

Let i denote the aggregate investment capital ratio: the ratio between aggregate investment $(I_0 + I_1)$ and aggregate capital $(K_0 + K_1)$, so that $i \equiv (I_0 + I_1)/(K_0 + K_1)$. Using the definitions of z and sectoral i_n, we have

$$i(z) = (1 - z)\, i_0(z) + z i_1(z). \tag{8}$$

Scaling the goods-market equilibrium market condition in equation (5) implies

$$c(z) + (1 - z) i_0(z) + z i_1(z) = A_0(1 - z) + A_1 z. \tag{9}$$

3.2.1 Investment and endogenous growth

Adjustment costs drive a wedge between gross investment I and expected change in the capital stock in the economy $\Phi(I, K)$. The function $\Phi(I_n, K_n)$, which controls the effectiveness of converting investment goods into installed capital, allows for both depreciation, so that there is a difference between gross and net investment, and also investment adjustment costs so that investment goods are used up in the installation process. The expected growth rate $\phi(i)$ of capital nets out both depreciation and installation costs, so that the growth in the capital stock is less than both gross investment i and the traditional notion of net investment.

Let $g_n(z)$ denote the expected growth rate of capital in sector n. Using (2), we have $g_n(z) = \phi(i_n(z))$, which differs from sectoral gross investment $i_n(z)$. Let $g(z)$ denote the expected growth rate of aggregate capital $(K_0 + K_1)$. We thus have

$$g(z) = (1 - z)\, g_0(z) + z g_1(z). \tag{10}$$

The concavity of $\phi(i)$ implies $g(z) \leq \phi(i(z))$. When $z = 0, 1$, the equality holds. Intuitively, *ceteris paribus,* the expected growth rate $g(z)$ is lower in a two-sector economy than the corresponding one-sector economy, since both sectors incur convex adjustment costs.

3.2.2 Endogenous capital reallocation

The equilibrium dynamics of z are given by:

$$dz_t = \mu_z(z_t)dt + \sigma z_t(1 - z_t)\left(dB_1(t) - dB_0(t)\right), \tag{11}$$

where the drift $\mu_z(z)$ is given by

$$\mu_z(z) = z(1 - z)\left[g_1(z) - g_0(z) + (1 - 2z)\sigma^2\right]. \tag{12}$$

Note that the volatility of dz is a quadratic function in z which attains its highest value at $z = 1/2$ and becomes zero at $z = 0, 1$ (i.e. the one-sector economy is absorbing), as in the two-tree pure exchange model of CLS. More interestingly in our model, the drift μ_z depends on $g_1(z) - g_0(z)$, the difference between the *endogenous* capital growth rates in the two sectors. The larger this difference, the more capital reallocation occurs in equilibrium. This component of growth, induced by the wedge between sectoral growth rates, fundamentally

differentiates the results in our model from CLS. The sectoral growth rates will endogenously differ between the two sectors because of the "imbalance" between the two capital stocks (i.e. $z \neq 1/2$) even when the two sectors have the same technology.

3.2.3 Investment and Tobin's q

We now turn to investment and the valuation of capital. The FOC for $i_1(z)$ is given by:

$$\frac{\alpha}{\phi'(i_1(z))} = c(z)\left(1 + (1-z)\frac{N'(z)}{N(z)}\right), \tag{13}$$

where $N(z)$, the log of the value function (per unit of aggregate capital), is given in the appendix. A similar first-order condition holds for $i_0(z)$ and is also in the appendix. Let $V_n(K_n; z)$ denote firm value in sector n. Using the homogeneity property, we have

$$V_n(K_n; z) = q_n(z)K_n, \quad n = 0, 1, \tag{14}$$

where Tobin's q in sector n is given by

$$q_n(z) = \frac{1}{\phi'(i_n(z))}\,. \tag{15}$$

Intuitively, the capital stock increases by $\phi'(i_n)$ per marginal unit of investment. Each unit of capital is valued at $q_n(z)$ Therefore, the firm optimally chooses i_n to equate $\phi'(i_n(z))q_n(z)$ to unity, the cost of investment.

The market value of aggregate capital is $V(z) = V_0(z) + V_1(z) = q(z)(K_0 + K_1)$, where Tobin's q for the aggregate capital stock is given by

$$q(z) = (1-z)q_0(z) + zq_1(z)\,. \tag{16}$$

3.2.4 Consumption and dividend yield

With complete markets and log utility, the aggregate consumption-wealth ratio $C(z)/V(z)$ is equal to the discount rate α, or equivalently $c(z) = \alpha q(z)$, as we noted in the one-sector setting. While the aggregate dividend yield (i.e. consumption/wealth ratio) is constant and equal to the discount rate α, the sectoral dividend yield dy_n is stochastic and is given by

$$dy_n(z) = \frac{A - i_n(z)}{q_n(z)}, \quad n = 0, 1. \tag{17}$$

The above formula implies that sectoral dividend yield can be negative if $i_n(z) > A$. Unlike a one-sector model, the sectoral dividend yield can be negative when the consumer's investment incentive is high for that sector. We show that this negative dividend yield may indeed occur when that sector is small and provides large diversification benefits.

4 An Exercise with Log Utility and Quadratic Adjustment Costs

Up to now, none of our results depend on the particular functional form of adjustment costs, $\phi(i)$. To further illustrate the properties of the model, we now specify a quadratic adjustment cost function as follows:

$$\phi(i) = i - \frac{\theta}{2} i^2 - \delta \,, \tag{18}$$

where $\theta \geq 0$ is the adjustment cost parameter. When $\theta = 0$, the expected growth rate of capital is $\phi(i) = i - \delta$. We may naturally interpret δ as the expected rate of depreciation in the special context without adjustment costs. By specifying the functional form, we first obtain additional analytic solutions, and then solve the model numerically.

In the appendix, we show that the equilibrium consumption-capital ratio $c(z)$ is given by

$$c^*(z) = \frac{1}{2}\left(A - \frac{1}{\theta}\right) + \frac{1}{2}\sqrt{\left(A - \frac{1}{\theta}\right)^2 + 4\frac{\alpha}{\theta}\left[\frac{(1-z)^2}{L_0(z)} + \frac{z^2}{L_1(z)}\right]} \tag{19}$$

where the functions $L_0(z)$ and $L_1(z)$ are defined as:

$$L_0(z) = (1-z)\left[1 - z\frac{N'(z)}{N(z)}\right] , \tag{20}$$

$$L_1(z) = z\left[1 + (1-z)\frac{N'(z)}{N(z)}\right] . \tag{21}$$

Note that $L_0(z) + L_1(z) = 1$. The investment-capital ratios $i_0(z)$ and $i_0(z)$ are given by

$$i_0^*(z) = \frac{1}{\theta}\left[1 - \frac{\alpha}{c^*(z)}\left(1 - z\frac{N'(z)}{N(z)}\right)^{-1}\right] , \tag{22}$$

$$i_1^*(z) = \frac{1}{\theta}\left[1 - \frac{\alpha}{c^*(z)}\left(1 + (1-z)\frac{N'(z)}{N(z)}\right)^{-1}\right] . \tag{23}$$

Note that $c(z)$, $i_0(z)$, and $i_1(z)$ are all explicit functions of $N(z)$ and its derivatives. Using these explicit expressions for decision rules, we obtain the following ODE for $N(z)$:

$$0 = \alpha \ln\left(\frac{c^*(z)}{N(z)}\right) + \phi(i_0^*(z))L_0(z) + \phi(i_1^*(z))L_1(z) - \frac{\gamma\sigma^2}{2}\left[L_0(z)^2 + L_1(z)^2\right] + \sigma^2 M(z) \,, \tag{24}$$

where $L_0(z)$ and $L_1(z)$ are given in (20) and (21), respectively, and

$$M(z) = \frac{z^2(1-z)^2 N''(z)}{N(z)}\,. \tag{25}$$

We can obtain $N(z)$ by solving the ODE (24) with the boundary conditions

$$N(0) = N(1) = p = (A - i)\exp\left[\frac{1}{\alpha}\left(\phi(i) - \frac{\sigma^2}{2}\right)\right], \tag{26}$$

where the constant value p is given in (6) and the optimal one-sector $i = I/K$ is given by

$$i = \frac{1}{2}\left[A + \frac{1}{\theta} - \sqrt{\left(A - \frac{1}{\theta}\right)^2 + \frac{4\alpha}{\theta}}\right]. \tag{27}$$

Our production model features endogenous growth and has direct implications for the levels of the interest rate $r(z)$ and the the expected return on the market portfolio $\mu^m(z)$, but not the systematic risk $\sigma_m^2(z)$ and the risk premium $rp^m(z)$.

The sectoral distribution of capital, z, is the key state variable. Adjustment costs makes z potentially slow moving as seen from (12). Moreover, the persistence of z induced by adjustment costs expose the consumer to more sector-specific risk, *ceteris paribus.* The only case in which prices are unaffected by the allocation of capital is at the absorbing boundaries when $z = 0, 1$, or when the adjustment costs are arbitrarily small.

4.1 Sectoral distribution of capital z and the value function

Recall that the production and adjustment technologies in the two sectors are identical. Despite the identical technologies, the two sectors price investment differently and carry different risk premia because of differences in their capital stocks, even with constant-returns-to-scale production.

We choose model parameters to generate sensible aggregate predictions and to highlight the impact of endogenous investment and growth on equilibrium pricing and capital reallocation. The annual subjective discount rate is $\alpha = 0.04$. Annual volatility is $\sigma = 0.10$ and the annual productivity parameter is $A = 0.10$. Finally, we choose the adjustment cost parameter $\theta = 10$ and $\delta = 0$. The correlation coefficient $\rho = 0$. In the one-sector economy (i.e. $z = 0, 1$), we have $i = 0.0368$, $q = 1.58$, and $g = \phi(i) = 0.03$. The equilibrium one-sector interest rate is $r = 0.06\%$ which tends to be high due to high precautionary saving.

The equilibrium risk premium is 1% because of low risk aversion ($\gamma = 1$) and low volatility ($\sigma = 10\%$).

Figure 1 has six panels, arranged in 3 rows and 2 columns. The upper left panel of Figure 1 shows $N(z)$, the logarithm of the representative consumer's value function per unit of aggregate capital ($K_0 + K_1$), as a function of $z = K_1/(K_0 + K_1)$. Intuitively, we expect that $N(z)$ is maximized at $z = 1/2$, where the consumer achieves the maximally attainable level of diversification between the two sectors.

The upper right panel of Figure 1 plots the drift of z, $\mu_z(z)$ given in equation (12). There is a natural tendency for z to move towards the center (i.e. when $z < 1/2$, $\mu_z(z) > 0$ and hence on average z increases towards 1/2.) This mean reversion effect of $\mu_z(z)$ in z is also present in CLS due to the definition of z. Unlike CLS, however, the "central tendency" is stronger in our production economy due to endogenous investment and growth. Controlling for size (that is, per unit of capital), the consumer has greater demand for the smaller sector, and hence invests more per unit of capital, *ceteris paribus*. For example, when $0 < z < 1/2$, sector 1 is the smaller one, so the firm invests and grows at a faster rate, and $i_1(z) > i_0(z)$ and $g_1(z) > g_0(z)$. The flexibility to adjust capital growth enhances the "central tendency" of $\mu_z(z)$ due to endogenous growth. In contrast, in the CIR model with no adjustment costs and hence unit marginal q, the economy frictionlessly responds to shocks and shifts capital between two sectors, always maintaining half of its capital stock in each sector.

4.2 Aggregate implications on quantities and prices

In the mid left and right panels of Figure 1, we plot the aggregate investment capital ratio $i(z)$ and the aggregate Tobin's q as functions of z. Note that neither $i(z)$ nor $q(z)$ are monotonic in z.

The symmetry between the two sectors allows us to focus on the region $0 \leq z \leq 1/2$. First, aggregate investment $i(z)$ decreases with z due to the adjustment cost. After reaching the lowest value at around $z = 0.10$, investment $i(z)$ starts to increase with z until it peaks at $z = 1/2$. Intuitively, when z is close to zero (i.e. sector 1 is effectively the only one), diversification has little value added, but the adjustment costs of having two sectors may be high. Aggregate investment therefore falls. However, for sufficiently high z, the diversification benefits outweigh the costs of adjusting capital stock. As a result, aggregate

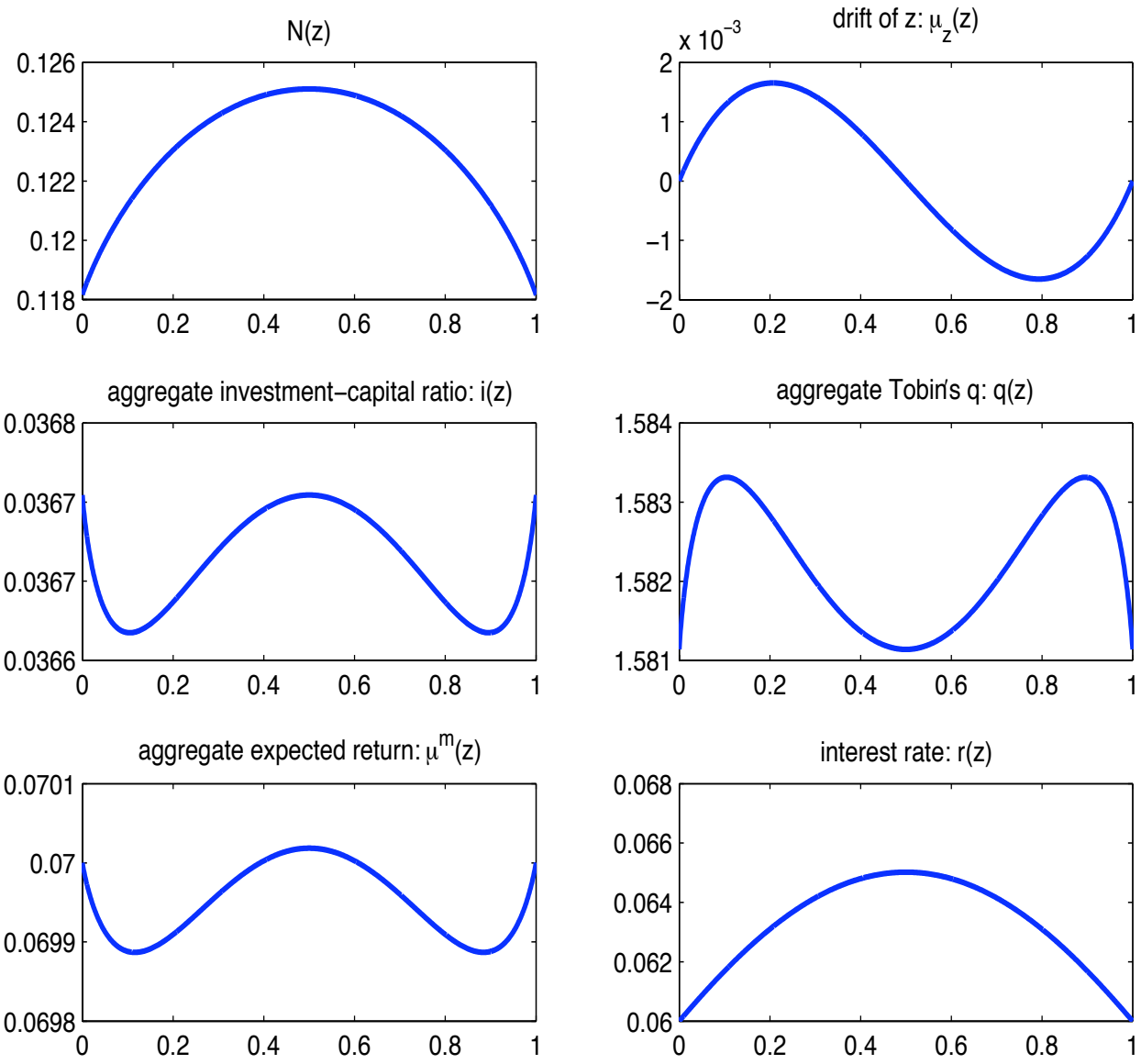

Figure 1: **Aggregate results (log utility).**

investment increases and peaks again at $z = 1/2$.

Perhaps surprisingly, aggregate Tobin's q moves in the *opposite* direction of aggregate investment $i(z)$ due to general equilibrium. With logarithmic utility, consumption is proportional to firm value, i.e. $c(z) = \alpha q(z)$, where α is the agent's subjective discount rate. In equilibrium with symmetry, we have $c(z) + i(z) = A$. Therefore, a unit increase in $i(z)$ implies a unit decrease in $c(z)$ and hence Tobin's q decreases α units. Aggregate consumption $c(z)$ and hence $q(z)$ first increase with z and then falls with z for $z \leq 1/2$. Therefore, our model predicts a negative (or weak) relation between i and q in aggregate data, even though the neoclassic q theory of investment hold perfectly in the model. Heterogeneity and equilibrium aggregation have first-order effects and potentially overturn the conventional wisdom. However, for the illustration, the quantitative effects of sectoral distribution z on aggregate investment are small. This is due to both consumption smoothing[8] and the investment adjustment cost which encourages the firm to smooth its investment. In fact, we show in Eberly and Wang (2009) in the deterministic case, and extend to uncertainty in the appendix, that aggregate values are immune to the sectoral distribution of capital in the case with log utility and log capital installation costs. Deviations from these assumptions exhibit larger distributional effects, which we explore when we depart from the log utility case in Section 5 and subsequent analysis.

Next, we turn to the asset pricing implications. In the bottom left and right panels of Figure 1, we plot the expected return of the market portfolio (aggregate wealth), $\mu^m(z)$, and the equilibrium interest rate $r(z)$.

The expected return on the market portfolio, $\mu^m(z)$, closely tracks the expected aggregate investment $i(z)$ and aggregate growth rate $g(z)$. This is consistent with the standard asset pricing result that growth increases the expected rate of return on the risky asset. Its shape again resembles a "W" as a function of z.

The risk-free rate $r(z)$ depends on both the expected growth rate of aggregate consumption and the volatility of aggregate consumption growth. Quantitatively, the precautionary saving motive, measured by the variance of the market portfolio $\sigma_m^2(z)$, varies much more with z than does aggregate growth. Hence, the precautionary motive determines the dependence of the equilibrium interest rate on z. Note that $r(z)$ reaches its maximum at $z = 1/2$,

[8]For log utility, the wealth effect offsets the substitution effect.

where diversification achieves the highest possible level and precautionary saving demand is lowest. A high interest rate is necessary to encourage saving in equilibrium when the economy is well-diversified and the precautionary saving motive is weak.

4.3 Sectoral implications

The next set of figures shows sectoral values; in each panel we graph results for sector 1 only for brevity since results are symmetric for sector 0. The top left and right panels of Figure 2 respectively plot the investment-capital ratio $i_1(z)$ and Tobin's q in sector 1.

Sectoral investment and q. Recall that Tobin's q in sector 1 is given by $q_1(z) = 1/\phi'(i_1(z)) = [1 - \theta i_1(z)]^{-1}$. Therefore, Tobin's q is monotonically increasing in its sectoral investment-capital ratio and hence Tobin's q and investment convey essentially the same information for a given adjustment cost function. Note that both investment-capital ratio $i_1(z)$ and Tobin's q, $q_1(z)$, decrease before z reaches 0.80. Sectoral Tobin's q and investment capital ratio i become significantly larger as the sector becomes smaller, because the consumer values the smaller sector more for diversification benefits, *ceteris paribus*. Recall that there are constant-returns-to-scale in production, so this relationship between q and sector size is not due to decreasing marginal returns in production. Rather, the diversification benefits of keeping the small sector "alive" with the potential to grow are very valuable. Upon vanishing, the sector will never be reborn, and the economy (with only the one surviving sector) will be significantly riskier thereafter.

Note that $i_1(z)$ and $q_1(z)$ increase with z for sufficiently high z. The representative agent's consumption smoothing motive in equilibrium requires consumption and hence aggregate investment to be relatively smooth and not too volatile. When sectors are sufficiently imbalanced, the contribution by the dwindling sector to total investment is negligible. Hence, to maintain a sufficient level of aggregate investment for the purpose of consumption smoothing, the investment-capital ratio in the larger sector (i.e. sector 1 when z is high) must rise as its share z increases. This explains the increasing behavior of the investment-capital ratio $i_1(z)$ and Tobin's q in z at the right side of the graph.

Sectoral risk premium and dividend yield. The mid left and right panels of Figure 2 graph the sectoral risk premium $rp_1(z)$ and dividend yield $dy_1(z)$. The risk premium of a

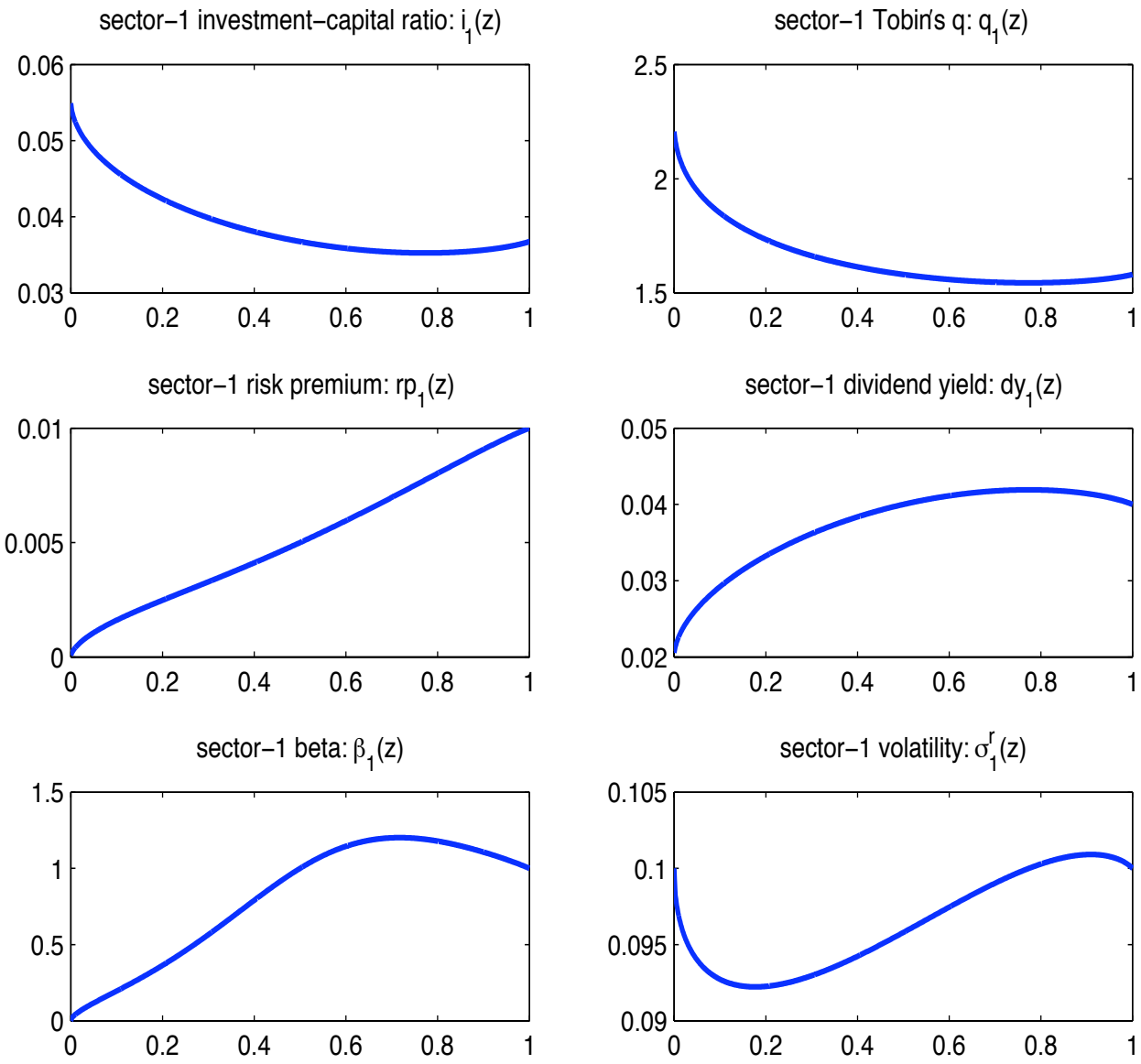

Figure 2: **Sectoral implications (log utility).**

miniscule sector is effectively zero, because this sector carries almost no weight in aggregate consumption, and the correlation ρ between the two shocks is zero. The same intuition applies in the pure-exchange economy (e.g. CLS). Recall that the interest rate is lowest at $z = 0$ and $z = 1$, therefore, the discount rate, the sum of the interest rate and the risk premium, for a sector is lowest when it is vanishing. Intuitively, in equilibrium, the preferences for consumption smoothing and risk diversification lower the risk premium and the discount rate for the shrinking sector. Since the physical production technology remains unchanged, the vanishing sector invests at the highest rate $i_1(0)$ to take advantage of its lowest cost of capital.

To finance this high level of investment around $z = 0$, the firm lowers its dividend yield. The dividend yield for the dwindling sector is positive in our example. However, for other parameter values, the firm may choose to issue equity and hence the dividend yield may be negative. Tobin's q reaches the maximal level $q_1(0)$ at $z = 0$ despite the low dividend yield. Note that the high valuation of capital for the vanishing sector is primarily driven by the discount rate effect induced by diversification benefits. Unlike the aggregate dividend yield, which is equal to the subject discount rate α for log utility, the sectoral dividend yield varies significantly with z.

Sectoral β and volatility. We now turn to sectoral risk measures. In the bottom left and right panels of Figure 2, we plot β and return volatility $\sigma_1^r(z)$ in sector 1. The β for sector 1 is given by

$$\beta_1(z) = \frac{\sigma}{(\sigma^m(z))^2}\left[\sigma_1^m(z) + (\,\sigma_1^m(z) - \sigma_0^m(z))\, z(1-z)\frac{q_1'(z)}{q_1(z)}\right], \tag{28}$$

where $\sigma_0^m(z)$ and $\sigma_1^m(z)$ are given by

$$\sigma_0^m(z) = \frac{\sigma_0(1-z)}{q(z)}\left[q_0(z) - z(1-z)q_0'(z) - z^2 q_1'(z)\right], \tag{29}$$

$$\sigma_1^m(z) = \frac{\sigma_1 z}{q(z)}\left[q_1(z) + (1-z)^2 q_0'(z) + z(1-z)q_1'(z)\right]. \tag{30}$$

The non-monotonic behavior of β can be understood by considering several benchmarks. First, zero risk premium for the disappearing sector implies $\beta_1(0) = 0$. Second, with increases in the share of capital z, more consumption is financed out of sector-1's output and hence $\beta_1(z)$ rises. Third, $\beta(1/2) = 1$, which follows from symmetry between the two sectors and

$\beta = 1$ for the market portfolio by definition. When z increases above 1/2, $\beta_1(z)$ exceeds one, because the other sector becomes smaller and carries smaller β (again by symmetry). Therefore, the bigger sector is riskier, *ceteris paribus*. Finally, when the sector becomes sufficiently large (i.e. low enough z), β has to fall as the sector becomes effectively the market portfolio, which has $\beta = 1$ by definition. Indeed, in the limit, when sector 1 comprises the whole economy ($z = 1$), $\beta_1 = 1$.

Now consider return volatility for sector 1, $\sigma_1^r(z)$. We have

$$\sigma_1^r(z) = \sigma\sqrt{\left(\frac{q_1'(z)}{q_1(z)}z(1-z)\right)^2 + \left(1 + \frac{q_1'(z)}{q_1(z)}z(1-z)\right)^2}. \tag{31}$$

While $\sigma_1^r(z)$ also varies non-monotonically with z, its behavior is rather different from $\beta_1(z)$. At $z = 0$, $\beta_1(z)$ is zero and hence all return volatility comes from the idiosyncratic component because the sector carries no weight in the aggregate. Since total return volatility is the same as capital stock growth volatility σ for the miniscule sector, we have $\sigma_1^r(0) = \sigma = 0.10$. When $z = 1$, sector 1 is the whole economy and hence the aggregate volatility is also $\sigma = 10\%$. It is thus natural to expect a non-monotonic relation between sectoral return volatility $\sigma_1^r(z)$ and sectoral distribution of capital stock z.

5 Equilibrium with Recursive Utility and Asymmetry

We now extend our baseline model to allow for more flexible preferences and sectoral asymmetry. Specifically, we consider the setting where the representative consumer has a homothetic preference featuring both constant relative risk aversion and constant elasticity of intertemporal substitution (Epstein and Zin (1989) and Weil (1990)). We use the continuous-time formulation of this recursive utility introduced by Duffie and Epstein (1992a). That is, the agent's has a recursive preference defined as follows:

$$J_t = \mathbb{E}_t\left[\int_t^\infty f(C_s, J_s)ds\right], \tag{32}$$

where $f(C, J)$ is known as the normalized aggregator for consumption C and the agent's continuation value J. Duffie and Epstein (1992a) show that $f(C, J)$ for Epstein-Zin non-expected (homothetic) utility is given by

$$f(C, J) = \frac{\alpha}{1 - \psi^{-1}}\frac{C^{1-\psi^{-1}} - ((1-\gamma)J)^\omega}{((1-\gamma)J)^{\omega-1}}, \tag{33}$$

where

$$\omega = \frac{1-\psi^{-1}}{1-\gamma}. \tag{34}$$

The parameter $\psi \geq 0$ measures the elasticity of intertemporal substitution, and the parameter $\gamma \geq 0$ is the coefficient of relative risk aversion. The parameter $\alpha > 0$ is the entrepreneur's subjective discount rate. The widely used time-additive separable constant-relative-risk-averse (CRRA) utility is a special case of the above Duffie-Epstein-Zin-Weil recursive utility specification where the coefficient of relative risk aversion γ is equal to the inverse of the elasticity of intertemporal substitution ψ, i.e. $\gamma = \psi^{-1}$ and hence $\omega = 1$.

For the general recursive utility given in equations (32) and (33), the scale-invariance property proves useful in keeping our model analysis tractable (see Duffie and Epstein (1992b) for example). Using this preference, we can quantify both the effect of intertemporal substitution and that of risk aversion on equilibrium resource allocation and asset pricing.

5.1 One-sector Economy

The one-sector economy defines the boundaries of the two-sector economy as one sector's capital stock shrinks to zero. We now allow for sector-specific values of sectoral parameters, so that the two sectors need not be *ex ante* identical. The one-sector equilibrium investment-capital ratio i_n^* for sector n solves the following non-linear implicit equation:

$$A_n - i_n^* = \frac{1}{\phi_n'(i_n^*)}\left[\alpha + (\psi^{-1}-1)\left(\phi_n(i_n^*) - \frac{\gamma\sigma_n^2}{2}\right)\right]. \tag{35}$$

Note that the left side of (35) is also the equilibrium consumption-capital ratio. Note that in equilibrium, the firm's optimality implies that Tobin's q is given by $q_n^* = 1/\phi_n'(i_n^*)$. Therefore, the marginal propensity to consume out of wealth C/V, i.e. the dividend yield on aggregate wealth, is given by

$$dy_n = \frac{A_n - i_n^*}{q_n^*} = \alpha + (\psi^{-1}-1)\left(\phi_n(i_n^*) - \frac{\gamma\sigma_n^2}{2}\right). \tag{36}$$

The equilibrium value function coefficient p_n for a one-sector economy is given by

$$p_n = \frac{\alpha}{\phi_n'(i_n^*)}\left[1 + \frac{\psi^{-1}-1}{\alpha}\left(\phi_n(i_n^*) - \frac{\gamma\sigma_n^2}{2}\right)\right]^{1/(1-\psi)}. \tag{37}$$

5.2 Two-sector Economy

For two-sector economy, sectoral investment-capital ratios i_0 and i_1 jointly solve the following implicit equations as functions of $z = K_1/(K_0 + K_1)$:

$$\left(\frac{c^*(z)}{N(z)}\right)^{1/\psi} = \frac{\alpha}{\phi_0'(i_0^*(z))}\frac{1}{N(z) - zN'(z)}\,, \tag{38}$$

$$\left(\frac{c^*(z)}{N(z)}\right)^{1/\psi} = \frac{\alpha}{\phi_1'(i_1^*(z))}\frac{1}{N(z) + (1-z)N'(z)}\,, \tag{39}$$

where $c^*(z)$ is the optimal aggregate consumption-capital ratio: $c^*(z) = C^*/(K_0 + K_1)$. Naturally, the goods-market equilibrium market condition given in (9) continues to hold.

Equations (38), (39), (9) and the following ODE jointly give the solution for sectoral investment-capital ratios:

$$\begin{aligned} 0 = {} & \frac{\alpha}{1-\psi^{-1}}\left[\left(\frac{c^*(z)}{N(z)}\right)^{1-\psi^{-1}} - 1\right] + \phi_0(i_0^*(z))L_0(z) + \phi_1(i_1^*(z))L_1(z) \\ & - \frac{\gamma}{2}\left[\sigma_0^2 L_0(z)^2 + \sigma_1^2 L_1(z)^2 + 2\rho\sigma_0\sigma_1 L_0(z)L_1(z)\right] + \frac{\sigma_0^2 - 2\rho\sigma_0\sigma_1 + \sigma_1^2}{2}M(z)\,, \end{aligned} \tag{40}$$

where $L_0(z)$, $L_1(z)$, and $M(z)$ are given in (20), (21), and (25), respectively.

Both $z = 0$ and $z = 1$ are absorbing barriers. They correspond to the one-sector model solution. The boundary conditions are

$$N(0) = p_0, \quad \text{and} \quad N(1) = p_1\,, \tag{41}$$

where p_n is the value function coefficient for the one-sector economy, and is given by (37), evaluated with parameters and the optimal investment-capital ratios in sectors 0 and 1.

Using Ito's formula, the dynamics of $z = K_1/(K_0+K_1)$, which govern endogenous capital reallocation, are given by

$$dz_t = \mu_z(z_t)dt + z_t(1-z_t)\sigma_1 dB_1(t) - z_t(1-z_t)\sigma_0 dB_0(t)\,, \tag{42}$$

where the drift of z, $\mu_z(z)$, is given by

$$\mu_z(z) = z(1-z)\left[\phi_1(i_1(z)) - \phi_0(i_0(z)) + (1-z)\sigma_0^2 - z\sigma_1^2 - (1-2z)\rho\sigma_0\sigma_1\right]. \tag{43}$$

5.3 Two-sector CIR Model

We next report the results for the setting with no adjustment costs in either sector, i.e. $\phi_n'(i) = 1$ for $n = 0, 1$. Because capital is perfectly liquid, we have a time-invariant steady-state sectoral capital distribution. That is, there is a single constant z which maximizes the representative consumer's welfare and is given by

$$z^* = \frac{(A_1 - \delta_1) - (A_0 - \delta_0) + \gamma\left(\sigma_0^2 - \rho\sigma_0\sigma_1\right)}{\gamma\left(\sigma_0^2 - 2\rho\sigma_0\sigma_1 + \sigma_1^2\right)}. \tag{44}$$

Aggregate consumption-capital ratio $c(z^*)$ is given by

$$c(z^*) = \alpha^{\psi} N(z^*)^{1-\psi}, \tag{45}$$

where $N(z^*)$ is given by

$$N(z^*) = \alpha\left[1 + (1-\psi)\left((A_0 - \delta_0)\, z^* + (A_1 - \delta_1)\,(1 - z^*) - \alpha - \Pi(z^*)\right)\right]^{1/(1-\psi)}, \tag{46}$$

and

$$\Pi(z^*) = \frac{\gamma}{2}\left[\sigma_0^2 (z^*)^2 + 2\rho\sigma_0\sigma_1 z^*\,(1 - z^*) + \sigma_1^2 (z^*)^2\right]. \tag{47}$$

6 Exercise with recursive utility: capital illiquidity

As in the baseline case, the analytic results with recursive utility are independent of the functional form for adjustment costs, $\phi_n(i)$. In order to calculate quantitative results we now use the baseline quadratic adjustment cost function:

$$\phi_n(i) = i - \frac{\theta_n}{2} i^2 - \delta_n. \tag{48}$$

We continue to use the parameter values we introduced in Section 4.1 for the baseline symmetric model with log utility.

To understand the role of capital liquidity, we now consider a comparative static change in the efficiency of reallocating capital. In standard equilibrium models, this experiment is not possible, since capital reallocation is either frictionless (CIR) or ruled out in pure-exchange settings (CLS). In this section, we analyze the aggregate and sectoral effects of changing the adjustment cost parameter θ. We choose three levels of the adjustment cost parameters: $\theta = 10, 20$, and $10,000$ for Figures 3-6. The higher the value of θ, the more

illiquid is physical capital. The extreme value of $\theta = 10,000$ corresponds to essentially completely illiquid capital. Without investment, the economy essentially behaves as a pure-exchange economy (CLS). We set the agent's coefficient of relative risk aversion $\gamma = 2$. All other parameter values are the same as in Section 4.

The adjustment costs impose direct resource costs (hence lowering the welfare) and also discourage savings and investment. The higher the adjustment cost parameter θ is, the lower welfare $N(z)$ is, and the higher consumption is. In fact, for the highest adjustment cost, the consumer consumes virtually the entire dividend and does not save; in this case, no direct resource costs are incurred at all, but the high adjustment cost gives rise to the misallocation of resources.

6.1 Aggregate implications

Figures 3 and 4 plot aggregate implications for two settings where the elasticity of intertemporal substitution ψ is set at $\psi = 0.5$ and $\psi = 2$, respectively. We consider these two values of elasticities because there is much debate about the magnitude of elasticity of intertemporal substitution. In the macro finance literature (where long-run risk is a key input), a high value of elasticity ψ is often chosen.[9]

First, we show that regardless of elasticity of intertemporal substitution, welfare $N(z)$ decreases with the adjustment cost. For a given θ, $N(z)$ is higher when the sectors are more balanced. Second, investment decreases with the adjustment cost and hence consumption must increase with the adjustment cost in the short run: aggregate output is either invested or consumed via dividends. Third, the higher the adjustment cost, the higher the rents to installed capital and hence the higher is Tobin's q.

Fourth, the higher the adjustment cost, the smaller is adjustment and thus on average the smaller is magnitude of the change in z, $|\mu_z(z)|$. When the adjustment cost is lower, the consumer actively reallocates capital to drive the allocation of capital back to the optimal value of $z = 0.5$, so the central tendency in Figures 3-4 is dramatically strengthened as capital becomes more liquid, i.e. the adjustment cost declines. Fifth, the aggregate risk

[9]Bansal and Yaron (2004) argue that the elasticity of intertemporal substitution is larger than one and use 1.5 in their long-run risk model. Attanasio and Vissing-Jorgensen (2003) estimate that the elasticity of intertemporal substitution is higher than unity for stockholders. Hall (1988) uses aggregate consumption data, obtains an estimate near zero. Using micro and macro evidence, Guvenen (2006) aims to reconcile the different estimates and finds that the elasticity depends on wealth.

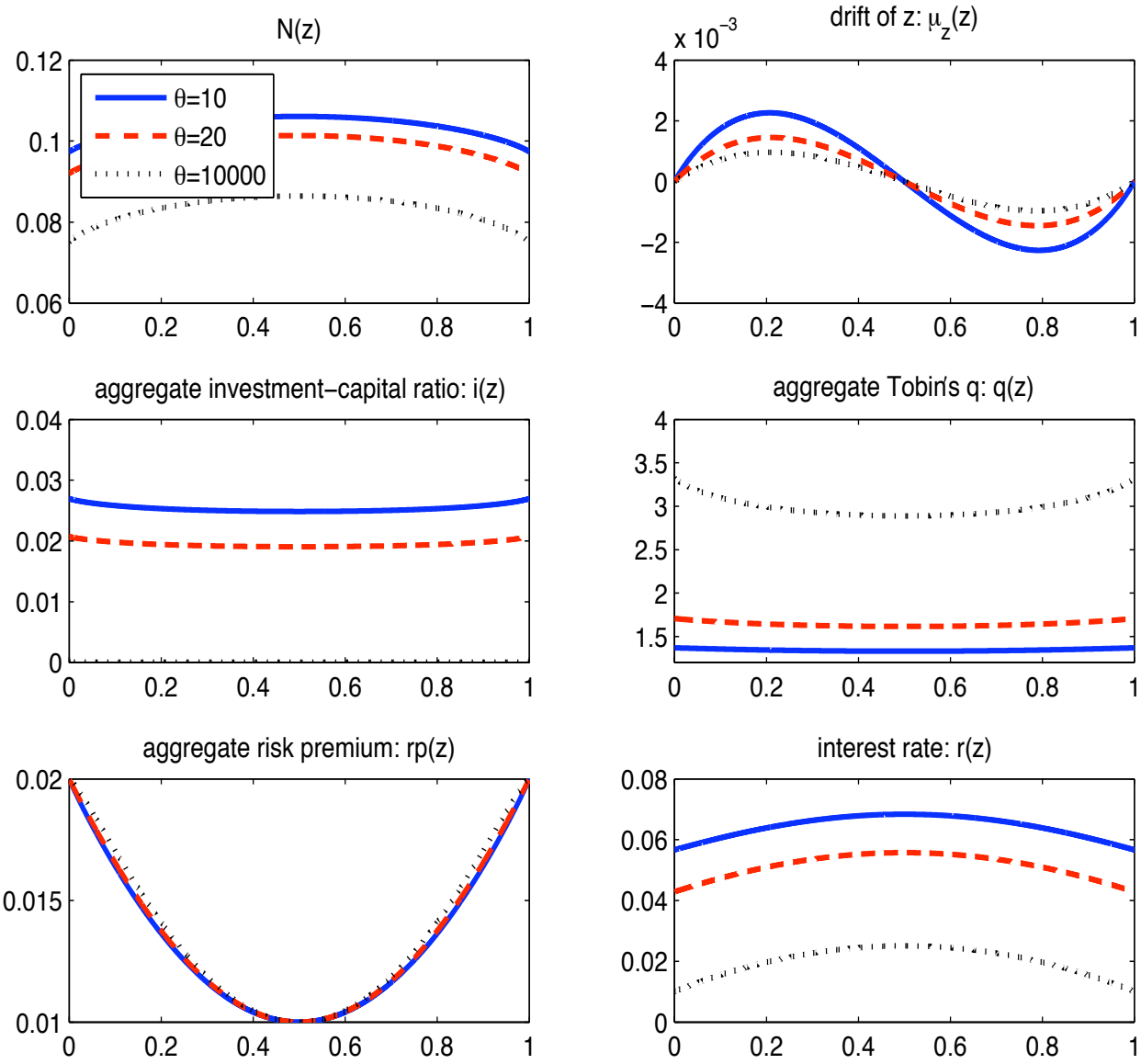

Figure 3: **Aggregate implications with risk aversion $\gamma = 2$ and elasticity $\psi = 0.5$.**

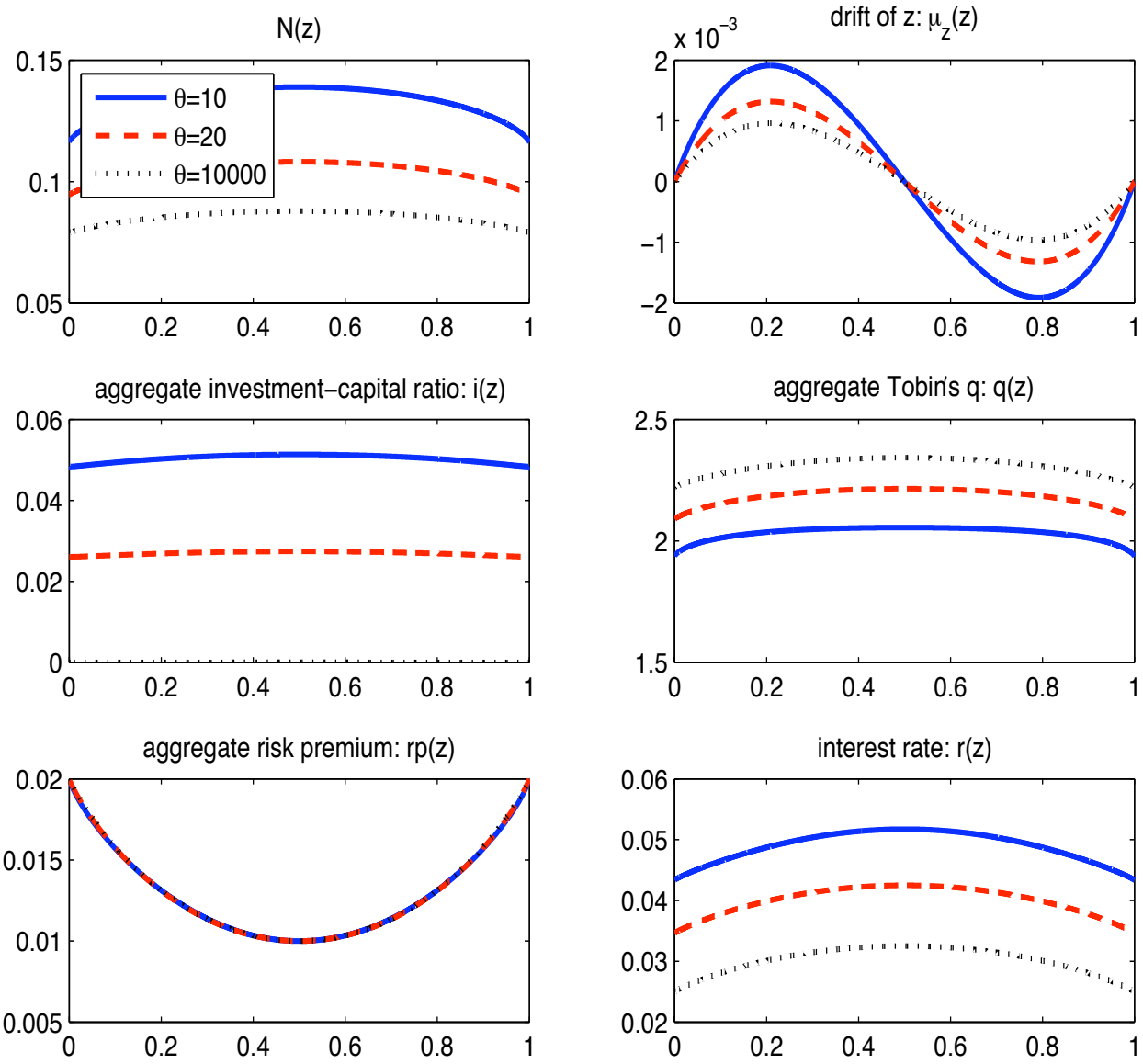

Figure 4: **Aggregate implications with risk aversion $\gamma = 2$ and elasticity $\psi = 2$.**

premium $rp(z)$ is virtually independent of the adjustment cost. This is perhaps counter-intuitive. We provide intuition in two steps: first, with one sector only (i.e. $z = 0, 1$), this is expected because the volatility of the shock to capital is exogenously given. Adjustment costs enter via the expected change, i.e. the drift, in capital accumulation. This in turn translates into the implication that only the drift $\mu_z(z)$, not the volatility, of the capital stock share z (the key state variable), depends on the adjustment cost specification. Since the risk premium depends on the volatility of z and the pricing kernel, we naturally do not expect much variation of the aggregate risk premium with respect to the adjustment cost. Moreover, for a given value of θ, the agent's incentive to consume is highest when the two sectors are more balanced because the systematic risk is smaller (due to diversification).

Finally, the adjustment cost has a significant effect on the level of the interest rate and hence also the expected aggregate market return.[10] Intuitively, the more liquid capital is, the more attractive and hence the higher is investment. In order to clear the goods market, we need to encourage the consumer to save so that investment can be financed. As a result, the equilibrium interest rate is higher in a more liquid economy, as we see for both levels of elasticities. This is often viewed as one undesirable effect of introducing production into equilibrium asset pricing models because it pushes up the equilibrium interest rate.

6.2 Sectoral implications

Figures 5 and 6 plot the corresponding sectoral results for the same two settings, i.e. elasticity of intertemporal substitution ψ is set at $\psi = 0.5$ and $\psi = 2$. In the low adjustment cost economy, the incentive to save and reallocate capital is strong, and hence investment is high at the sectoral level. This effect is also reflected in the lower value of Tobin's q when adjustment costs are low. When adjustment costs are so high as to prohibit investment almost entirely, the value of Tobin's q for a vanishing sector increases sharply, as the marginal value of reviving the shrinking sector skyrockets.

The sectoral risk premium and sectoral β are almost independent of the adjustment costs provided that the sector is not too small. However, when the sector is small enough (low z for sector 1), the properties of the sectoral risk premium and β differ depending on the elasticity of intertemporal substitution and the adjustment costs. In those situations, the

[10]This is because aggregate risk premium is effectively independent of the adjustment cost as we have argued in the preceding paragraph and documented in Figures 3-4.

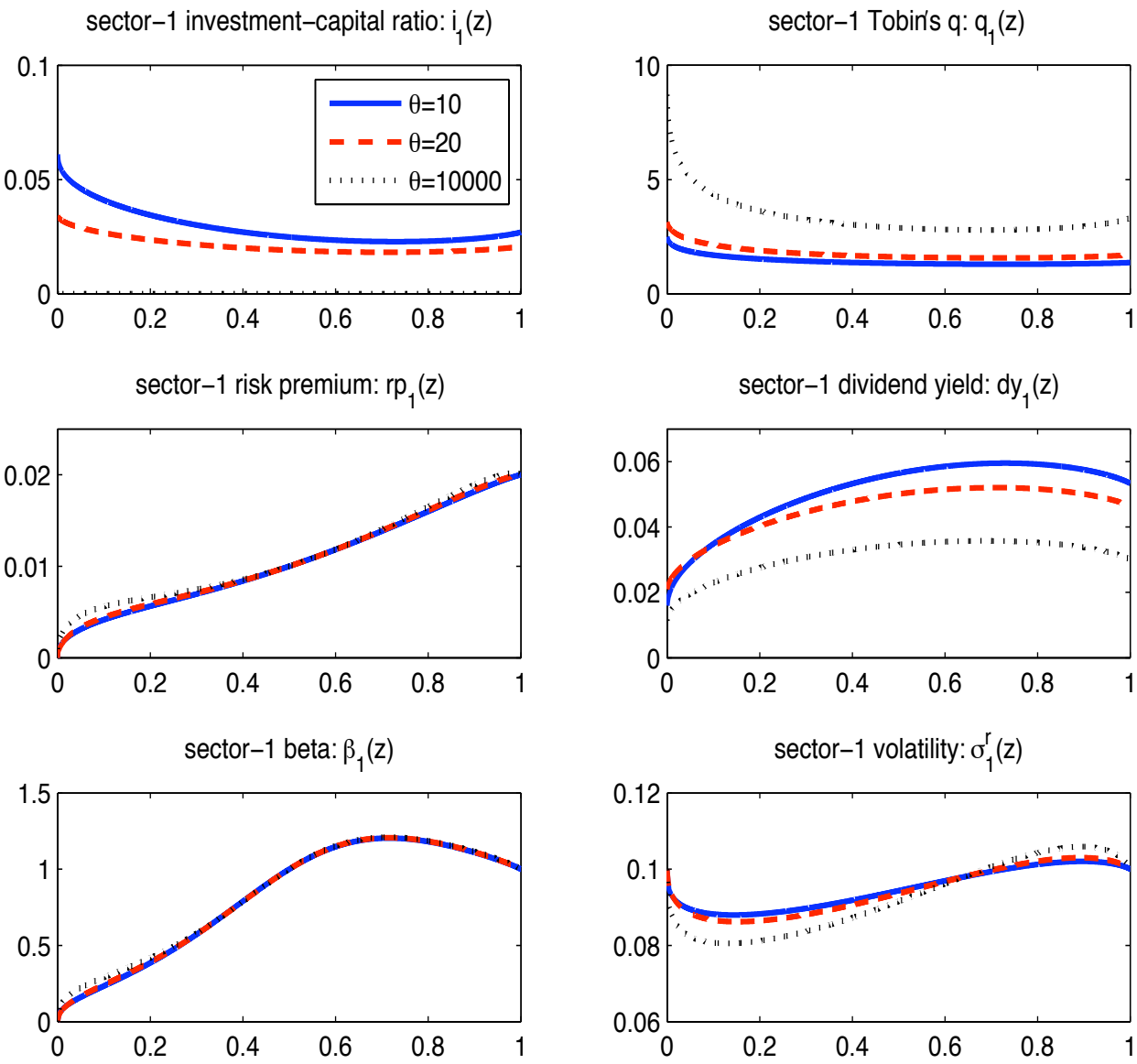

Figure 5: **Sectoral implications with risk aversion $\gamma = 2$ and elasticity $\psi = 0.5$.**

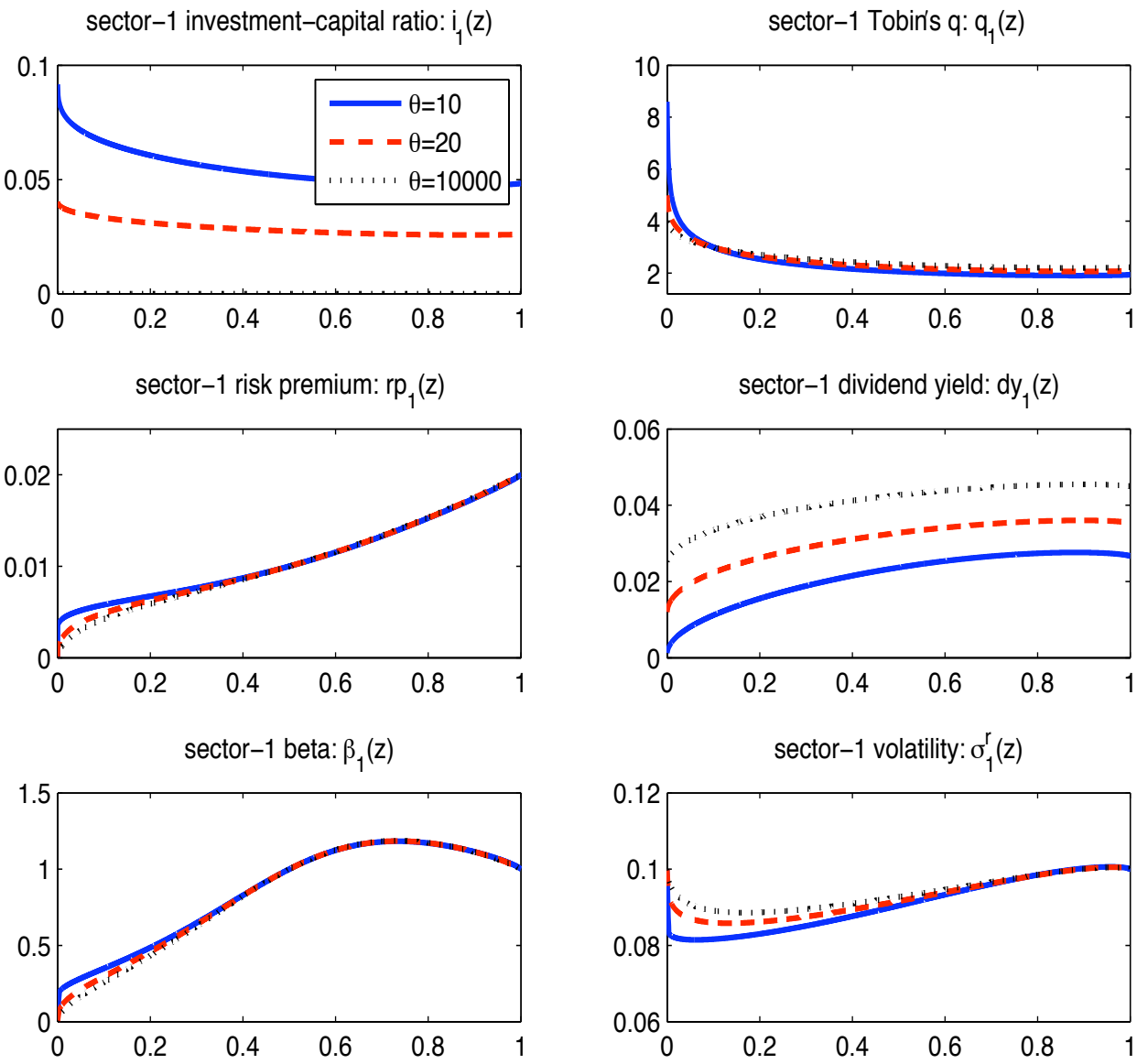

Figure 6: **Sectoral implications with risk aversion $\gamma = 5$ and elasticity $\psi = 2$.**

diversification incentive is very strong.

With a relatively small elasticity of intertemporal substitution (e.g. $\psi = 0.5$), for low values of z, the higher the adjustment cost is, the larger the sectoral risk premium and β are. Intuitively, a more costly adjustment process makes the smaller sector riskier.

With a large elasticity of intertemporal substitution (e.g. $\psi = 2$), the representative agent's incentive to smooth consumption over time is very high. As a result, the value of the dwindling sector skyrockets and investment increases substantially when the adjustment cost is relatively low. Intuitively, if ψ is high (e.g. $\psi = 2$), for the dwindling sector, the sensitivity of Tobin's q with respect to changes in sectoral distribution z is quite high when the adjustment cost is low. As a result, the sectoral risk premium and sectoral β are higher in dwindling sectors when the adjustment cost is low.

7 Exercises: Recursive Utility with Quadratic Adjustment Costs

In this section, we continue to use the baseline quadratic adjustment cost function, and explore the implications of changing risk aversion and asymmetric sectors. Otherwise, we continue to use the parameter values that we introduced in Section 4.1 for the baseline symmetric model with log utility.

7.1 Varying Risk Aversion

Since diversification across the two sectors plays an important role in capital allocation, we now consider changing risk aversion in order to explore the quantitative impact of risk. We already discussed the log utility case ($\gamma = 1$) in Figures 1 and 2, so now we graph the results for different values of risk aversion, $\gamma = 2, 5$, holding the other parameters of the model fixed. With non-expected utility, we can hold the intertemporal elasticity of substitution, ψ, fixed at 0.5, while changing risk aversion. The case with $\gamma = 2$ is thus the standard CRRA case.

Aggregate implications. In Figure 7, we plot the aggregate implications of the model with risk aversion $\gamma = 2, 5$. In all cases, higher risk aversion lowers welfare $N(z)$ and raises investment. Note, however, the greater curvature in welfare $N(z)$ as a function of z, the distribution of capital, for higher values of γ. The more risk averse consumer responds more to changes in the distribution of capital, which determine how well-diversified the household

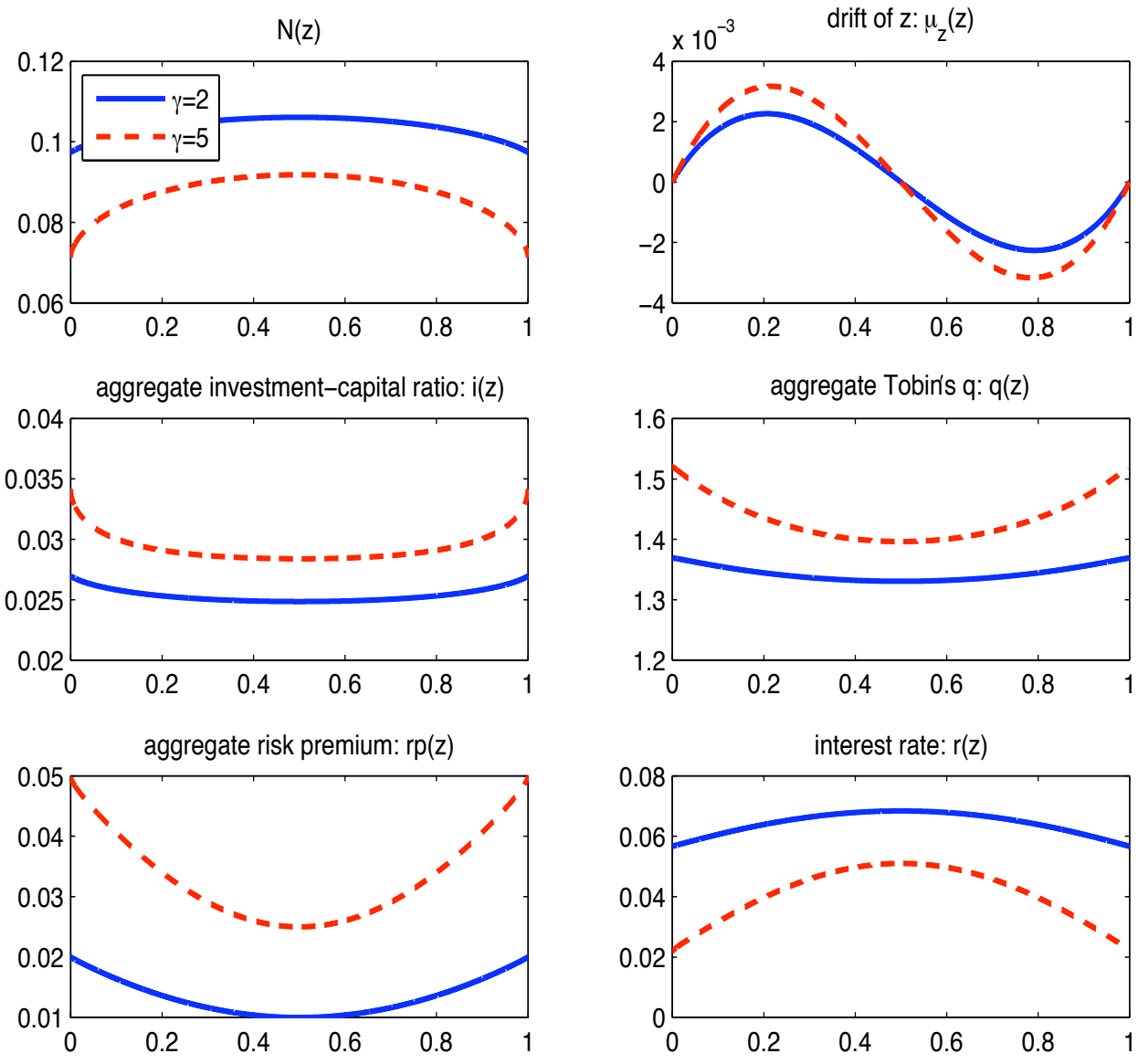

Figure 7: **Aggregate implications with different levels of risk aversion $\gamma = 2, 5$ and elasticity of intertemporal substitution $\psi = 0.5$.**

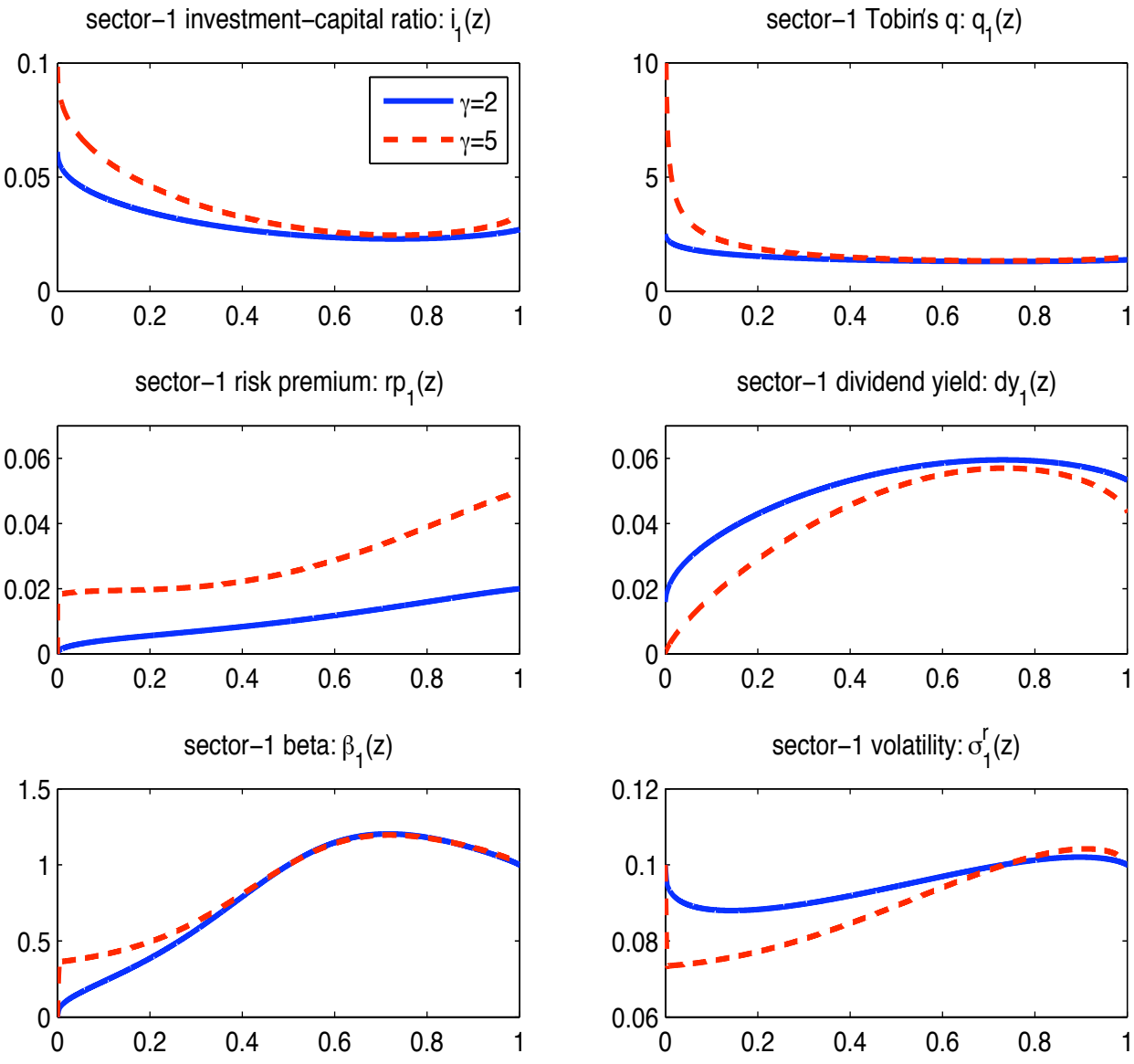

Figure 8: **Sectoral implications with different levels of risk aversion $\gamma = 2, 5$ and elasticity of intertemporal substitution $\psi = 0.5$.**

is. When his risk aversion is high, the consumer cuts consumption more as he becomes less diversified (z closer to zero or one), and instead engages in more precautionary savings. The middle panel of Figure 7 shows that this savings response translates into higher investment and growth near the boundary values of z, compared to $z = 0.5$. This greater response of investment to the distribution of capital, z, increases the mean reversion in the model, evidenced in the drift for the dynamics of z, $\mu_z(z)$. The higher is risk aversion, the greater is the central tendency in z. These effects are also evident in the interest rate, which also shows more curvature in the high risk aversion case. With higher risk aversion, the model also generates higher expected returns and risk premia, consistent with the lower asset prices and investment, compared to the low risk aversion case.

Sectoral implications. In Figure 8, we plot the sectoral implications for the model with risk aversion $\gamma = 2, 5$. As in the aggregate case, the higher value of risk aversion is associated with higher investment and Tobin's q in each sector, as well as a higher risk premium. The dividend yield is lower when risk aversion is higher because of the strong investment response to higher risk aversion, depleting the dividend.

These results indicate that risk aversion, even for modest values of γ, substantially enhances the effect of the distribution of capital on aggregate variables. In particular, the central tendency in the model is much stronger as risk aversion increases, since the consumer is more sensitive when the economy is less diversified. The enhanced central tendency drives greater investment and growth for extreme values of z. As in the single sector model, greater risk aversion increases risk premia, but lowers expected returns owing to the lower interest rate resulting from greater precautionary saving. Hence, asset prices rise with risk aversion in equilibrium.

7.2 Asymmetric sectors, Varying Risk Aversion

So far, we have only considered cases with two symmetric sectors, so the economy achieves maximum diversification and highest welfare at $z = 0.5$. Now we allow for the two sectors to have different values of the adjustment cost parameter θ, so that one sector is relatively liquid and the other illiquid. In this case, shocks to z not only change the degree of diversification, but also the economy's liquidity composition. In Figures 9 through 11, we plot results for

the model with $\theta_0 = 10$, and $\theta_1 = 10,000$.[11] By choosing $\theta_1 = 10,000$, we effectively make the sector 1 illiquid (Lucas tree) sector.

Aggregate Implications. When z is low, sector 1 (the illiquid sector) is small and the overall economy is relatively liquid. As z rises, the economy has relatively more illiquid capital. Now, the value function $N(z)$ in Figure 9 achieves its maximum at a value of z less than 0.5, since the consumer prefers to hold more of the relatively liquid capital, *ceteris paribus*. Because sector 0 is significantly more attractive than sector 1, the liquidity preference dominates the diversification concern so that the drift in z is negative for all z. Additionally, there is more reallocation for higher values of risk aversion. The middle panel of Figure 9 shows that investment falls monotonically as z increases: as capital becomes less liquid on average, savings and aggregate investment fall monotonically. Similarly, consumption rises monotonically as z increases and capital is less liquid. Paradoxically, Tobin's q rises with z as investment falls. This effect is consistent with the fact that the value of installed capital rises as adjustment costs generate rents to installed capital. Again, changes in liquidity cause investment and Tobin's q to move in opposite directions: higher liquidity (higher values of z) increases investment but decreases Tobin's q. The bottom panel of Figure 9 shows that the aggregate risk premium is highest when the economy is less-well-diversified, especially for higher risk aversion. The interest rate reaches its peak to the left of $z = 0.5$, where it is also relatively flat. In this region, the economy is relatively liquid so there is little change in precautionary savings as z varies. On the right-hand side, however, where the economy is illiquid, precautionary savings and hence the interest rate are more sensitive to changes in z, especially for higher risk aversion.

Sectoral Implications. The next two figures, Figures 10 and 11, show the sectoral values as functions of z in the economy; since the two sectors are no longer symmetric, we now graph the two sectors separately. Figure 10 shows values for sector 1, the illiquid sector with high adjustment costs. The first panel of Figure 10 shows that investment is always near zero (because of the prohibitively high adjustment costs), so Tobin's q varies with z, especially as sector 1 becomes very large. The middle panel shows that the risk premium in sector 1 rises as that sector becomes a larger share of the economy, consistent with our earlier discussion of the symmetric sectors. Since investment is zero for all values of z, the

[11] In our earlier paper, Eberly and Wang (2009), we studied asymmetric productivity, A, in a deterministic setting.

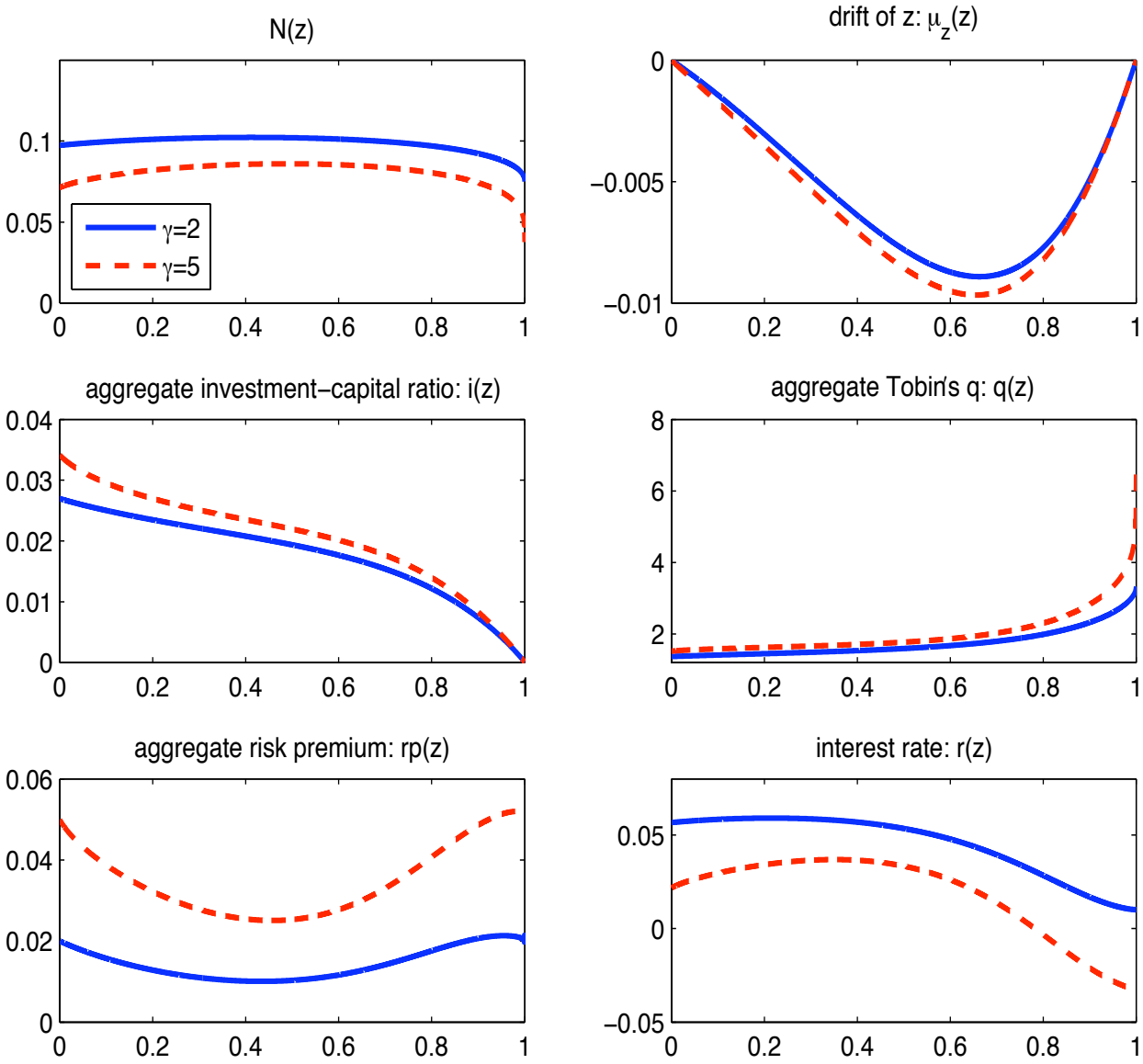

Figure 9: **Aggregate implications with asymmetric sectors:** $\theta_0 = 10$ **and** $\theta_1 = 10,000$**.**

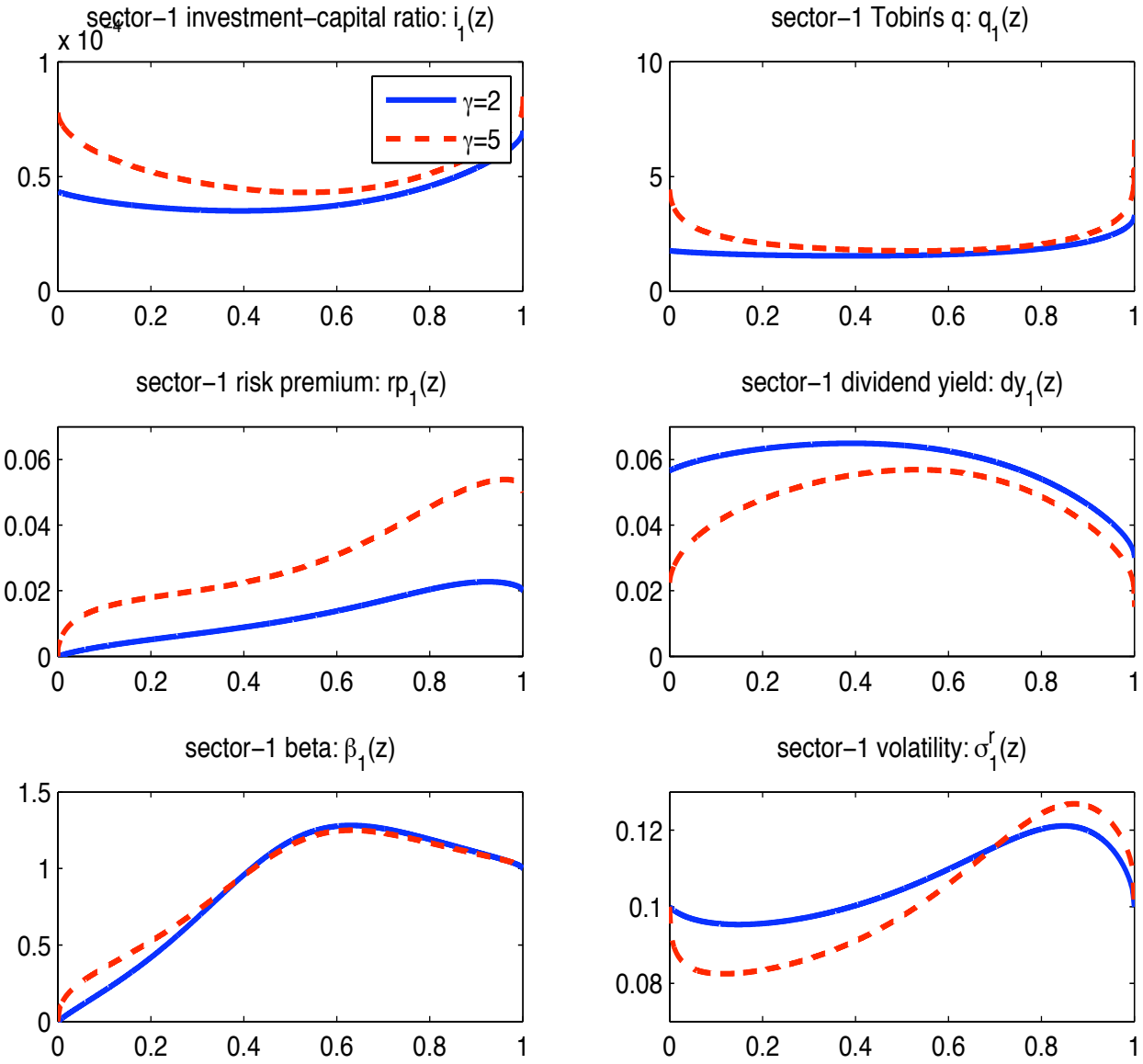

Figure 10: **Sector-1 implications with asymmetric sectors:** $\theta_0 = 10$ **and** $\theta_1 = 10,000$**.**

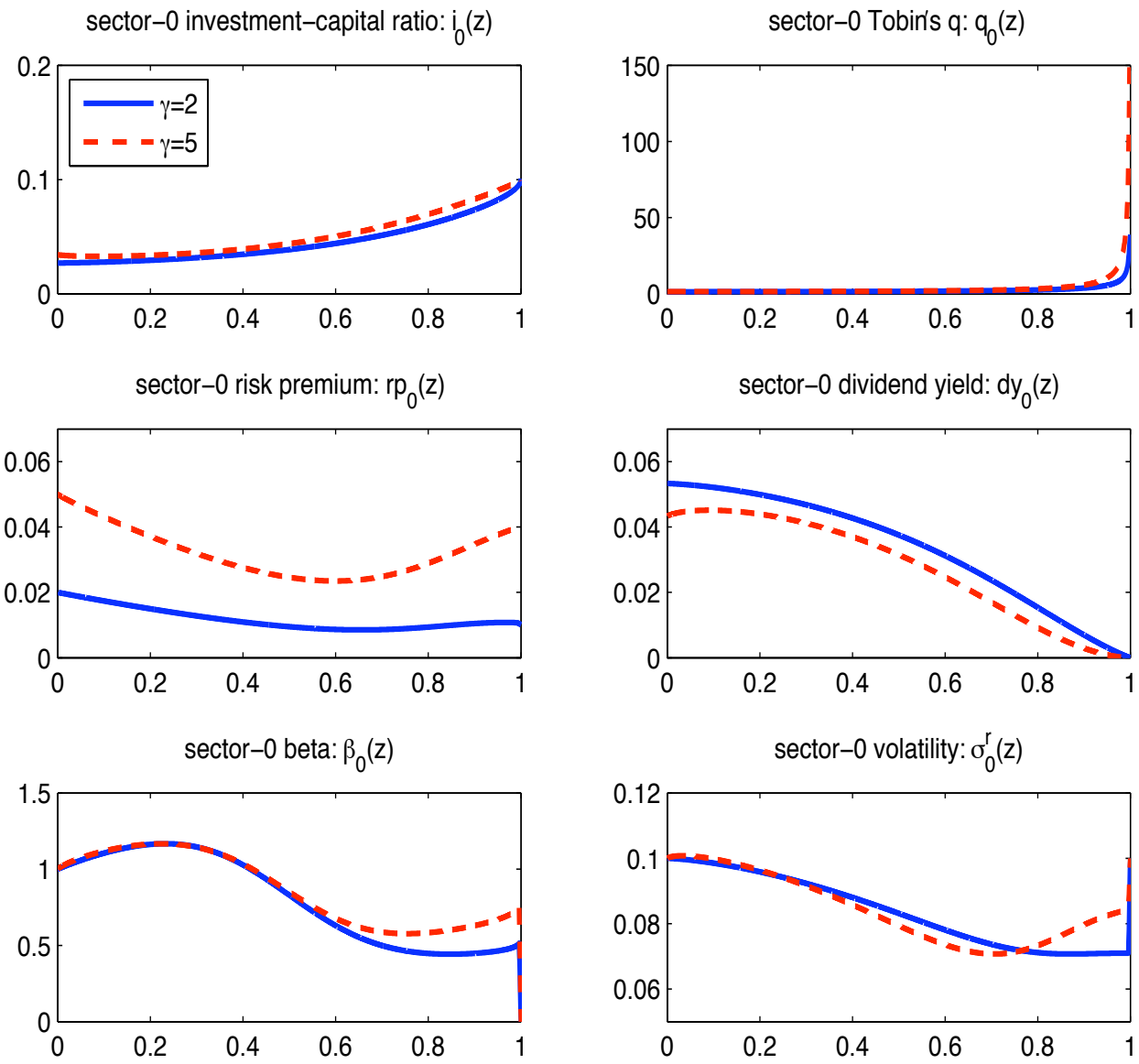

Figure 11: **Sector-0 implications with asymmetric sectors:** $\theta_0 = 10$ **and** $\theta_1 = 10,000$**.**

dividend yield simply reflects the ratio of A to value, or the inverse of Tobin's q. Similar to what we saw in the symmetric case, the bottom panel shows that sectoral β rises and then falls as sector 1 becomes larger.

Figure 11 for the liquid sector shows the properties of liquid capital in the model. The first panel of Figure 11 shows that both investment and Tobin's q in sector 0 rise with z, as the economy becomes less liquid on average. In this example, as z approaches unity and the liquid sector tends to disappear, the value of liquid capital (measured by Tobin's q) exceeds 100, since the agent places such a high value on resuscitating the liquid sector. The second panel of the charts shows that the risk premium in the liquid sector does not generally increase with the size of the sector, especially for higher risk aversion. As z rises and sector 0 shrinks, its risk premium initially falls, as expected, but then it rises again (before falling abruptly when the sector becomes insignificant), especially when risk aversion is high. In the range where the liquid sector is a relatively small part of the economy, sector 0 is nonetheless the only sector with an "adjustable" capital stock. Thus, it buffers all shocks (even shocks to the other sector) to provide consumption smoothing. This can be seen in the bottom panel of the figure, where sector 0 volatility increases to the right of the graph. Moreover, sector 0's β also falls, and then increases again (before going to zero) as sector 0 shrinks. Thus, as sector 0 becomes very small, the value of its capital (Tobin's q) shoots up because the overall cost of capital falls, but sector 0 itself becomes riskier and more volatile.

This version of the model allows us to consider the properties of an economy with an unbalanced capital stock. Of course, an overall negative shock to the capital stock reduces wealth and consumption. But if an economy finds itself with a relative excess of illiquid capital, the effects are worse because the economy is not only poorer, but also undiversified and illiquid. Relative to setting z at its utility-maximizing value, a high value of z is associated with lower aggregate investment and growth, even though the investment rate in liquid capital increases to rebuild the liquid sector. Because the overall economy is less liquid, the aggregate risk premium rises and so does aggregate return volatility. The interest rate falls because the consumer has a greater incentive to save in the illiquid economy. In the model, the results are endogenously generated by the equilibrium response to an excess of illiquid capital relative to liquid capital. Such a shock causes the overall economy to

become less liquid: this is an endogenous change in liquidity in contrast to the comparative static change in θ we considered in Section 6.

8 Conclusion

We have developed an analytical framework that extends and nests standard equilibrium models in macroeconomics and finance. The two sector structure allows us to consider both *ex ante* and *ex post* heterogeneity. This is especially important in assessing the role of equilibrium in models with a representative firm. By examining the dynamics of the model when one sector is relatively small and has little impact on the aggregates, researchers can examine a "partial equilibrium" exercise in an equilibrium setting. In the limit, as a sector becomes an increasingly large share of the economy, our results converge to those of a single sector equilibrium model - which provides the boundary conditions for our model. Similarly, by allowing for investment with adjustment costs, we nest both the "two trees" approach of CLS and the frictionless model of CIR, as adjustment costs go to infinity and to zero, respectively, in our model. Not only does our framework allow for intermediate cases between CLS and CIR, it also allows for asymmetric capital adjustment across the two sectors, which is implicitly ruled out by both polar cases.

The model is driven by the tension between diversification and adjustment costs. The agent is most diversified by a balanced capital stock, but maintaining this balance requires incurring costs of reallocating capital. The efficiency cost of reallocating capital is a drag on growth in the economy, so the agent gives up some efficiency and growth in order to diversify risk.

After developing the general framework, we use it to demonstrate both when heterogeneity is not relevant in the aggregate and when it is - and for the latter case, how heterogeneity affects the equilibrium. In the symmetric economy with log utility, the distribution of capital across sectors has very little effect on aggregate values. This immunity is exact when the adjustment cost function is also log, as we show in a deterministic case in Eberly and Wang (2009) and extend here to the stochastic case. However, even with quadratic adjustment costs, the distributional effects are negligible when the utility function is log. When we depart from log utility, greater risk aversion generates a much larger consequence of the distribution of capital.

Even in the benchmark case, the two sector economy exhibits endogenous capital reallocation. The desire for diversification generates mean reversion in the capital distribution, counterbalanced by costs of adjustment, so reallocation is slow and time-varying. For example, when a sector is small, the desire to invest and restore diversification is very large. Conversely, the cost of capital is very low because the sector is virtually risk-free. Hence, Tobin's q and investment are very large in the dwindling sector, and the dividend yield may be negative, even with constant-returns-to-scale in production and identical technologies in the two sectors.

When risk aversion is higher, the economy is more sensitive to the distribution of capital, exhibiting greater reallocation and greater price variation at both the aggregate and sectoral levels. These results are further enhanced when liquidity differs between the two sectors, so that one sector has relatively liquid (low adjustment cost) capital. In this case shocks to the distribution of capital change both diversification and liquidity in the economy. In particular, a low-liquidity economy is both undiversified and faces costly rebalancing, so the agent is especially sensitive to risk. An economy with too much illiquid capital has low investment and low growth, but high risk premia and high volatility even while the interest rate is low. This situation persists while the economy rebuilds liquidity by investing in the low adjustment cost sector.

These results demonstrate the impact of shifting the balance between risk diversification and costly reallocation. Higher risk aversion focuses on the value of diversification, while altering the reallocation technology changes the costs of moving capital in the desired direction. Because liquidity is valuable in a stochastic economy, the more so the higher is risk aversion, these shifts significantly alter the dynamic equilibrium prices and quantities. Put differently, shocks in the model can unbalance the desired "match" of capital to sectors, and the adjustment cost function controls how readily new matches can be made. Hence, the model generates endogenous shifts in the matching function governing reallocation, as suggested in the labor literature by Shimer (2007) and in current data by Kocherlakota (2010).

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A Appendix: Technical details (Not for publication)

We provide technical details for the general case with recursive utility. We first solve the central planner's resource allocation problem. Then, by using the standard welfare theorem, we use equilibrium allocation to derive both aggregate and sectoral asset pricing implications.

A.1 The social planner's resource allocation solution.

We conjecture that representative agent's value function $J(K_0, K_1)$ has the homogeneity property in sectoral capital stocks K_0 and K_1 and can be written in the following form:

$$J(K_0, K_1) = \frac{1}{1-\gamma} \left((K_0 + K_1) N(z) \right)^{1-\gamma}, \tag{A.1}$$

where $z = K_1/(K_0 + K_1)$ and $N(z)$ is a function to be determined. We use $J_n(K_0, K_1)$ to denote the first-order derivative with respect to capital stock K_n in sector $n = 0, 1$. Similarly, we use $J_{mn}(K_0, K_1)$ to denote the second-order derivatives with respect to capital stocks in sectors n and m.

The following Hamilton-Jacobi-Bellman (HJB) equation describes the planner's problem:

$$0 = \max_{I_0, I_1} f(C, J) + \phi_0(i_0) K_0 J_0 + \phi_1(i_1) K_1 J_1 + \frac{1}{2}\sigma_0^2 K_0^2 J_{00} + \rho\sigma_0\sigma_1 K_0 K_1 J_{01} + \frac{1}{2}\sigma_1^2 K_1^2 J_{11}. \tag{A.2}$$

Using the conjectured value function (A.1), we have the following two first-order conditions (FOCs) with respect to the sectoral investment-capital ratios i_0 and i_1:

$$\left(\frac{c^*(z)}{N(z)} \right)^{1/\psi} = \frac{\alpha}{\phi_0'(i_0^*(z))} \frac{1}{N(z) - zN'(z)}, \tag{A.3}$$

$$\left(\frac{c^*(z)}{N(z)} \right)^{1/\psi} = \frac{\alpha}{\phi_1'(i_1^*(z))} \frac{1}{N(z) + (1-z)N'(z)}, \tag{A.4}$$

where $c(z) = C/(K_0 + K_1)$ is the aggregate consumption-capital ratio. With the optimal $c^*(z)$, the normalized aggregator of Duffie-Epstein utility for the agent is given by

$$f(C^*, J) = \frac{\alpha}{1 - \psi^{-1}} \left[\left(\frac{c^*(z)}{N(z)} \right)^{1-\psi^{-1}} - 1 \right] \left((K_0 + K_1) N(z) \right)^{1-\gamma}. \tag{A.5}$$

As a special case, if $\psi = 1$, we have $f(C, J) = \alpha \left[\ln c(z) - \ln N(z) \right] \left((K_0 + K_1) N(z) \right)^{1-\gamma}$.

Substituting the normalized aggregator (A.5) and the FOCs (A.3) and (A.4) for i_0 and i_1 into the HJB (A.2), we obtain the nonlinear differential equation (40) in the text. Similarly,

we apply our two-sector results to $z = 0$ and $z = 1$, we obtain the boundary conditions given by (41) and (37) at the absorbing boundaries: $z = 0, 1$.

Using the Ito's formula, we show that the dynamics of $z = K_1/(K_0 + K_1)$ are given by

$$dz_t = \mu_z(z_t)dt + z_t(1 - z_t)\sigma_1 dB_1(t) - z_t(1 - z_t)\sigma_0 dB_0(t)\,, \tag{A.6}$$

where the drift of z, $\mu_z(z)$, is given by

$$\mu_z(z) = z(1 - z)\left[\phi_1(i_1^*(z)) - \phi_0(i_0^*(z)) + (1 - z)\sigma_0^2 - z\sigma_1^2 - (1 - 2z)\rho\sigma_0\sigma_1\right]. \tag{A.7}$$

The aggregate capital accumulation dynamics is

$$\frac{d(K_0(t) + K_1(t))}{K_0(t) + K_1(t)} = g(z_t)dt + (1 - z_t)\sigma_0 dB_0(t) + z_t\sigma_1 dB_1(t)\,, \tag{A.8}$$

where the aggregate growth (capital accumulation) rate $g(z)$ is given by

$$g(z) = (1 - z)\phi_0(i_0^*(z)) + z\phi_1(i_1^*(z))\,. \tag{A.9}$$

The volatility of the aggregate growth (capital accumulation) rate is given by

$$\sigma(z) = \sqrt{\sigma_0^2(1 - z)^2 + 2\rho\sigma_0\sigma_1(1 - z)z + \sigma_1^2 z^2}\,. \tag{A.10}$$

The aggregate Tobin's q is

$$q(z) = \frac{V(K_0, K_1)}{K_0 + K_1} = (1 - z)q_0(z) + zq_1(z)\,. \tag{A.11}$$

A.2 Asset pricing implications.

Let ξ denote the equilibrium stochastic discount factor (SDF). Using the results in Duffie and Epstein (1992), we have

$$\xi_t = \exp\left[\int_0^t f_J(C_s^*, J_s)\,ds\right] f_C(C_t^*, J_t). \tag{A.12}$$

We have

$$f_J(C^*, J) = \left(\frac{\alpha}{1 - \psi^{-1}}\right)\left[\left(\psi^{-1} - \gamma\right)\left(\frac{c^*(z)}{N(z)}\right)^{1-\psi^{-1}} - (1 - \gamma)\right], \tag{A.13}$$

$$f_C(C^*, J) = \frac{\alpha(C^*)^{-\psi^{-1}}}{((1 - \gamma)J(K_0, K_1))^{\omega-1}} = \frac{\delta(N(z)(K_0 + K_1))^{\psi^{-1}-\gamma}}{(C^*)^{\psi^{-1}}}\,. \tag{A.14}$$

The equilibrium dynamics of the SDF is given by

$$d\xi_t = -r(z_t)\xi_t dt - \eta_0(z_t)\xi_t dB_0(t) - \eta_1(z_t)\xi_t dB_1(t)\,, \tag{A.15}$$

where the equilibrium interest rate as a function of z is given by

$$\begin{aligned} r(z) = {} & \alpha + \alpha\left(\frac{\psi^{-1}-\gamma}{1-\psi^{-1}}\right)\left[1-\left(\frac{c^*(z)}{N(z)}\right)^{1-\psi^{-1}}\right] + \gamma g(z) \\ & - (\gamma+1)\left[(1-z)\sigma_0^2 - z\sigma_1^2 - (1-2z)\rho\sigma_0\sigma_1\right]\epsilon(z) - \epsilon(z)\left(\phi_1(i_1^*(z)) - \phi_0(i_0^*(z))\right) \\ & + \frac{\psi^{-1}}{2}\left[\frac{d^2}{dz^2}\ln c(z) - (1-\gamma\psi)\frac{d^2}{dz^2}\ln N(z)\right] z^2(1-z)^2(\sigma_0^2 - 2\rho\sigma_0\sigma_1 + \sigma_1^2) \\ & - \frac{(\gamma+1)\gamma}{2}\left(\sigma_0^2(1-z)^2 + \sigma_1^2 z^2 + 2\rho\sigma_0\sigma_1 z(1-z)\right) - \frac{(\sigma_0^2 - 2\rho\sigma_0\sigma_1 + \sigma_1^2)\epsilon^2(z)}{2}\,, \end{aligned} \tag{A.16}$$

and the equilibrium market prices of risk for two diffusion risks $B_0(z)$ and $B_1(z)$, $\eta_0(z)$ and $\eta_1(z)$, are respectively given by

$$\eta_0(z) = \sigma_0\epsilon(z) + \gamma\sigma_0(1-z), \tag{A.17}$$

$$\eta_1(z) = -\sigma_1\epsilon(z) + \gamma\sigma_1 z, \tag{A.18}$$

where

$$\epsilon(z) = \psi^{-1}\left(-\frac{c^{*\prime}(z)}{c^*(z)} + (1-\gamma\psi)\frac{N'(z)}{N(z)}\right) z(1-z)\,. \tag{A.19}$$

The market-to-book ratio, average q, is also equal to marginal q, for sector n is given by:

$$q_n(z) = \frac{V_n}{K_n} = \frac{1}{\phi_n'(i_n^*(z))}, \quad n = 0, 1. \tag{A.20}$$

The dividend yields in sector n, dy_n, is given by

$$dy_n(z) = \frac{D_n}{V_n} = \frac{A_n - i_n^*(z)}{q_n(z)}. \tag{A.21}$$

Next, we derive the dynamics for the rate of return from investing in sector 0, $dR_0(t)$, which is given by the sum of sector-0 dividend yield $D_0(t)dt/V_0(t) = dy_0(z_t)dt$ and the expected rate of capital gains $dV_0(t)/V_0(t)$. Using Ito's formula, we obtain:

$$\begin{aligned} dR_0(t) &= \frac{D_0(t)dt + dV_0(t)}{V_0(t)} = dy_0(z_t)dt + \frac{dq_0(z_t)}{q_0(z_t)} + \frac{dK_0(t)}{K_0(t)} + \frac{dq_0(z_t)}{q_0(z_t)}\frac{dK_0(t)}{K_0(t)}\,, \\ &= \mu_0^r(z_t)dt + \frac{q_0'(z_t)}{q_0(z_t)} z_t(1-z_t)\left(\sigma_1 dB_1(t) - \sigma_0 dB_0(t)\right) + \sigma_0 dB_0(t), \end{aligned} \tag{A.22}$$

where the expected rate of return in sector 0, $\mu_0^r(z)$, is given by

$$\begin{aligned}\mu_0^r(z) &= dy_0(z) + \phi_0(i_0(z)) + z(1-z)\left[\phi_1(i_1(z)) - \phi_0(i_0(z))\right]\frac{q_0'(z)}{q_0(z)} \\ &\quad -z^2(1-z)\left(\sigma_0^2 - 2\rho\sigma_0\sigma_1 + \sigma_1^2\right)\left[\frac{q_0'(z)}{q_0(z)} - \frac{1}{2}\frac{q_0''(z)}{q_0(z)}(1-z)\right].\end{aligned} \tag{A.23}$$

Let $\sigma_0^r(z)$ denote the return volatility in sector 0. We may calculate $\sigma_0^r(z)$ as follows:

$$\sigma_0^r(z) = \left[\left(\frac{q_0'(z)}{q_0(z)}z(1-z)\right)^2\left(\sigma_0^2 - 2\rho\sigma_0\sigma_1 + \sigma_1^2\right) - 2z(1-z)\frac{q_0'(z)}{q_0(z)}\left(\sigma_0^2 - \rho\sigma_0\sigma_1\right) + \sigma_0^2\right]^{1/2} \tag{A.24}$$

Similarly, the instantaneous rate of return $dR_1(t)$ including both the dividend yield and capital gains in sector 1 is given by

$$\begin{aligned}dR_1(t) &= \frac{D_1(t)dt + dV_1(t)}{V_1(t)} = dy_1(z_t) + \frac{dq_1(z_t)}{q_1(z_t)} + \frac{dK_1(t)}{K_1(t)} + \frac{dq_1(z_t)}{q_1(z_t)}\frac{dK_1(t)}{K_1(t)}, \\ &= \mu_1^r(z_t)dt + \frac{q_1'(z_t)}{q_1(z_t)}z_t(1-z_t)\left(\sigma_1 dB_1(t) - \sigma_0 dB_0(t)\right) + \sigma_1 dB_1(t),\end{aligned} \tag{A.25}$$

where the expected rate of return in sector 1, $\mu_1^r(z)$, is given by

$$\begin{aligned}\mu_1^r(z) &= dy_1(z) + \phi_1(i_1(z)) + z(1-z)\left[\phi_1(i_1(z)) - \phi_0(i_0(z))\right]\frac{q_1'(z)}{q_1(z)} \\ &\quad +z(1-z)^2\left(\sigma_0^2 - 2\rho\sigma_0\sigma_1 + \sigma_1^2\right)\left[\frac{q_1'(z)}{q_1(z)} + \frac{1}{2}\frac{q_1''(z)}{q_1(z)}z\right].\end{aligned} \tag{A.26}$$

Let $\sigma_1^r(z)$ denote the return volatility in sector 1. We may calculate $\sigma_1^r(z)$ as follows:

$$\sigma_1^r(z) = \left[\left(\frac{q_1'(z)}{q_1(z)}z(1-z)\right)^2\left(\sigma_0^2 - 2\rho\sigma_0\sigma_1 + \sigma_1^2\right) + 2z(1-z)\frac{q_1'(z)}{q_1(z)}\left(\sigma_1^2 - \rho\sigma_0\sigma_1\right) + \sigma_1^2\right]^{1/2} \tag{A.27}$$

The sectoral risk premium is then given by $rp_n(z) = \mu_n^r(z) - r(z)$, for $n = 0, 1$.

Using the portfolio argument, we obtain the following dynamics for the rate of return on the market portfolio $dR^m(t)$:

$$dR^m(t) = \mu^m(z_t)dt + \sigma_0^m(z_t)dB_0(t) + \sigma_1^m(z_t)dB_1(t), \tag{A.28}$$

where the expected return of the market portfolio is then given by

$$\mu^m(z) = \frac{1}{q(z)}\left[(1-z)q_0(z)\mu_0^r(z) + zq_1(z)\mu_1^r(z)\right], \tag{A.29}$$

and the volatility functions are given by

$$\sigma_1^m(z) = \frac{\sigma_1 z}{q(z)}\left[q_1(z) + (1-z)^2 q_0'(z) + z(1-z)q_1'(z)\right], \tag{A.30}$$

$$\sigma_0^m(z) = \frac{\sigma_0(1-z)}{q(z)}\left[q_0(z) - z(1-z)q_0'(z) - z^2 q_1'(z)\right]. \tag{A.31}$$

The market return volatility is therefore given by

$$\sigma^m(z) = \sqrt{\left(\sigma_0^m(z)\right)^2 + 2\rho\sigma_1^m(z)\sigma_0^m(z) + \left(\sigma_1^m(z)\right)^2}. \tag{A.32}$$

The aggregate market risk premium is given by $rp^m(z) = \mu^m(z) - r(z)$.

Sectoral betas are defined in the standard way, i.e. $\beta_0(z_t) = \mathrm{Cov}_t(dR_0, dR^m)/\mathrm{Var}_t(dR^m)$. The betas for sectors 0 and 1, $\beta_0(z)$ and $\beta_1(z)$, are given by

$$\beta_0(z) = \frac{\sigma_0\sigma_0^m(z) + \rho\sigma_0\sigma_1^m(z)}{\left(\sigma^m(z)\right)^2} + \frac{\left(\sigma_1 - \rho\sigma_0\right)\sigma_1^m(z) - \left(\sigma_0 - \rho\sigma_1\right)\sigma_0^m(z)}{\left(\sigma^m(z)\right)^2} z(1-z)\frac{q_0'(z)}{q_0(z)}, \tag{A.33}$$

$$\beta_1(z) = \frac{\sigma_1\sigma_1^m(z) + \rho\sigma_1\sigma_0^m(z)}{\left(\sigma^m(z)\right)^2} + \frac{\left(\sigma_1 - \rho\sigma_0\right)\sigma_1^m(z) - \left(\sigma_0 - \rho\sigma_1\right)\sigma_0^m(z)}{\left(\sigma^m(z)\right)^2} z(1-z)\frac{q_1'(z)}{q_1(z)}. \tag{A.34}$$

The instantaneous correlation between $dR_0(t)$ and $dR_1(t)$ is calculated as follows:

$$\begin{aligned}\chi(z) = {} & \frac{z(1-z)}{\sigma_0^r(z)\sigma_1^r(z)}\left[\sigma_0^2\frac{q_1'(z)}{q_1(z)}\left(z(1-z)\frac{q_0'(z)}{q_0(z)} - 1\right) + \sigma_1^2\frac{q_0'(z)}{q_0(z)}\left(z(1-z)\frac{q_1'(z)}{q_1(z)} + 1\right)\right] \\ & - \frac{\rho\sigma_0\sigma_1}{\sigma_0^r(z)\sigma_1^r(z)}\left[2z^2(1-z)^2\frac{q_0'(z)}{q_0(z)}\frac{q_1'(z)}{q_1(z)} + z(1-z)\left(\frac{q_0'(z)}{q_0(z)} - \frac{q_1'(z)}{q_1(z)}\right) - 1\right].\end{aligned} \tag{A.35}$$

For the special case where both sectors are liquid (i.e. two-sector CIR version), we can solve the ODE and decision rules in closed form. The results for this important special case are summarized in Section 5.3. A key result is that the steady-state capital ratio between the two sectors is a constant z^* given by (44) as implied by perfect liquidity of capital.