

Implementing an Optimal Tax Plan with Short Term Debt*

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Abstract

Debortoli, Nunes, and Yared (2021) showed that when initial government debt is too high, Lucas and Stokey's (1983) debt management policy fails to implement a Ramsey plan for flat-rate taxes on labor. We show that using short-term debt to finance accumulated primary deficits and preventing continuation Ramsey planners from resetting current-period tax rates implements all Ramsey plans, including those with the high debt levels that trouble Lucas and Stokey's implementation. Our implementation equalizes Lagrange multipliers on distinct implementability constraints that face the Ramsey planner and the continuation planners, an indicator of a successful implementation. We provide examples with different initial debt structures and government spending patterns.

Keywords: Ramsey planner, continuation Ramsey planner, implementability, short term debt

JEL Classification: E3, E6, C7

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1 Introduction

Lucas and Stokey (1983) implemented an optimal plan for flat-rate taxes on labor by managing the term structure of government debt under a timing protocol that represents an “intermediate situation” of “partial commitment”.¹ In their model, a government finances exogenous sequences $\{G_i\}_{i=0}^\infty$ of government expenditures and initial debt service coupons $\{b_{-1,i}\}_{i=0}^\infty$ with a sequence of flat-rate taxes on labor, the only factor of production. At time 0, a planner chooses a sequence of flat-rate taxes $\{\tau_{0,i}\}_{i=0}^\infty$ and restructures an initial debt coupon process $\{b_{-1,i}\}_{i=0}^\infty$ to become $\{b_{0,i}\}_{i=0}^\infty$. In each period $i > 0$, a continuation planner honors the debt service sequence that it inherits from a period $i - 1$ planner,² but is free to redesign the continuation of the flat-rate tax sequence and to reschedule government debt service from period i onward.³ Lucas and Stokey said that an optimal plan for setting tax rates is “time consistent”⁴ if all continuation planners choose to continue it. Lucas and Stokey (1983, Sec. 3) provided eight examples in which properly rescheduling government debts and issuing consols each period implements a Ramsey plan for flat-rate taxes on labor.

Debortoli, Nunes, and Yared (2021) constructed examples in which initial government debts are so high that in Lucas and Stokey’s way restructuring government debt fails to motivate continuation planners to continue the Ramsey tax plan; coincidentally, the Ramsey plan sets a tax rate above the peak of the Laffer curve; the continuation Ramsey planner instead sets that tax rate below the peak of the Laffer curve.

We modify Lucas and Stokey’s timing protocol by withdrawing from each continuation planner the authority to reset today’s tax rate and adding the authority to set tomorrow’s tax rate. Under our timing protocol, there exist debt management policies, different from Lucas and Stokey’s, that implement a Ramsey plan whenever one exists. We focus on one such policy that each period involves minimally rescheduling the initial debt sequence. This policy uses short-term debt to finance primary government deficits accumulated under the Ramsey tax plan and payments due each period under the initial debt structure $\{b_{-1,i}\}_{i=0}^\infty$.⁵

To appreciate this debt management policy, it is useful to recall how Aguiar, Amador,

¹Because they are not implemented under sequential timing protocols that actually characterize government decision-making processes, Kydland and Prescott (1977) and Calvo (1978) argued that optimal plans are implausible.

²This is the “commitment” part of Lucas and Stokey’s “partial commitment.”

³This is the “partial” part of Lucas and Stokey’s “partial commitment.”

⁴We say that it is “implemented.”

⁵Subsection 5.3 describes other possible policies.

Hopenhayn, and Werning (2019) contrasted Lucas and Stokey's (1983) model with theirs:

Lucas and Stokey (1983) studied optimal fiscal policy with complete markets and discussed at length how maturity choice is a useful tool to provide incentives to a government that lacks commitment to taxes and debt issuance, but cannot default. The government has an incentive to manipulate the risk-free real interest rate, by changing taxes which affects investors' marginal utility, to alter the value of outstanding long-term bonds, something ruled out by our small open-economy framework with risk-neutral investors. Their main result is that the maturity of debt should be spread out, resembling the issuance of consols. Our model instead emphasizes default risk, something absent from their work. Our main result is also the reverse, providing a force for the exclusive use of short-term debt.

Short-term debt takes center stage in our model too, but for different reasons because our economic environment differs from Aguiar et al.'s. Unlike Aguiar, Amador, Hopenhayn, and Werning (2019), we retain almost all parts of Lucas and Stokey's and Debortoli et al.'s structure including complete markets, a closed economy with endogenous interest rates, the presence of incentives that continuation Ramsey planners have to use tax rates to manipulate interest rates, obligations of governments to pay their debts, and Lucas and Stokey's "partial commitment" assumption:

We focus on a situation in which there are no binding commitments on future taxes but in which debt commitments are fully binding. . . . (this) seems to come closest to the institutional arrangement we observe in any stable, democratically governed countries. . . . Our main finding, in this intermediate situation, is that being unable to make commitments about future tax rates is not a constraint. In the absence of any ability to bind choices about tax rates directly, each government restructures the debt in a way that induces its successors to continue with the optimal tax policy. For this to be possible, a rich enough mix of debt instruments must be available, where a rich enough means, roughly, one security for each dated, state-contingent good being traded.

Lucas and Stokey (1983, p. 69)

Lucas and Stokey concluded that commitment to servicing debt suffices to induce successor governments to implement the Ramsey tax plan, but Debortoli et al.'s counterexample shows that actually it may not be enough. We show that if we slightly alter Lucas and Stokey's timing protocol and substantially modifies their debt management policy, commitment to servicing debt is enough.

There generally exist multiple debt management strategies that feature different combinations of short-term and longer maturity debts that implement the Ramsey plan in our setting. We show that, relative to the debt structure of the Lucas and Stokey's implementation, all of them are more tilted towards the short-end of the debt maturity structure. This occurs because by using short-term debt to reduce the average maturity of the debt structure, a continuation planner weakens its successor's temptation to manipulate the equilibrium marginal-rate-of-substitution process and thereby induces its successors to continue the Ramsey plan. This indicates the source of a key advantage of short-term debt as an implementation tool. The more short-term debt a continuation planner uses, the more effectively it addresses this incentive problem. Because the incentive problem worsens as initial debt increases, the fraction of short-term debt required to implement the Ramsey plan (weakly) increases with initial debt (see Proposition 6.) Ultimately, this is why a short-term-debt-only strategy always implements the Ramsey plan, while other strategies do not implement it when initial debt is high enough.

Differences in outcomes between our paper and Aguiar, Amador, Hopenhayn, and Werning (2019) arise from their different settings. Ours is a general-equilibrium setting in which the government cannot default, as in Lucas and Stokey (1983), while theirs is a small open economy as in Eaton and Gersovitz (1981), augmented with term debt, in which the government can default. The difference in settings implies that the problems that short-term debt addresses differ. The basic problem in their setting is the sovereign's incentive to dilute the values of existing *term* debts via default. This default-induced debt-dilution problem resembles a durable-good monopolist's problem. So the key is (1) to leave aside the debt-dilution problem caused by the existing term debts and (2) to avoid creating new debt-dilution problems. This is why it is a good idea to use only short-term debt in their setting. Doing that creates a new exclusive relationship between the sovereign and each new batch of short-term creditors, fully aligning their incentives over time subject to the sovereign's default option. By way of contrast, in our setting, the basic problem is a continuation Ramsey planer's incentive to deviate from the Ramsey plan by manipulating the equilibrium marginal-rate-of-substitution process in a way that will alter the value of

existing term debts without defaulting. This is why we find that it is necessary to use enough short-term debt in our setting.

Related literature

Barro (1979), Lucas and Stokey (1983), and Debortoli, Nunes, and Yared (2021) are the most pertinent references for us, but many other papers discuss closely related issues. Angeletos (2002) shows that by choosing the maturity structure of non-state-contingent bonds, a government can smooth tax rates in response to shocks to its expenditures as effectively as if it had access to complete markets in state-contingent government debt. He provides an example in which the government does that by trading a risk-free one-period bond and a consol. Buera and Nicolini (2004) showed quantitatively that such a government portfolio policy entails large trades, hundreds of times GDP, that would be very sensitive to a model's parameter values. Aiyagari et al. (2002) study optimal taxation in a setting in which a government can issue only one-period risk-free debt. They show that an optimal policy combines features of Barro's (1979) model (a near-unit root component in taxes and debt) and Lucas and Stokey's (1983) (strong dependence of taxes and deficits on current shocks). Like Angeletos (2002) and Buera and Nicolini (2004), Shin (2007) shows that a government can implement complete-market Ramsey allocations by actively managing just a few bonds. Shin uses his model to explain why in the 18th century Britain issued long-term bonds during peacetime and short-term bonds during wars. Bassetto (2014) studies how heterogeneity among economic agents affects optimal government responses to war-time surges in expenditures. If a government favors taxpayers, it runs larger deficits during wars, while if a government favors rentiers who earn most of their income from assets, it raises taxes immediately. He uses his model to understand differences between British and French fiscal policies during the 17th and 18th centuries.

Bhandari et al. (2017) provide conditions under which government debt under a Ramsey plan converges to an invariant distribution. They construct an approximation to that asymptotically invariant distribution as well as an approximation to the rate of convergence to it. They show how binding implementability constraints push government debt in a direction that eventually lets the government use fluctuations in equilibrium interest rates to insure against shocks to government expenditures, rather than using fluctuations in the par value of debt. Aparisi de Lannoy et al. (2025) study optimal taxation and government debts of different maturities, specifying preferences and shock process to match macro and

asset pricing facts, something that previous papers had not done. They provide formulas that express an optimal government portfolio policy as a function of statistics formed from financial and macroeconomic data. For US data, they find that it is optimal for the government to issue bonds in amounts that decline approximately exponentially with increases in their times to maturity.

Zhu (1992) constructs a complete markets model with capital. He finds that while optimal capital income tax rates are indeterminate, their one-period ahead conditional expectation equals zero. Chari, Christiano, and Kehoe (1994) construct a quantitative version of a similar model. They find that in a stationary equilibrium the optimal *ex ante* tax rate on capital income is approximately zero, that optimal labor tax rates exhibit the same persistence as exogenous economic shocks and have low volatility, and that most welfare gains come from setting the initial tax rate at an arbitrary exogenous upper bound. Chari, Nicolini, and Teles (2020) also study optimal capital taxation. They find that when a government has access to enough instruments, capital should not be taxed. Barro and Chari (2024) show that zero taxation of capital income is optimal when a government can't impose a capital levy.

Our work is related to Clymo and Lanteri (2020) who study how to implement a Ramsey plan under “Limited-Time Commitment” (LTC). They show that N periods of commitment are required to implement a Ramsey plan in a setting with N -period debt. Since we let the initial debt structure $\{b_{-1,i}\}_{i=0}^{\infty}$ be an arbitrary sequence as in Lucas and Stokey (1983), Clymo and Lanteri's LTC implementation requires the number of periods of commitment equal the longest debt maturity. By way of contrast, under commitment to a tax rate just one-period ahead, our implementation of the Ramsey plan using a short-term debt management strategy works for any initial term debt structure.

We follow Lucas and Stokey in assuming arbitrary initial term debt structures. Many other papers have instead assumed exponentially maturing term debts with constant amortization rates.⁶ For example, using such exponential term debt structures, DeMarzo and He (2021) extended the classic capital-structure model of (Leland, 1994) in a corporate finance context, while DeMarzo, He, and Tourre (2023) studied a sovereign finance environment. A borrower's incentive to dilute the value of existing debt claims shape dynamics in these papers: a borrower continuously ratchets its term debt upward until it eventually defaults.⁷ DeMarzo, He, and Tourre (2023) specified an exogenous discount factor pro-

⁶See Arellano and Ramanarayanan (2012) and Chatterjee and Eyigungor (2012) for examples in sovereign debt settings.

⁷This debt ratcheting mechanism resembles a mechanism that drives the price of a durable good to zero

cess in the tradition of “small open economy” models. In contrast, in Lucas and Stokey (1983), Aiyagari et al. (2002), and Debortoli, Nunes, and Yared (2021) and our paper, the government manipulates an equilibrium discount factor process within a closed economy.

Organization

Section 2 describes a setting with an infinite horizon and a discrete time increment $\Delta > 0$. A representative household and benevolent government participate in competitive markets with distorting flat-rate taxes on labor. The household supplies labor that produces goods that are allocated between consumption and government expenditures. The government finances an exogenous sequence of government expenditures and an exogenous initial sequence of debt payments by collecting a flat-rate tax on labor. A manifold of competitive equilibria with distorting taxes is indexed by sequences of budget-feasible flat-rate taxes on labor.

Subsection 3.1 describes how a Ramsey planner chooses tax rates to maximize household utility subject to resource and implementability constraints implied by the household’s first-order conditions. Subsection 3.2 introduces continuation Ramsey planners who must honor debt structures inherited from previous planners when they decide whether to confirm the Ramsey planner’s intertemporal plan for setting tax rates. Understanding that, the Ramsey planner chooses how to restructure the exogenous initial government debt structure. This subsection also introduces a different timing protocol that lets continuation planners set taxes one period in advance and prevents them from resetting current-period tax rates.

Section 4 presents a debt management policy that under the new timing protocol implements the optimal plan. Our debt management policy “takes a short route” by leaving longer-term debts unchanged and using short-term debt to finance the resulting accumulated unpaid government liabilities under the optimal plan, defined as $\Pi_j^* = \sum_0^{j-1} \frac{q_{0,i}^*}{q_{0,j}^*} (b_{-1,i} - S_{0,i}^*)$, where $\{q_{0,i}^*\}_{i=0}^\infty$ and $\{S_{0,i}^*\}_{i=0}^\infty$ are bond price and primary surplus processes associated with the Ramsey plan. Note how a period i flow liability $(b_{-1,i} - S_{0,i}^*)$ times $\frac{q_{0,i}^*}{q_{0,j}^*}$ contributes to the period j value of government liabilities. The $\{\Pi_j^*\}_{j=0}^\infty$ process consolidates all equilibrium price and quantity information about the exogenous initial term debt structure that concerns the Ramsey planner. We use forward induction to prove that this debt management policy implements the optimal plan under our timing protocol.

Subsection 4.2 connects conditions for implementing the optimal plan with the invari-

in a durable goods monopoly setting without commitment studied by Coase (1972), Stokey (1981).

ance and equality of Lagrange multipliers on the implementability constraints that confront the planner and the continuation planners. Subsection 4.3 conducts a reverse engineering exercise that reveals mechanics that underlie our implementation. We work backwards from decision rules that express the tax rates chosen by the Ramsey planner and continuation Ramsey planners as functions of period- i government expenditures and debt payments as well as the Lagrange multipliers for their respective implementability constraints. While the functional forms of these decision rules are identical, for the Ramsey planner and for continuation Ramsey planners, different arguments enter these functions. For a Ramsey plan to be implemented, these arguments must be aligned appropriately. This logic puts restrictions on debt management policies and Lagrange multipliers that imply the “short route” debt management policy together with the invariance of the respective Lagrange multipliers.

Section 5 studies Debortoli et al.’s counterexamples in which Lucas and Stokey’s debt management policy fails to implement Ramsey plans when initial debts are too high. We show how our modified timing protocol and debt management policy implement a Ramsey plan regardless of whether Ramsey tax rates fall above or below the peak of a Laffer curve. It is invariance of Lagrange multipliers for Ramsey and continuation plans that is essential for implementability (see subsection 5.2). Subsection 5.3 shows that for some initial debt structures, but not for others, policies that issue both short and longer term debts also implement a Ramsey plan. Using both short- and long-term debt can fail to implement a Ramsey plan, for example, when initial debts are too high. Subsection 5.4 uses numerical examples to compare our implementation with other implementations including Lucas and Stokey’s.

Section 6 describes additional examples that alter initial debt structures and government expenditure patterns. We designed these examples to illustrate covariation of outcomes under different exogenous government expenditure and initial term debt structures. By slightly enriching initial debt structures in Debortoli et al.’s counterexamples, subsection 6.1 shows that whether Lucas-Stokey’s implementation works does not depend on whether the Ramsey planner sets a tax rate on the ‘bad’ side of the Laffer curve. Subsection 6.2 studies Ramsey tax plans and debt management policies that implement them when initial debt payments decline exponentially toward zero, while subsection 6.3 periodic initial debt sequences, and subsection 6.4 studies periodic government spending sequences. Associated optimal plans display different patterns of government accumulations of one-period liabilities and consequent issues of one-period debt that illustrate how our implementation

works.

Section 7 discusses possible extensions. A technical appendix ends our paper.

2 Setting

There is no uncertainty.⁸ Time t_i is discrete with increment $\Delta > 0$, so $\{t_i = i\Delta, i = 0, 1, \dots\}$. Period i means time t_i . A representative household and a benevolent government participate in a complete set of perfectly competitive markets. The representative household supplies a labor supply sequence $\{N_{0,i}\Delta\}_{i=0}^{\infty}$ that produces a sequence of a single nonstorable good that must be allocated between a consumption sequence $\{C_{0,i}\Delta\}_{i=0}^{\infty}$ and an exogenous government expenditure sequence $\{G_i\Delta\}_{i=0}^{\infty}$:

$$C_{0,i}\Delta + G_i\Delta = N_{0,i}\Delta, \text{ for all } i \geq 0, \quad (1)$$

where a first subscript indicates a time that a variable is chosen and a second subscript indicates a time that a variable is realized. The representative household orders consumption and labor supply streams $\{C_{0,i}\}_{i=0}^{\infty}$ and $\{N_{0,i}\}_{i=0}^{\infty}$ according to

$$\sum_{i=0}^{\infty} e^{-\rho i\Delta} U(C_{0,i}, N_{0,i})\Delta, \quad (2)$$

where $\rho > 0$ and $U(\cdot, \cdot)$ is strictly increasing in consumption C , strictly decreasing in labor supply N , globally concave, and continuously differentiable. Along with Debortoli, Nunes, and Yared (2021), we assume that labor supply $\{N_{0,i}\}$ has no upper bound.

In period 0, the government inherits an initial term-debt structure $\{b_{-1,i}\Delta\}_{i=0}^{\infty}$ that it must service: the government must pay its creditors an amount $b_{-1,i}\Delta$ during time interval $[i\Delta, (i+1)\Delta)$. Let $\tau_{0,i}$ denote a period i flat tax rate that the government sets at period $i = 0$ and let $q_{0,i}$ be the period 0 value of a zero-coupon bond with a unit payoff at time i .

⁸Lucas and Stokey (1983, ftn. 1) said that many, if not most, of the issues analyzed in their paper could be analyzed in a context without uncertainty, like the setting analyzed by Turnovsky and Brock (1980). Lucas and Stokey's footnote 2 about the consequences of Turnovsky and Brock's restricting continuation governments to issue only short term debt is especially interesting in view of the substantial difference between Lucas and Stokey's debt management policy and ours.

The representative household faces a single intertemporal budget constraint:

$$\sum_{i=0}^{\infty} (C_{0,i}\Delta) \cdot q_{0,i} \leq \sum_{i=0}^{\infty} (b_{-1,i}\Delta) \cdot q_{0,i} + \sum_{i=0}^{\infty} ((1 - \tau_{0,t_i})N_{0,i}\Delta) \cdot q_{0,i}. \quad (3)$$

The left side of (3) is the present value of the household's consumption stream $\{C_{0,i}\}_{i=0}^{\infty}$ and the right side is the sum of the household's financial wealth and its human wealth, the present value of after-tax labor income.

Given tax rate sequence $\{\tau_{0,i}\}_{i=0}^{\infty}$ and bond price sequence $\{q_{0,i}\}_{i=0}^{\infty}$, the household chooses $\{C_{0,i}\Delta, N_{0,i}\Delta\}_{i=0}^{\infty}$ to maximize (2) subject to constraint (3). First-order necessary conditions for $N_{0,i}$ and $C_{0,i}$ are

$$1 - \tau_{0,i} = -\frac{U_{N,i}}{U_{C,i}} \quad (4)$$

$$q_{0,i} = e^{-\rho i\Delta} \frac{U_{C,i}}{U_{C,0}}. \quad (5)$$

Let

$$S_{0,i}\Delta = (\tau_{0,i}N_{0,i} - G_i) \Delta \quad (6)$$

denote the primary government budget surplus over time interval $[i\Delta, (i+1)\Delta)$. Constraint (3) becomes⁹

$$\sum_{i=0}^{\infty} (S_{0,i}\Delta) \cdot q_{0,i} \geq \sum_{i=0}^{\infty} (b_{-1,i}\Delta) \cdot q_{0,i}. \quad (7)$$

The left side is the time 0 value of primary government budget surplus $\{S_{0,i}\Delta\}_{i=0}^{\infty}$ and the right side is the time 0 value of the government debt payout sequence $\{b_{-1,i}\Delta\}_{i=0}^{\infty}$.

Definition 1. *Given the government's initial term-debt structure $\{b_{-1,i}\Delta\}_{i=0}^{\infty}$ and expenditure sequence $\{G_i\Delta\}_{i=0}^{\infty}$, and a flat-rate tax sequence $\{\tau_{0,i}\}_{i=0}^{\infty}$, a competitive equilibrium is a feasible allocation $\{C_{0,i}\Delta, N_{0,i}\Delta\}_{i=0}^{\infty}$, and a bond price sequence $\{q_{0,i}\}_{i=0}^{\infty}$ for which*

- *the government's budget constraint (1) is satisfied, and*
- *the allocation $\{C_{0,i}\Delta, N_{0,i}\Delta\}_{i=0}^{\infty}$ solves the household's optimization problem.*

We follow Lucas and Stokey (1983) and use first-order conditions (4) and (5) together with feasibility constraint (1) to eliminate tax rates and bond prices from the government

⁹Competitive equilibrium and Walras' Law underlies this assertion.

budget constraint (7) and thereby to deduce the following implementability constraint on competitive equilibrium allocations $\{C_{0,i}\Delta, N_{0,i}\Delta\}_{i=0}^{\infty}$:

$$\sum_{i=0}^{\infty} e^{-\rho i \Delta} [C_{0,i}U_{C,i} + (C_{0,i} + G_i)U_{N,i} - b_{-1,i}U_{C,i}] \cdot \Delta \geq 0. \quad (8)$$

Proposition 1. *Given an initial term-debt structure $\{b_{-1,i}\Delta\}_{i=0}^{\infty}$ and government spending sequence $\{G_i\Delta\}_{i=0}^{\infty}$, an allocation $\{C_{0,i}\Delta, N_{0,i}\Delta\}_{i=0}^{\infty}$ is a competitive equilibrium allocation if and only if it satisfies (8) at $i = 0$ and (1) for all $i \geq 0$.*

3 Ramsey and Continuation Ramsey Plans

A Ramsey planner and a sequence of continuation Ramsey planners are linked together by a structure of ‘partial commitments’ like those described by Lucas and Stokey (1983, p. 69) in which debts incurred by previous planners must be paid or rescheduled at equilibrium prices, but sequences of tax rates can be redesigned each period.

3.1 Ramsey Plan

A Ramsey planner chooses the best competitive equilibrium. It can accomplish this by choosing a sequence $\{C_{0,i}\Delta\}_{i=0}^{\infty}$ that maximizes (2) subject to resource constraint (1) and implementability constraint (8). Given that sequence, we can use first-order necessary conditions (4) and (5) to compute associated equilibrium tax rate and bond price sequences.

We denote competitive equilibrium objects associated with a Ramsey plan as

$$\{C_{0,i}^*\Delta, N_{0,i}^*\Delta, \tau_{0,i}^*, q_{0,i}^*\}_{i=0}^{\infty}.$$

An equivalence class \mathcal{D} of continuation debt structures $\{b_{0,i}\}_{i=1}^{\infty}$ supports a Ramsey tax plan.¹⁰

Characterizing Ramsey plan. Attach a nonnegative multiplier Φ_0 to implementability constraint (8) and form a Lagrangian:

$$\mathcal{L}_0 = \sum_{i=0}^{\infty} e^{-\rho i \Delta} [U(C_{0,i}, C_{0,i} + G_i) + \Phi_0 (C_{0,i}U_{C,i} + (C_{0,i} + G_i)U_{N,i} - b_{-1,i}U_{C,i})] \Delta. \quad (9)$$

¹⁰In complete market structures like ours, this is a typical portfolio irrelevance outcome.

The Ramsey planner maximizes the right side of (9) over consumption plans $\{C_{0,i}\}_{i=0}^{\infty}$ and minimizes over multiplier Φ_0 . Call the extremizing values: $\{C_{0,i}^*\}_{i=0}^{\infty}$ and Φ_0^* . For a given Φ_0 , the optimal consumption plan satisfies

$$C(\Phi_0; b_{-1,i}, G_i) := \arg \max_{C_{0,i}} [U(C_{0,i}, C_{0,i} + G_i) + \Phi_0 (C_{0,i} U_{C,i} + (C_{0,i} + G_i) U_{N,i} - b_{-1,i} U_{C,i})]. \quad (10)$$

Substituting (10) into implementability condition (8), we obtain

$$\sum_{i=0}^{\infty} e^{-\rho i \Delta} [C(\Phi_0; b_{-1,i}, G_i) U_{C,i} + (C(\Phi_0; b_{-1,i}, G_i) + G_i) U_{N,i} - b_{-1,i} U_{C,i}] \Delta \geq 0. \quad (11)$$

Substituting $C(\Phi_0; b_{-1,i}, G_i)$ into the right side of (9) lets us write \mathcal{L}_0 as a function $\mathcal{L}_0(\Phi_0)$. An optimal Φ_0^* that maximizes $\mathcal{L}_0(\Phi_0)$ must satisfy (11). The Ramsey allocation can be represented with two functions, namely, $C_{0,i}^* = C(\Phi_0^*; b_{-1,i}, G_i)$ and $N_{0,i}^* = C_{0,i}^* + G_i = N(\Phi_0^*; b_{-1,i}, G_i)$.

Proposition 2. *Under regularity conditions 1 and 2 described in Appendix A, (11) holds with equality and there exists a positive root Φ_0^* of (11) that solves*

$$\Phi_0^* = \arg \min_{\Phi_0} \mathcal{L}_0(\Phi_0),$$

where the Ramsey allocation, tax rate, and bond price sequence satisfy

$$C_{0,i}^* = C(\Phi_0^*; b_{-1,i}, G_i); \quad N_{0,i}^* = C_{0,i}^* + G_i; \quad (12)$$

$$\tau_{0,i}^* = 1 + \frac{U_{N,i}(C_{0,i}^*, N_{0,i}^*)}{U_{C,i}(C_{0,i}^*, N_{0,i}^*)}; \quad q_{0,i}^* = e^{-\rho i \Delta} \frac{U_{C,i}(C_{0,i}^*, N_{0,i}^*)}{U_{C,0}(C_{0,0}^*, N_{0,0}^*)}. \quad (13)$$

The value \mathcal{L}_0^* of a Ramsey plan is $\sum_{i=0}^{\infty} e^{-\rho i \Delta} U(C_{0,i}^*, C_{0,i}^* + G_i) \Delta$.

See Appendix A for a proof and supporting technical details.

3.2 Continuation Ramsey Plans

There is a sequence of continuation Ramsey planners indexed by $j \in \{1, 2, \dots\}$, each of whom manipulates a continuation representative household. A period j continuation planner confronts a debt structure $\{b_{j-1,i}\}_{i=j}^{\infty}$ chosen by the preceding planner ($j-1$) and the tail government spending rate process $\{G_i\}_{i=j}^{\infty}$. As Section 5 notes, there exist many debt

term structures $\{b_{0,t}\}_{t=0}^{\infty}$ that support a Ramsey plan whenever one exists, so the term debt structure is indeterminate. There also exist many debt maturity structures open to the period $(j - 1)$ continuation planner.

Facing $\{b_{j-1,i}\}_{i=j}^{\infty}$ and $\{G_i\}_{i=j}^{\infty}$, the period j continuation planner faces an intertemporal budget constraint:

$$\sum_{i=j}^{\infty} (S_{j,i}\Delta) \cdot q_{j,i} \geq \sum_{i=j}^{\infty} (b_{j-1,i}\Delta) \cdot q_{j,i}, \quad (14)$$

where $S_{j,i}\Delta = (\tau_{j,i}N_{j,i} - G_i)\Delta$ is the primary surplus for period j government over time interval $[i\Delta, (i + 1)\Delta)$. The period j continuation planner seeks continuation sequences $\{C_{j,i}, N_{j,i}\}_{i=j}^{\infty}$ that maximize

$$\sum_{i=j}^{\infty} e^{-\rho(i-j)\Delta} U(C_{j,i}, N_{j,i})\Delta \quad (15)$$

subject to continuation implementability constraint (CIC):

$$\sum_{i=j}^{\infty} e^{-\rho(i-j)\Delta} (C_{j,i}U_{C,i} + (C_{j,i} + G_i)U_{N,i} - b_{j-1,i}U_{C,i})\Delta \geq 0. \quad (16)$$

Our period j continuation planner confronts the tail government spending rate process of $\{G_i\}_{i=j}^{\infty}$ and a debt structure $\{b_{j-1,i}\}_{i=j}^{\infty}$ inherited from the preceding planner. The period j continuation planner cannot reset $\tau_{j-1,j}$. It sets a flat-rate tax $\tau_{j,j+1}$ that the period $j + 1$ continuation planner cannot reset and sets (actually, just “recommends”) a continuation tax plan $\{\tau_{j,i}\}_{i=j+2}^{\infty}$ that a period $j + 1$ continuation planner is free to reset. Thus, the period j continuation Ramsey planner cannot reset the period j tax rate.¹¹ Knowing the period j continuation government’s tax plan, a period j continuation household orders $\{C_{j,i}, N_{j,i}\}_{i=j}^{\infty}$ according to the continuation utility functional (15).

Definition 2. *A continuation of a Ramsey plan in period j is the tail of the Ramsey plan for $i \geq j$: $\{C_{0,i}^*\Delta, N_{0,i}^*\Delta, \tau_{0,i}^*, q_{0,i}^*\}_{i=j}^{\infty}$. A continuation Ramsey planner in period j must administer tax rate $\tau_{j-1,j}$ for period j and finance an inherited continuation debt structure $\{b_{j-1,i}\Delta\}_{i=j}^{\infty}$ but is free to choose a continuation tax rate plan $\{\tau_{j,i}\}_{i=j+1}^{\infty}$. It also passes on a*

¹¹We thus require that the tax-setting authority have a commitment horizon that is (weakly) longer than the shortest debt maturity. While we have not conducted a thorough study, a quick look at the history of US Federal income tax legislation indicates that most, but not all, of the time, tax rates were reset infrequently relative to debt reschedulings. We acknowledge that exceptions did occur – e.g., some of the Bush tax rate changes in the early 2000’s, some of which were actually retroactive.

rescheduled continuation debt structure $\{b_{j,i}\Delta\}_{i=j+1}^{\infty}$ to a period $i + 1$ continuation planner. A Ramsey plan is said to be **implemented**¹² if each continuation Ramsey planner confirms the Ramsey planner's tax rate plan $\{\tau_{0,i}^*\}_{i=0}^{\infty}$.

Constructing continuation Ramsey plan. Attach a nonnegative multiplier Φ_j to CIC (16) and form a Lagrangian:

$$\mathcal{L}_j = \sum_{i=j}^{\infty} e^{-\rho(i-j)\Delta} [U(C_{j,i}, C_{j,i} + G_i) + \Phi_j (C_{j,i}U_{C,i} + (C_{j,i} + G_i)U_{N,i} - b_{j-1,i}U_{C,i})] \Delta. \quad (17)$$

Call the extremizing values: $\{C_{j,i}^*\}_{i=j}^{\infty}$ and Φ_j^* . Note that under our timing protocol, continuation households' period j consumption satisfies $C_{j,j}^* = C_{j-1,j}^*$ because of the FOCs. For a given Φ_j , the optimal consumption plan for $i \geq j + 1$ satisfies

$$C(\Phi_j; b_{j-1,i}, G_i) := \arg \max_{C_{j,i}} [U(C_{j,i}, C_{j,i} + G_i) + \Phi_j (C_{j,i}U_{C,i} + (C_{j,i} + G_i)U_{N,i} - b_{j-1,i}U_{C,i})]. \quad (18)$$

Substituting (10) into the CIC (16), we obtain

$$\begin{aligned} & \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} [C(\Phi_j; b_{j-1,i}, G_i)U_{C,i} + (C(\Phi_j; b_{j-1,i}, G_i) + G_i)U_{N,i} - b_{j-1,i}U_{C,i}] \Delta \\ & + [C_{j,j}^*U_{C,j} + (C_{j,j}^* + G_j)U_{N,j} - b_{j-1,j}U_{C,j}] \Delta \geq 0. \end{aligned} \quad (19)$$

Similarly, we obtain the continuation Ramsey plan:

Proposition 3. *Under regularity conditions 1 and 3 described in Appendix A and timing protocol given in Definition 2, (19) holds with equality and there exists a positive root Φ_j^* of (19) that solves*

$$\Phi_j^* = \arg \min_{\Phi_j} \mathcal{L}_j(\Phi_j),$$

where the continuation Ramsey allocation and tax rate sequence for $i \geq j + 1$ satisfy

$$C_{j,i}^* = C(\Phi_j^*; b_{j-1,i}, G_i); \quad N_{j,i}^* = C_{j,i}^* + G_i; \quad \tau_{j,i}^* = 1 + \frac{U_{N,i}(C_{j,i}^*, N_{j,i}^*)}{U_{C,i}(C_{j,i}^*, N_{j,i}^*)}. \quad (20)$$

The value \mathcal{L}_j^* of a continuation Ramsey plan is $\sum_{i=j}^{\infty} e^{-\rho(i-j)\Delta} U(C_{j,i}^*, C_{j,i}^* + G_i) \Delta$.

¹²Chari and Kehoe (1993) and Chari, Nicolini, and Teles (2020) characterize an implementation of the Ramsey plan in the context of a Markov equilibrium.

Choices of Ramsey planner and continuation Ramsey planners. While the Ramsey planner chooses both current and future tax rates ($\tau_{0,0}$ and $\tau_{0,i}; i \geq 1$), the period- j continuation planner chooses future taxes ($\tau_{j,i}; j \geq i$) but not the current tax $\tau_{j,j}$. In Appendix C we describe a setup with an altered timing protocol in which the Ramsey planner's choice set looks like a continuation Ramsey planner's. There, we assume that the Ramsey planner inherits a time-0 tax rate that it cannot alter by imposing $\tau_{0,0} = \tau_{-1,0}$.

3.3 Lucas and Stokey's Timing Protocol

In Lucas and Stokey's (1983) timing protocol, a period j planner sets $\tau_{j,j}$ as well as a continuation tax plan $\{\tau_{j,i}\}_{i=j+1}^{\infty}$ that a period $j+1$ continuation planner can reset. For instance, Lucas and Stokey (1983) constructed examples of government expenditure processes and restructured debt processes $\{b_{0,i}\Delta\}_{i=1}^{\infty}$ that induce period 1 continuation planner to adhere to continuations of a Ramsey tax plan $\{\tau_{0,i}^*\}_{i=1}^{\infty}$ and its associated allocation and price system $\{C_{0,i}^*\Delta, N_{0,i}^*\Delta, q_{0,i}^*\Delta\}_{i=1}^{\infty}$. Debortoli, Nunes, and Yared (2021) constructed examples in which continuation term debt structures $\{b_{0,i}\Delta\}_{i=1}^{\infty}$ like those recommended by Lucas and Stokey (1983) fail to induce continuation Ramsey planners to continue a Ramsey plan. We want to implement the Ramsey plan even examples like Debortoli et al.'s. To do that we shall modify Lucas and Stokey's timing protocol and then construct a different debt-management strategy.

4 Implementation

4.1 Debt Management

A sequence of primary surplus $S_{0,i}^*\Delta$ for period i given in (6) is an important component of our implementation of a Ramsey plan. Another key component is a scalar Π_j^* that tracks unpaid government liabilities accumulated under a Ramsey plan from period 0 to $j-1$, conditional on no initial term debts before period $j-1$, $\{b_{-1,i}\}_{i \leq j-1}$, having been rescheduled. It is defined by the following equation:

$$\Pi_j^* = \sum_{i=0}^{j-1} \frac{q_{0,i}^*}{q_{0,j}^*} (b_{-1,i} - S_{0,i}^*) \Delta \quad \text{with} \quad \Pi_0^* = 0. \quad (21)$$

Note that $(b_{-1,i} - S_{0,i}^*)$ describes the part of the initial term debt in period i , $b_{-1,i}$, that

is not paid by the contemporaneous period's primary surplus $S_{0,i}^*$. Evaluating it at bond price ratio $\frac{q_{0,i}^*}{q_{0,j}^*}$, we obtain its period- j value. The stock Π_j^* sums period- j values of these unpaid liabilities from periods 0 through $j - 1$. The time $j - 1$ continuation planner must finance Π_j^* . Let $r_{0,j}$ denote the equilibrium interest rate in period j . Along the Ramsey plan, $r_{0,j}^* = \frac{1}{\Delta} \log(q_{0,j}^*/q_{0,j+1}^*)$ and Π_j^* follows a recursion:

$$\Pi_{j+1}^* = (\Pi_j^* + (b_{-1,j} - S_{0,j}^*)\Delta) e^{r_{0,j}^*\Delta}. \quad (22)$$

We now describe a debt management policy that implements the Ramsey plan under our timing protocol. First, this policy leaves term debts with periods to maturity weakly larger than two unchanged, so that

$$b_{j-1,i}\Delta = b_{-1,i}\Delta, \quad j \geq 1, i \geq j + 1. \quad (23)$$

Second, to finance the sum of (i) the accumulated liability Π_j^* given in (21) and (ii) the initial term debt $b_{-1,j}\Delta$ due in period j , the debt management strategy tells the period $j - 1$ government to issue just enough one-period (short-term) debt that will fall due in period j . Consequently, one-period debt issuance satisfies

$$b_{j-1,j}\Delta = \Pi_j^* + b_{-1,j}\Delta, \quad j \geq 1. \quad (24)$$

Equation (24) evidently “takes the short route.” In subsection 4.2, we focus on this “short-route” policy. Later in section 5 we shall show that there exist other debt maturity structures that a period $(j - 1)$ continuation planner can use to implement the Ramsey plan.

4.2 Implementation Theorem

We use forward induction starting with the first continuation Ramsey planner. Assume that the period $j - 1$ planner confirmed the Ramsey tax plan by setting $\{\tau_{j-1,i} = \tau_{0,i}^*\}_{i=j}^\infty$. Consider the period $j \geq 1$ continuation problem. The period j government inherits longer term debt $\{b_{j-1,i}\Delta = b_{-1,i}\Delta\}_{i=j+1}^\infty$ and one-period debt $b_{j-1,j}\Delta$. Because the period j continuation planner cannot reset the period j tax rate $\tau_{j-1,j}$, the period j 's government's tax rate equals $\tau_{j-1,j} = \tau_{0,j}^*$ and consequently $C_{j,j} = C_{j-1,j} = C_{0,j}^*$. The period j continuation planner is free to choose a new continuation consumption sequence $\{C_{j,i}\Delta\}_{i=j+1}^\infty$ that need

not equal $\{C_{0,i}^* \Delta\}_{i=j+1}^\infty$. Substituting the debt management policy described in equations (23) and (24) into CIC (16), we obtain:

$$\begin{aligned} & \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} (C_{j,i} U_{C,i} + (C_{j,i} + G_i) U_{N,i} - b_{-1,i} U_{C,i}) \Delta \\ & + (C_{0,j}^* U_{C,j} + (C_{0,j}^* + G_j) U_{N,j} - b_{-1,j} U_{C,j}) \Delta - \Pi_j^* U_{C,j} \geq 0. \end{aligned} \quad (25)$$

Attach a nonnegative Lagrange multiplier Φ_j to (25), take the period j continuation government's objective function (15), and form a continuation Lagrangian \mathcal{L}_j :

$$\begin{aligned} \mathcal{L}_j = & \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} [U(C_{j,i}, N_{j,i}) + \Phi_j (C_{j,i} U_{C,i} + (C_{j,i} + G_i) U_{N,i} - b_{-1,i} U_{C,i})] \Delta \\ & + [U(C_{0,j}^*, N_{0,j}^*) + \Phi_j (C_{0,j}^* U_{C,j} + (C_{0,j}^* + G_j) U_{N,j} - b_{-1,j} U_{C,j})] \Delta - \Phi_j \Pi_j^* U_{C,j}. \end{aligned} \quad (26)$$

The continuation planner maximizes (26) with respect to $\{C_{j,i}\}_{i=j+1}^\infty$ and minimizes it with respect to Φ_j . For $i \geq j + 1$, the continuation planner sets $C_{j,i}$ to solve

$$\max_{C_{j,i}} \{U(C_{j,i}, C_{j,i} + G_i) + \Phi_j (C_{j,i} U_{C,i} + (C_{j,i} + G_i) U_{N,i} - b_{-1,i} U_{C,i})\}, \quad (27)$$

so the maximizer takes the form $C_{j,i} = C(\Phi_j; b_{-1,i}, G_i)$ for $i \geq j + 1$, where $C(\cdot; \cdot, \cdot)$ is given in (10).

Proposition 4. *Under regularity conditions 1 and 4 described in Appendix A, (25) holds with equality and $I_j(\Phi_j^*) = 0$ where*

$$\begin{aligned} I_j(\Phi_j) = & \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} (C(\Phi_j; b_{-1,i}, G_i) U_{C,i} + (C(\Phi_j; b_{-1,i}, G_i) + G_i) U_{N,i} - b_{-1,i} U_{C,i}) \Delta \\ & + (C_{0,j}^* U_{C,j} + (C_{0,j}^* + G_j) U_{N,j} - b_{-1,j} U_{C,j}) \Delta - \Pi_j^* U_{C,j}. \end{aligned} \quad (28)$$

The continuation Ramsey allocation for period j satisfies

$$C_{j,i}^* = C(\Phi_j^*; b_{-1,i}, G_i); \quad N_{j,i}^* = C_{j,i}^* + G_i; \quad i \geq j + 1 \quad (29)$$

See Appendix A for a proof.

Lemma 1. *The Lagrange multiplier Φ_0^* associated with the Ramsey plan satisfies $I_j(\Phi_0^*) = 0$.*

See Appendix A for a proof.

Problem (27) that confronts a period j continuation planner who is choosing $C_{j,i}$ for $i \geq j + 1$ is static, as is problem (10) that had confronted the Ramsey planner who chose $C_{0,i}$ for $i \geq 0$. Consequently, $C_{j,i} = C_{0,i}$ for $i \geq j + 1$ if and only if Φ_j in (27) equals Φ_0^* in (10).

Theorem 1. *Suppose that*

$$U(C, N) = \log C - \eta \frac{N^\gamma}{\gamma} \quad \text{with } \eta > 0 \quad \text{and} \quad \gamma \geq 1. \quad (30)$$

A Ramsey plan exists under Conditions 1 and 4 described in Appendix A. Under the timing protocol described in Definition 2, the plan can be implemented by using the debt management strategy provided in (23) and (24). Lagrange multipliers of continuation planners equal the Lagrange multiplier of the Ramsey planner: $\Phi_j^ = \Phi_0^*$ for $j \geq 1$.*

Proof. We use forward induction. We suppose that the Ramsey plan has been implemented up to period $j - 1$ and then verify that it can also be implemented in period j . We begin by verifying that (30) satisfies Condition 1. Together with Condition 4, Proposition 4 implies that the continuation planner's optimal Lagrange multiplier $\Phi_j^* > 0$ is positive and that the CIC (25) must hold with equality so that $I_j(\Phi_j^*) = 0$. Substituting (30) into (27), we obtain $C_{j,i} = C(\Phi_j; b_{-1,i}, G_i)$ for $i \geq j + 1$, where $C(\Phi; b, G)$ satisfies the following first-order necessary condition for the problem defined on the right side of equation (27):

$$F(C, \Phi_j) := \frac{1}{C} - \eta(C + G)^{\gamma-1} + \Phi_j \left(\frac{b}{C^2} - \eta\gamma(C + G)^{\gamma-1} \right) = 0. \quad (31)$$

The period j continuation planner confronts the tail of the initial term debt service sequence $\{b_{-1,i}\Delta; i \geq j + 1\}$ as well as a short-term debt balance of $b_{j-1,j}\Delta = \Pi_j^* + b_{-1,j}\Delta$. Using $N(\Phi_j; b_{-1,i}, G_i) = C(\Phi_j; b_{-1,i}, G_i) + G_i$ to simplify the CIC (28), we obtain:

$$I_j(\Phi_j) = \left(1 - \eta(C_{j,j} + G_j)^\gamma - \frac{b_{-1,j}}{C_{j,j}} \right) \Delta - \frac{\Pi_j^*}{C_{j,j}} + \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} \left(1 - \eta(C_{j,i} + G_i)^\gamma - \frac{b_{-1,i}}{C_{j,i}} \right) \Delta, \quad (32)$$

where $C_{j,j} = C_{j-1,j} = C_{0,j}^*$ under our timing protocol.

To show that $I_j(\Phi_j) = 0$ has a unique positive root where $I_j(\Phi_j) = 0$ is defined by CIC (32), it is necessary and sufficient to show that $\sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} \left(1 - \eta(C_{j,i} + G_i)^\gamma - \frac{b_{-1,i}}{C_{j,i}}\right)$, the last term in (32), is strictly monotone in Φ_j . Denote $P(\Phi_j; b, G) = 1 - \eta(C(\Phi_j; b, G) + G)^\gamma - b/C(\Phi_j; b, G)$. It is then sufficient to prove that $P(\Phi_j; b, G)$ is a strictly monotone function of Φ_j . Recognizing that C is a function of Φ_j and differentiating $F(C, \Phi_j) = 0$ given in (31), we obtain

$$\frac{\partial F}{\partial C} \frac{\partial C}{\partial \Phi_j} + \frac{\partial F}{\partial \Phi_j} = 0, \quad (33)$$

where $\frac{\partial F}{\partial C} = -1/C^2 - \eta(\gamma - 1)(C + G)^{\gamma-2} - \Phi_j \eta \gamma (\gamma - 1)(C + G)^{\gamma-2} - 2\Phi_j b/C^3 < 0$ and $\frac{\partial F}{\partial \Phi_j} = (b/C^2 - \eta \gamma (C + G)^{\gamma-1})$. Differentiate $P(\Phi_j; b, G)$ and use (33) and $\frac{\partial F}{\partial C} < 0$ to obtain

$$\frac{\partial P}{\partial \Phi_j} = \left[b/C^2 - \eta \gamma (C + G)^{\gamma-1} \right] \frac{\partial C}{\partial \Phi_j} = -\frac{\partial F}{\partial C} \left(\frac{\partial C}{\partial \Phi_j} \right)^2 > 0. \quad (34)$$

We conclude that $P(\Phi_j; b, G)$ and therefore $\sum_{i=j+1}^{\infty} P(\Phi_j; b_{-1,i}, G_i)$ are both strictly monotone in Φ_j . Since $I_j(\Phi_j)$ is strictly monotone in Φ_j , $I_j(\Phi_j) = 0$ has a unique root. Applying Lemma 1 lets us conclude that the Ramsey plan is implemented. □

Two remarks about implementation. *Remark 1:* Because expenditures G are exogenous in our setting, as in Lucas and Stokey (1983) and Debortoli, Nunes, and Yared (2021), for implementation we require commitment to a tax rate one period ahead. In a more general setting with endogenous G , governments must commit not only to taxes one period ahead but also to expenditures and transfers one period ahead. This is because a continuation planner must ensure that the equilibrium marginal rate of substitution process that prices all financial claims be the same one that the Ramsey planner chose. *Remark 2:* Our short-term-debt strategy also implements the modified Ramsey plan described at the end of subsection 3.2 in which the Ramsey planner confronts a time-0 tax rate that it cannot alter.

4.3 Mechanics

Our implementation induces a period $j \geq 1$ planner to set continuation allocation $\{C_{j,i}^*\}_{i \geq j}$ to the tail of the Ramsey allocation $\{C_{0,i}^*\}_{i \geq j}$, so continuation tax rates, asset prices, and debt management plans align to support the Ramsey tax plan and associated outcomes.

To understand mechanics underlying these outcomes, first recall from Propositions 2 and 4 that for $i \geq j + 1$

$$C_{0,i}^* = C(\Phi_0^*; b_{-1,i}, G_i) \quad \text{and} \quad C_{j,i}^* = C(\Phi_j^*; b_{j-1,i}, G_i). \quad (35)$$

In (35), different *arguments* in the same *function* determine $C_{0,i}^*$ and $C_{j,i}^*$. A good way to understand our implementation is to find arguments of (35) that guarantee that $C_{0,i}^* = C_{j,i}^*$ for all $j \geq 1$. To do that, we reverse engineer (1) continuation debt-management policies that satisfy $b_{j-1,i} = b_{-1,i}$ for $i \geq j + 1$, and (2) assure that $\Phi_j^* = \Phi_0^*$ for all $j \geq 1$.

If we impose reverse engineering specification (1) and also assume that the Ramsey plan has been followed up to period $j - 1$, at the beginning of period $j - 1$ the government has accumulated Π_j^* given by (21), evaluated at time j prices. The period $j - 1$ planner must finance both Π_j^* and the initial term debt due $b_{-1,j}\Delta$. To do that, the period $j - 1$ planner issues one-period debt equal to $b_{j-1,j}\Delta = \Pi_j^* + b_{-1,j}\Delta$. This is the short-term-debt-only policy given in (24).

To establish that $\Phi_j^* = \Phi_0^*$, recall that period j 's continuation planner's Lagrange multiplier Φ_j^* satisfies

$$\begin{aligned} & \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} (C(\Phi_j^*; b_{-1,i}, G_i)U_{C,i} + (C(\Phi_j^*; b_{-1,i}, G_i) + G_i)U_{N,i} - b_{-1,i}U_{C,i}) \Delta \\ & + (C_{j,j}^*U_{C,j} + (C_{j,j}^* + G_j)U_{N,j} - b_{-1,j}U_{C,j}) \Delta - \Pi_j^*U_{C,j} = 0 \end{aligned} \quad (\text{CIC})$$

and that the Ramsey planner's Lagrange multiplier Φ_0^* satisfies

$$\begin{aligned} & \sum_{i=j+1}^{\infty} e^{-i\rho\Delta} (C(\Phi_0^*; b_{-1,i}, G_i)U_{C,i} + (C(\Phi_0^*; b_{-1,i}, G_i) + G_i)U_{N,i} - b_{-1,i}U_{C,i}) \Delta \\ & + (C_{0,j}^*U_{C,j} + (C_{0,j}^* + G_j)U_{N,j} - b_{-1,j}U_{C,j}) \Delta - \Pi_j^*U_{C,j} = 0, \end{aligned} \quad (\text{IC})$$

Setting $C_{j,j}^* = C_{0,j}^*$ makes the second term in (CIC) for period j 's continuation planner identical to the second term in (IC) for the Ramsey planner. Furthermore, if $\Phi_j^* = \Phi_0^*$, the first term in (CIC) for period j 's continuation planner equals the first term in (IC) for the Ramsey planner. Thus, both terms agree, so both equations hold.

We can now verify that $\Phi_j^* = \Phi_0^*$ and $C_{j,j}^* = C_{0,j}^*$ for all $j \geq 1$ under our timing protocol for setting taxes, which requires that the period j continuation planner must administer tax rate $\tau_{j-1,j}$ for period j which is set by the period $j - 1$ planner. This induces continuation

households in period j to choose $C_{j,j}^* = C_{j-1,j}^* = C_{0,j}^*$, which follows from the household's first-order necessary condition $1 - \tau_{j-1,j} = -U_{N,j}(C_{j,j}, C_{j,j} + G_j)/U_{C,j}(C_{j,j}, C_{j,j} + G_j)$.¹³ Thus, taken together, our timing protocol and debt management strategy imply $\Phi_j^* = \Phi_0^*$.¹⁴ We have thus reverse engineered a “take the short route” debt-management policy that implements the Ramsey plan.

5 Revisiting Debortoli et al.’s Examples

5.1 Analysis

Debortoli et al. constructed examples in which Lucas and Stokey’s debt management policy does not implement the Ramsey plan under their timing protocol. These examples feature an initial debt structure that takes the form

$$b_{-1,0} = b > 0 \text{ and } b_{-1,i} = 0 \text{ for all } i \geq 1 \quad (36)$$

a high value of b , a constant government spending process $G_i = G$ for all $i \geq 0$, and the utility function (30). After first reviewing why Lucas and Stokey’s debt management policy fails to implement the Ramsey plan in Debortoli et al.’s examples under their timing protocol, we explain how our debt management policy succeeds in implementing the Ramsey plan under our timing protocol.

To begin, let $\{C_{0,0}^* = C_0^*, C_{0,i}^* = C_1^*; i \geq 1\}$ denote the Ramsey plan and let C^{Laffer} denote consumption at the peak of the Laffer curve (see Figure 1.) Debortoli et al. show (i) that there exists $b^* \in (0, \bar{b})$ such that the Ramsey planner sets $C_1^* > C^{\text{Laffer}}$ if $b < b^*$ and $C_1^* < C^{\text{Laffer}}$ if $b > b^*$ (their Proposition 1);¹⁵ and (ii) that if $b < b^*$, then Lucas and Stokey’s debt management policy implements the Ramsey plan, but that if $b > b^*$, then Lucas and Stokey’s debt management policy does not implement the Ramsey plan (their Proposition 2.) Debortoli et al. highlight a coincidence that occurs between C_1^* being tied

¹³Note that for $j = 1$, our timing protocol implies $C_{1,1}^* = C_{0,1}^*$, which ensures the forward induction.

¹⁴This follows from Lemma 1, which states that Ramsey planner’s Lagrange multiplier Φ_0^* is a root of (CIC), and from Theorem 1, which states that (CIC) has a unique root for the utility function given in (30).

¹⁵Debortoli et al. show that a Ramsey plan exists if and only if $b < \bar{b}$, where

$$\bar{b} = \max_{\tilde{C}} \tilde{C} \left[[1 - \eta(\tilde{C} + G)^\gamma] + \frac{e^{-\rho\Delta}}{1 - e^{-\rho\Delta}} (1 - \eta G^\gamma) \right].$$

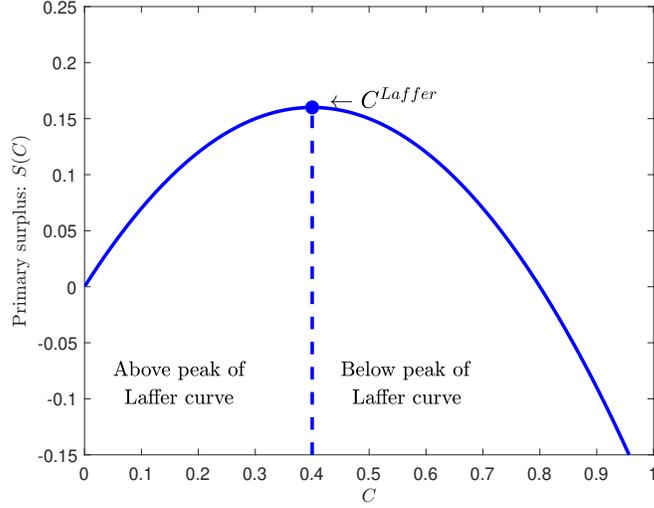


Figure 1: **Laffer curve.** Parameter values: $\gamma = 1$, $\eta = 1$, and $G = 0.2$.

to a tax rate that is above the peak of the Laffer curve and the failure of Lucas and Stokey's debt management policy to implement the Ramsey plan.

Does that coincidence persist for other initial debt structures under Lucas and Stokey's timing protocol? Subsection 6.1 shows that the answer is no. Does that coincidence persist under our timing protocol and debt management policy? The following proposition shows that the answer to this question is also no.

Proposition 5. *For all $b \in (0, \bar{b})$, the following debt management policy implements the Ramsey plan under our Definition 2 timing protocol:*

$$b_{0,1}\Delta = \Pi_1^* \quad \text{and} \quad b_{0,i}\Delta = b_{-1,i}\Delta = 0; \quad i \geq 2, \quad (37)$$

where $\Pi_1^* = \frac{e^{\rho\Delta}C_1^*}{C_0^*} (b - C_0^* (1 - \eta(C_0^* + G)^\gamma)) \Delta$. The Lagrange multiplier for the continuation planner's problem equals the Lagrange multiplier for the Ramsey problem: $\Phi_1^* = \Phi_0^*$.

We can establish this proposition directly by applying Theorem 1, but to highlight the irrelevance for implementability of being on the bad side of the Laffer curve, we instead provide the following alternative proof.

Proof. Under our timing protocol, the Ramsey planner sets $C_{1,1} = C_1^*$. Confronting debt

structure (37), the period 1 continuation planner chooses $\{C_{1,i}, i \geq 2\}$ to maximize

$$\sum_{i=1}^{\infty} e^{-\rho(i-1)\Delta} \left(\log C_{1,i} - \eta \frac{N_{1,i}^{\gamma}}{\gamma} \right) \Delta$$

subject to the CIC:

$$\left(1 - \eta(C_{1,1} + G)^{\gamma} - \frac{b_{0,1}}{C_{1,1}} \right) \Delta + \sum_{i=2}^{\infty} e^{-\rho(i-1)\Delta} \left(1 - \eta(C_{1,i} + G)^{\gamma} - \frac{b_{0,i}}{C_{1,i}} \right) \Delta = 0. \quad (38)$$

The associated Lagrangian is extremized by setting $C_{1,i} = C_{1,2}$ for all $i \geq 3$. Consequently, the CIC (38) and the debt management strategy (37) jointly imply a quadratic equation for $C_{1,2}$:

$$\frac{1}{1 - e^{-\rho\Delta}} \frac{1}{C_{1,2}} (S(C_{1,2}) - b_{-1,2}) = - \left[\frac{e^{2\rho\Delta}}{C_{0,0}^*} (S(C_{0,0}^*) - b_{-1,0}) + \frac{e^{\rho\Delta}}{C_{0,1}^*} (S(C_{0,1}^*) - b_{-1,1}) \right], \quad (39)$$

where $S(C) = C(1 - \eta(C + G)^{\gamma})$ is the primary surplus rate. Equation (13) for setting taxes confirms that a quadratic equation $S(C)$ implies a Laffer curve.

Now rewrite the Ramsey plan's IC as

$$\frac{1}{1 - e^{-\rho\Delta}} \frac{1}{C_{0,2}^*} (S(C_{0,2}^*) - b_{-1,2}) = - \left[\frac{e^{2\rho\Delta}}{C_{0,0}^*} (S(C_{0,0}^*) - b_{-1,0}) + \frac{e^{\rho\Delta}}{C_{0,1}^*} (S(C_{0,1}^*) - b_{-1,1}) \right]. \quad (40)$$

Since the right sides of (39) and (40) are identical, their left sides must be identical too. Consequently, $C_{1,2} = C_1^*$ is a root of the quadratic equation (40). It must be the larger root because if it were not, there would exist another root of (39), \hat{C}_1 , larger than C_1^* . But if that were true, the Ramsey planner would have chosen $\{C_{0,0} = C_0^*, C_{0,1} = C_1^*, C_{0,i} = \hat{C}_1; i \geq 2\}$, contradicting the hypothesis that $\{C_{0,0} = C_0^*, C_{0,1} = C_1^*, C_{0,i} = C_1^*; i \geq 2\}$ is the Ramsey plan and thus verifying Proposition 5. \square

The validity of Proposition 5 does not depend on whether the Ramsey planner sets the tax rate on the good or the bad side of the Laffer curve. Even though a high period 1 tax rate and associated low period 1 consumption are associated with large accumulated government liabilities, under our timing protocol the period 1 continuation planner cannot reset period 1 consumption. This makes the right side of the CIC (39) equal to the right side of the IC (40). By managing only short-term debt and leaving inherited debts maturities weakly larger than two periods unchanged, the period 1 continuation planner continues the

Ramsey tax plan, aligning the left side of its CIC (39) with the IC (40) under the Ramsey plan.

5.2 Three Lagrange Multipliers

Lagrange multipliers of the Ramsey planner and continuation Ramsey planners behave differently under our timing protocol and debt management policy, on the one hand, and under Lucas and Stokey's, on the other hand. Here we focus on the $\eta = \gamma = 1$ case, in which case the Lagrange multiplier for the Ramsey plan satisfies

$$\Phi_0^* = \frac{C_1^* - (C_1^*)^2}{(C_1^*)^2 - b_{-1,i}}. \quad (41)$$

For both Lucas and Stokey's setup and our setup, the Lagrangian for a period-1 continuation planner is

$$\mathcal{L}_1 = \sum_{i=1}^{\infty} e^{-\rho(i-1)\Delta} \left[\log C_{1,i} - (C_{1,i} + G_i) + \Phi_1 \left(1 - (C_{1,i} + G_i) - \frac{b_{0,i}}{C_{1,i}} \right) \right] \Delta, \quad (42)$$

where $\{b_{0,i}\}_{i \geq 1}$ is the debt structure that the Ramsey planner passes on to the period 1 continuation planner. The period 1 continuation planner's first-order condition for $C_{1,i}$ is

$$\frac{1}{C_{1,i}} - 1 + \Phi_1 \left(\frac{b_{0,i}}{C_{1,i}^2} - 1 \right) = 0. \quad (43)$$

We first study outcomes under Lucas and Stokey's timing protocol and debt management policy. Debortoli et al. showed that the Ramsey planner sets

$$b_{0,i} = \frac{(C_1^*)^2 - C_1^*}{\Phi_1} + (C_1^*)^2 \quad \text{for all } i \geq 1, \quad (44)$$

an equation that can be obtained by substituting $C_{1,i} = C_{0,i}^* = C_1^*$, which has to hold for the Ramsey plan to be implemented, into the first-order necessary condition (43). Substituting (44) into the CIC (38), we can obtain what Debortoli et al. call a 'constructed' Lagrange multiplier associated with the period 1 continuation plan, namely,

$$\hat{\Phi}_1 = \frac{1}{2} \frac{1 - C_1^*}{C_1^* - C^{\text{Laffer}}}, \quad (45)$$

where $C^{\text{Laffer}} = (1 - G)/2$. The Lagrange multiplier Φ_0^* associated with the Ramsey plan given by (41) does not equal the “constructed” Lagrange multiplier $\hat{\Phi}_1$ associated with the period 1 continuation Ramsey plan given by (45).

Equation (45) implies that $\hat{\Phi}_1 < 0$ if and only if $C_1^* < C^{\text{Laffer}}$, which by virtue of (13) means that the Ramsey plan puts $\tau_{0,1}^*$ on the ‘wrong’ side of the Laffer curve. When $\hat{\Phi}_1 < 0$, it is not optimal for the continuation planner to continue the Ramsey plan. In this way, Debortoli, Nunes, and Yared (2021) establish that the period 1 continuation planner would never choose a tax rate $\tau_{1,1}^*$ above the peak of the Laffer curve. This verifies that if the initial debt is too high, then under Lucas and Stokey’s timing protocol and debt management strategy, the Ramsey plan is not implemented. When $\hat{\Phi}_1 < 0$, the Lagrange multiplier Φ_1^* associated with the continuation Ramsey plan does not equal the constructed Lagrange multiplier $\hat{\Phi}_1$ associated with the period 1 continuation Ramsey plan given by (45), an inequality that indicates that the Ramsey plan will not be implemented under this timing protocol and debt management policy.

Turning now to our timing protocol given in Definition 2, the period 1 continuation Ramsey planner sets $C_{1,1} = C_{0,1}^*$ and does not adjust term debts with times to maturity (weakly) larger than two, so $b_{0,i} = b_{-1,i}$ for all $i \geq 2$. Substituting these two conditions into (43) delivers the following ‘constructed’ Lagrange multiplier for the period 1 planner:

$$\hat{\Phi}_1 = \frac{C_1^* - (C_1^*)^2}{(C_1^*)^2 - b_{-1,i}}, \quad i \geq 2. \quad (46)$$

Evidently, $\hat{\Phi}_1$ is positive and equals the Ramsey planner’s Lagrange multiplier Φ_0^* given in (41), so $\hat{\Phi}_1 = \Phi_0^* > 0$. Consequently, the Lagrange multiplier Φ_1^* associated with the continuation Ramsey plan equals the constructed Lagrange multiplier $\hat{\Phi}_1$ associated with the period 1 continuation Ramsey plan given by (46).

Taking stock, under Lucas and Stokey’s timing protocol, when b is too high, $\hat{\Phi}_1 < 0$ and the Ramsey plan is not implemented. However, under our timing protocol, $\hat{\Phi}_1 = \Phi_0^* > 0$ and our “short route” debt management policy implements the Ramsey plan.

5.3 Other Debt-Management Policies Sometimes Work

We have established that if a Ramsey plan exists, our short-term debt management strategy implements it. We can also show that for some initial debt structures, but not for others, debt management policies that issue a mixture of short and some longer term debts also

implement a Ramsey plan. We explore other possible implementation here and in Appendix B.

Suppose that in period 0 a Ramsey planner issues one-period debt $b_{0,1}\Delta$ to finance the sum of (i) a fraction $\alpha \in [0, 1]$ of the accumulated liability up to period 1, Π_1^* given in (21), and (ii) the initial term debt $b_{-1,1}\Delta$ due in period 1:

$$b_{0,1}\Delta = \alpha\Pi_1^* + b_{-1,1}\Delta. \quad (47)$$

To finance the remaining liabilities, which include (1) the fraction $(1 - \alpha)$ of Π_1^* and (2) the initial longer-term debts $\{b_{-1,i}\}_{i \geq 2}$, the Ramsey planner issue a portfolio $\{b_{0,i}\}_{i \geq 2}$ of longer-term debts.

Facing $\{b_{0,i}\}_{i \geq 1}$, the period-1 continuation planner's first-order-necessary condition requires that the continuation Lagrange multiplier Φ_1 satisfy (43). Because confirming the Ramsey plan requires $C_{1,i} = C_1^*$ for $i \geq 2$, $b_{0,i}$ must satisfy

$$b_{0,i}\Delta = \frac{C_1^*}{\Phi_1} [-1 + C_1^* + \Phi_1 C_1^*] \Delta. \quad (48)$$

The following proposition states that a minimal amount α^* of short-term debt is required to implement the Ramsey plan.

Proposition 6. *For a given $b \in (0, \bar{b})$, a Ramsey plan can be implemented with debt strategy (47)-(48), with $\alpha \in (\alpha^*(b), 1]$ for some $\alpha^*(b) \in [0, 1)$, where $\alpha^*(b)$ is non-decreasing in b .*

Proof. Substitute (48) into CIC (38) and solve for Φ_1 to obtain the 'constructed' Lagrange multiplier $\hat{\Phi}_1$:

$$\hat{\Phi}_1 = \frac{-(1 - C_1^*)}{-(1 - \alpha)e^{\rho\Delta}(C_1^* - (1 - G)) - C_1^*}. \quad (49)$$

As pointed out by Debortoli, Nunes, and Yared (2021), if the constructed Lagrange multiplier $\hat{\Phi}_1$ is positive, then a Ramsey plan is implemented.

Let $D(\alpha; b)$ denote the denominator of (49). As $C_1^* < 1 - G < 1$ in this Debortoli, Nunes, and Yared (2021) economy, the numerator of (49) is always negative and $D(\alpha; b)$ is a strictly decreasing function of α . If $\alpha = 1$, then $D(1) < 0$ and $\hat{\Phi}_1 = \Phi_0^* = \frac{1 - C_1^*}{C_1^*} > 0$. Consider the two cases: 1) $D(0; b) \geq 0$, and 2) $D(0; b) < 0$. In case 1), there exists $1 > \alpha^*(b) \geq 0$ such that $D(\alpha^*(b); b) = 0$ and therefore, for any $\alpha > \alpha^*(b)$, $D(\alpha; b) < 0$ and $\hat{\Phi}_1 > 0$. In case 2), $\alpha^*(b) = 0$, and for any $\alpha > \alpha^*(b) = 0$, $D(\alpha; b) < 0$ and $\hat{\Phi}_1 > 0$.

Evidently $D(\alpha; b)$ decreases in C_1^* . Because C_1^* is strictly decreasing in b (see Proposition

1 in Debortoli, Nunes, and Yared (2021)), $D(\alpha; b)$ increases in b . Note that $D(\alpha; b)$ is a strictly decreasing function of α . For $0 \leq b_1 < b_2 < \bar{b}$, consider the following cases: 1) $\alpha^*(b_1) > 0$ and $\alpha^*(b_2) > 0$; 2) $\alpha^*(b_1) > 0$ and $\alpha^*(b_2) = 0$; and 3) $\alpha^*(b_1) = 0$ and $\alpha^*(b_2) = 0$. For cases 1) and 2), if $\alpha^*(b_1) > \alpha^*(b_2)$, contradiction arises because $0 \geq D(\alpha^*(b_2); b_2) > D(\alpha^*(b_2); b_1) > D(\alpha^*(b_1); b_1) = 0$. Therefore, we have $\alpha^*(b_1) \leq \alpha^*(b_2)$. For case 3), $\alpha^*(b_1) = \alpha^*(b_2) = 0$. Therefore, $\alpha^*(b)$ is non-decreasing in b . \square

Now consider two special cases. If $\alpha = 1$, then the debt management strategy corresponds to our short-term debt only strategy. If $\alpha = 1 - e^{-\rho\Delta}$, then the debt management strategy corresponds to Lucas and Stokey's consol-based implementation strategy in which $b_{0,1} = b_{0,i}$ for $i \geq 2$ and $\hat{\Phi}_1$ given in (49) simplifies to (45). According to Proposition 6, Lucas and Stokey's consol-based implementation strategy fails for an initial debt b so high that $\alpha^*(b) > 1 - e^{-\rho\Delta}$.

In general, a strategy associated with a given α fails to implement the Ramsey plan if initial debt b is too high. A necessary condition for a successful implementation is a positive 'constructed' Lagrange multiplier: $\hat{\Phi}_1 > 0$. This condition requires that $-(1 - \alpha)e^{\rho\Delta} + 1)C_1^* + (1 - \alpha)e^{\rho\Delta}(1 - G) < 0$ or equivalently

$$C_1^* > \frac{(1 - \alpha)e^{\rho\Delta}(1 - G)}{(1 - \alpha)e^{\rho\Delta} + 1}. \quad (50)$$

As C_1^* is strictly decreasing in b (see Debortoli et al.) and the right side of (50) is increasing in α , there exists a threshold $\tilde{b}(\alpha)$ that is an increasing function of α for all $\alpha \leq 1$ such that inequality (50) prevails if $b < \tilde{b}(\alpha)$. Thus, for a given α , $\hat{\Phi}_1 > 0$; a mixed short- and long-term debt issuance policy (47)-(48) implements a Ramsey plan only when $b < \tilde{b}(\alpha)$.¹⁶

We note that under a debt-management strategy with $\alpha \in (\alpha^*(b), 1)$, $\hat{\Phi}_1 = \Phi_1^* \neq \Phi_0^*$. Such equality of Lagrange multipliers is a tell-tale sign that this strategy does not work when initial debt is high, as noted in the preceding paragraph. In contrast, the short-term-debt-only strategy has the advantage over strategies with $\alpha \in (\alpha^*(b), 1)$ that it works whenever a Ramsey plan exists. This occurs because our $\alpha = 1$ strategy always implies the equalities $\hat{\Phi}_1 = \Phi_1^* = \Phi_0^*$.

¹⁶For our short-term debt only strategy, $\alpha = 1$, and the Lucas and Stokey's strategy, $\alpha = 1 - e^{-\rho\Delta}$, $\tilde{b}(1) = \bar{b}$ and $\tilde{b}(1 - e^{-\rho\Delta}) = b^*$, respectively.

5.4 Quantitative Example

We turn briefly to Debortoli et al.’s numerical example of a situation in which the Ramsey plan exists but is not implemented. Debortoli, Nunes, and Yared (2021) set an initial debt structure: $b_{-1,0} = 0.6$ and $b_{-1,i} = 0$ for $i \geq 1$ with $G = 0.2$ and $\eta = \gamma = 1$.¹⁷ The Ramsey plan is $C_{0,0}^* = 0.8172$, $\tau_{0,0}^* = 0.1828$ and $C_{0,i}^* = 0.3125$, $\tau_{0,i}^* = 0.6875$ for all $i \geq 1$. At the peak of the Laffer curve, $C^{\text{Laffer}} = 0.4$ and $\tau^{\text{Laffer}} = 0.6$, so the period-0 Ramsey tax rate is on the ‘right’ side of the Laffer curve and the period-1 Ramsey tax rate is on the ‘wrong’ side of the Laffer curve. The Ramsey plan’s Lagrange multiplier is $\Phi_0^* = 2.19$.

Lucas and Stokey’s Ramsey planner leaves the period-1 continuation planner with $b_{0,i} = 0.1523$ for all $i \geq 1$, which differs from the tail $b_{-1,i} = 0$ for all $i \geq 1$ of the initial debt structure. Consequently, the continuation planner’s ‘constructed’ Lagrange multiplier $\hat{\Phi}_1 = -3.93 < 0$, explaining why Lucas and Stokey’s implementation fails in this case. Facing debt structure $\{b_{0,i} = 0.1523\}_{i=1}^\infty$, the continuation planner sets $C_{1,i} = 0.4874$ instead of continuing the Ramsey plan $C_{0,i}^* = 0.3125$ for all $i \geq 1$. The associated Lagrange multiplier for the continuation planner is $\Phi_1^* = 2.93 > 0$, which differs from both the ‘constructed’ Lagrange multiplier $\hat{\Phi}_1 = -3.93$ and the Ramsey planner’s Lagrange multiplier $\Phi_0^* = 2.19$.

Turning to our timing protocol and debt management policy, the Ramsey planner sets $b_{0,1} = 0.3871$ and $b_{0,i} = b_{-1,i} = 0$ for all $i \geq 2$, leaving term debts with time to maturities (weakly) than two unchanged. The continuation planner sets $C_{1,i} = C_{0,i}^* = 0.3125$ for $i \geq 1$, confirming the tail of the Ramsey plan. The associated ‘constructed’ Lagrange multiplier for the continuation planner, $\hat{\Phi}_1$, now equals the resulting Lagrange multiplier for the continuation planner, $\Phi_1^* = 2.19 > 0$, which equals the Lagrange multiplier for Ramsey planner: $\hat{\Phi}_1 = \Phi_1^* = \Phi_0^* = 2.19$. This (counter)example to Lucas and Stokey’s implementation in Debortoli et al. thus confirms that equality of the Lagrange multipliers for the Ramsey plan and the continuation Ramsey plan under our short-term debt management strategy is a tell-tale sign of implementability.

We now turn to other debt management strategies that implement the Ramsey plan for this numerical example. Applying Proposition 6 with $b = 0.6$ yields $\alpha^*(0.6) = 0.61$. Consequently, the debt management strategy described in (47)-(48) works if and only if at least 61% of the accumulated liability $\Pi_1^* = 0.3871$ is financed with short-term debt, i.e., $b_{0,1} \geq 0.3871$.

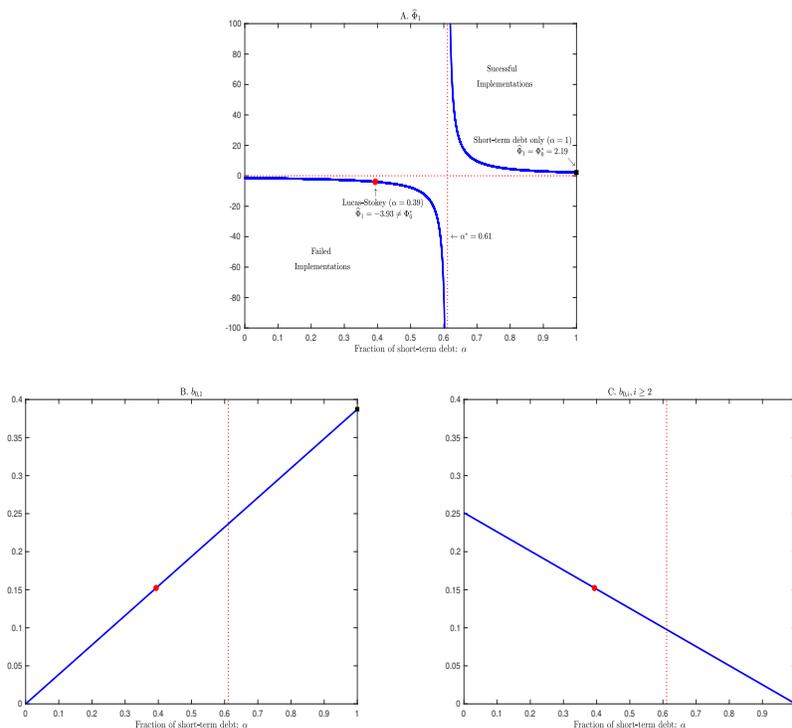
Panel A of Figure 2 plots the constructed Lagrange multiplier $\hat{\Phi}_1$ as a function of α

¹⁷See their section 3 and their figure 3.

(see (49)) and shows that a debt strategy works if and only if $\alpha > \alpha^* = 61\%$ as $\hat{\Phi}_1 > 0$. Lucas and Stokey's implementation strategy falls in the 'failed implementation' region since $\alpha = 39\% < \alpha^* = 61\%$.

Panels B and C of Figure 2 respectively plot $b_{0,1}$ and $\{b_{0,i}\}_{i \geq 2}$ for the debt strategy described by (47)-(48). As expected, $b_{0,1}$ increases with α and $\{b_{0,i}\}_{i \geq 2}$ decreases with α . The black and red dots correspond to our short-term debt only strategy and the Lucas and Stokey's (failed) implementation strategy. The vertical dotted magenta line in the three panels indicate $\alpha = \alpha^* = 61\%$.

Figure 2: Debt-management policies and constructed Lagrange multiplier. Initial debt: $b_{-1,0} = 0.6$, $b_{-1,i} = 0$ for $i \geq 1$ and parameter values: $\eta = \gamma = 1$, $\rho = 0.5$, $G = 0.2$, and $\Delta = 1$.



6 Richer Expenditure and Initial Debt Processes

This section adds numerical examples to those presented in subsection 5.4. Subsection 6.1 studies consequences of modestly perturbing Debortoli et al.'s initial debt structure (36) while retaining their assumption that government expenditures are constant. Our calcula-

tions in this setting show that an optimal tax rate on the wrong side of the Laffer curve is not a tell-tale sign that Lucas and Stokey’s debt management policy fails to implement the Ramsey plan, but that variations in pertinent Lagrange multipliers on the Ramsey and continuation problems are a tell-tale sign. Remaining subsections turn to looking under the hood of our implementation, in the same spirit of the suite of examples provided in Lucas and Stokey (1983, Sec. 3). Subsection 6.2 studies implementable debt and tax sequences that emerge when government expenditures are constant but initial debt payments $\{b_{-1,i}\}$ decline exponentially. Subsection 6.3 studies implementable debt and tax sequences that emerge when government expenditures are constant but initial debt payments $\{b_{-1,i}\}$ fluctuate periodically. Subsection 6.4 studies implementable debt and tax sequences that emerge when initial debt payments $\{b_{-1,i}\}$ are constant but government expenditures fluctuate periodically. Like example 4 of Lucas and Stokey (1983, sec. 3), this example presents an optimal tax sequence and an equilibrium interest rate sequence that matches prescriptions and other aspects of Barro (1979).

6.1 Laffer Curve Not Informative, Lagrange Multipliers Are

We now parameterize a class of initial debt structures in the following way:

$$b_{-1,0} = b_0, \quad b_{-1,1} = b_1, \quad b_{-1,i} = b_2, \quad i \geq 2. \quad (51)$$

Comparing debt structure (51) with Debortoli et al.’s initial debt structure (36) that we studied in section 5, notice that now $b_{-1,i} = b_2 \neq b_1$ for periods $i \geq 2$. Let $\{C_{0,0}^* = C_0^*, C_{0,1}^* = C_1^*, C_{0,i}^* = C_2^*; i \geq 2\}$ denote the Ramsey plan.

Under Lucas and Stokey’s timing protocol and debt management policy, a Ramsey planner that confronts initial debt structure (51) restructures debt according to

$$b_{0,1} = \frac{(C_1^*)^2 - C_1^*}{\Phi_1} + (C_1^*)^2 \quad \text{and} \quad b_{0,i} = \frac{(C_2^*)^2 - C_2^*}{\Phi_1} + (C_2^*)^2, \quad i \geq 2. \quad (52)$$

We have reverse engineered this restructuring plan from choices that the period 1 continuation planner must make to continue the Ramsey plan. We must verify that Lagrangian (42) is maximized when the consumption sequence $\{C_{1,i}\}_{i \geq 1}$ satisfies $C_{0,1}^* = C_1^*$ and $C_{0,i}^* = C_2^*$ for $i \geq 2$.¹⁸ Substituting (52) into the CIC (38), we obtain the following ‘constructed’

¹⁸Subsection 5.2 describes an analogous reverse-engineering exercise for Debortoli et al.’s initial debt structure (36).

Lagrange multiplier for the period 1 continuation planner:

$$\hat{\Phi}_1 = \frac{1}{2} \frac{(1 - C_1^*) + (1 - C_2^*) \frac{1}{1 - e^{-\rho\Delta}}}{(C_1^* - C^{\text{Laffer}}) + (C_2^* - C^{\text{Laffer}}) \frac{1}{1 - e^{-\rho\Delta}}}, \quad (53)$$

where $C^{\text{Laffer}} = (1 - G)/2$. Compare (53) under Debortoli et al.'s initial debt structure (51) with (45) under the debt structure (36) and notice that the sign of the ‘constructed’ Lagrange multiplier $\hat{\Phi}_1$ shapes Laffer curves for all periods $i \geq 1$. Evidently, $\frac{1}{1 - e^{-\rho\Delta}}$ represents a discounted value of $(C_2^* - C^{\text{Laffer}})$ from period 2 onward.

If $\hat{\Phi}_1 > 0$, the period 1 continuation planner chooses to continue the Ramsey tax plan and sets $\Phi_1^* = \hat{\Phi}_1$. However, if $\hat{\Phi}_1 < 0$, the period 1 continuation planner chooses not to continue the Ramsey plan but instead chooses a new plan that extremizes its Lagrangian (42) and provides a positive period 1 continuation plan Lagrange multiplier $\Phi_1^* > 0 > \hat{\Phi}_1$.

If the Ramsey planner sets all tax rates on the ‘wrong’ side of the Laffer curve, so that $C_1^* < C^{\text{Laffer}}$ and $C_2^* < C^{\text{Laffer}}$, then $\hat{\Phi}_1 < 0$. If $C_i^* > C^{\text{Laffer}}$ for all $i \geq 1$, $\hat{\Phi}_1 > 0$.

Depending on values of the initial debt structure parameters b_0, b_1, b_2 in specification (51), a Ramsey planner may choose to set some tax rates on the ‘wrong’ side of the Laffer curve and other tax rates on the ‘good’ side, while $\hat{\Phi}_1$ can be either positive or negative. Whether a Ramsey plan’s tax rates are on the ‘wrong’ side of a Laffer curve does not indicate the plan can be implemented. Table 1 illustrates this. Case I sets $b_{-1,0} = 0.3$, $b_{-1,1} = 0$, and $b_{-1,i} = 0.15$ for $i \geq 2$, while case II increases $b_{-1,0}$ to 0.4 and keeps all future debts at their case I values. Under Lucas and Stokey’s timing protocol and debt management policy, the Ramsey plan is implemented in case I but not in case II. With our timing protocol, our debt management policy implements a Ramsey plan in both cases.

Panel A in Table 1 reports Ramsey allocations and associated Ramsey Lagrange multipliers $\Phi_0^* > 0$ for both cases. The peak of the Laffer curve $C^{\text{Laffer}} = 0.4$ so that $\tau^{\text{Laffer}} = 0.6$ and the Ramsey allocation in period 1 is on the ‘wrong’ side of the Laffer curve, while the Ramsey plan consumptions for all future periods ($i \geq 2$) are on the ‘right’ side of the Laffer curve. This opens the possibility that the ‘constructed’ Lagrange multiplier $\hat{\Phi}_1$ for the two cases under Lucas and Stokey’s implementation can be either positive or negative, as Panel B of Table 1 confirms. In case I, the Ramsey planner restructures debt to $b_{0,1} = 0.0710$ and $b_{0,i} = 0.2357$ for all $i \geq 2$. Facing this debt structure, the continuation planner chooses to continue the Ramsey plan and $\Phi_1^* = \hat{\Phi}_1 = 13.96$. Although the period 1 Ramsey allocation is on the ‘wrong’ side of the Laffer curve, the continuation planner chooses to continue the Ramsey plan because it recognizes that the benefits of devaluing its future debts, $b_{0,i}$ for all

Table 1: Lucas and Stokey's and our implementations

Parameter values, $\eta = \gamma = 1, G = 0.2, \rho = 0.5$, and $\Delta = 1$, which imply $C_{\text{Max}}^{\text{Laffer}} = 0.4$.

Case	I			II		
Period i	0	1	≥ 2	0	1	≥ 2
$b_{-1,i}$	0.3	0	0.15	0.4	0	0.15
A. Ramsey plan						
$C_{0,i}^*$	0.6297	0.2930	0.5035	0.6459	0.0702	0.4102
LM Φ_0^*	2.41			13.24		
B. LS implementation and continuation Ramsey plan						
$b_{0,i}$	n.a.	0.0710	0.2357	n.a.	0.0272	0.2509
LM $\hat{\Phi}_1$	13.96			-2.92		
$C_{1,i}^*$	n.a.	0.2930	0.5035	n.a.	0.3877	0.6116
LM Φ_1^*	13.96			1.92		
Works?	Yes: $\hat{\Phi}_1 = \Phi_1^*$			No: $\hat{\Phi}_1 \neq \Phi_1^*$		
C. Our implementation						
Π_i^*	0	0.1478	-0.0018	0	0.0538	0.0251
$b_{0,i}$	n.a.	0.1478	0.15	n.a.	0.0538	0.15
LMs	$\hat{\Phi}_1 = \Phi_1^* = \Phi_0^* = 2.41$			$\hat{\Phi}_1 = \Phi_1^* = \Phi_0^* = 13.24$		
Works?	Yes			Yes		

periods $i \geq 2$, outweighs the costs of being on the ‘wrong’ side of the Laffer curve in period 1. The Lagrange multipliers $\Phi_1^* = \hat{\Phi}_1 > 0$ express how the continuation Ramsey planner accepts this tradeoff and continues the Ramsey plan. Consequently, Lucas and Stokey’s debt management policy implements the Ramsey plan in case I.

In case II, the Ramsey planner restructures initial debts $\{b_{-1,i}\}_{i \geq 1}$ to $b_{0,1} = 0.0272$ and $b_{0,i} = 0.2509$ for all $i \geq 2$. The planner backloads its term debt more than that in case I: $b_{0,1}$ decreases from 0.0710 in case I to 0.0272 and $b_{0,i}$ for all $i \geq 2$ increases from 0.2357 in case I to 0.2509. The period 1 continuation planner’s ‘constructed’ Lagrange multiplier is negative in case II: $\hat{\Phi}_1 = -2.92$. Facing the restructured term debt, instead of continuing the Ramsey plan the continuation planner sets $C_{1,1} = 0.3877$ and $C_{1,i} = 0.6116$ for $i \geq 2$. The resulting Lagrange multiplier for the continuation planner is $\Phi_1^* = 1.92$.

Since $\Phi_1^* > 0 > \hat{\Phi}_1$, Lucas and Stokey’s debt management policy does not implement the Ramsey plan in case II. Notice that although it doesn’t continue the Ramsey plan, the continuation planner chooses to stay on the ‘wrong’ side of the Laffer curve in period 1: $C_{1,1} = 0.3877 < C^{\text{Laffer}}$. Relative to case I, in case II future (restructured) debts $b_{0,i} = 0.2509$ for all $i \geq 2$ are so high that keeping $C_{1,1}$ on the ‘wrong’ side of the Laffer curve is optimal because that reduces the value of those debts.

Panel C Table 1 confirm that under our timing protocol, our debt management policy implements the Ramsey plan in both cases. By keeping term debts with times to maturity (weakly) than two unchanged, i.e., $b_{0,i} = b_{-1,i} = 0.15$ for all $i \geq 2$, while using short-term (one-period) debt to track accumulated liabilities so that $b_{0,1} = \Pi_1^* = 0.1478$ in case I and $b_{0,1} = \Pi_1^* = 0.0538$ in case II, the period 1 continuation planner’s Lagrange multiplier equals its ‘constructed’ Lagrange multiplier, which also equals Ramsey planner’s Lagrange multiplier: $\Phi_1^* = \hat{\Phi}_1 = \Phi_0^*$.

Key takeaways.

1. The examples in Table 1 confirm Theorem 1: a Ramsey plan can always be implemented under our timing protocol and debt management policy, and Lagrange multipliers for all continuation Ramsey planners, i.e., Φ_i^* for all $i \geq 1$, equal the Ramsey planner’s Lagrange multiplier $\Phi_0^* > 0$.
2. tax rates on the “wrong” side of the Laffer curve are not always coincident with failure of Lucas and Stokey’s debt management policy to implement a Ramsey plan under their timing protocol. Even if the Ramsey planner sets some tax rates above the peak of the Laffer curve, under our timing protocol the plan can be implemented with our debt management policy.
3. Under our short-term-debt-only strategy, $\hat{\Phi}_i = \Phi_i^* = \Phi_0^*$ for all $i \geq 1$. This Lagrange-multiplier invariance implies that continuation Ramsey planners have no incentive to deviate from the continuation of the Ramsey plan. In contrast, because Debortoli et al. rely on consol bonds as in Lucas and Stokey (1983), even with the simple initial debt structure $b_{-1,0} = b > 0$ and $b_{-1,i} = 0$ for all $i \geq 1$ the continuation Ramsey planner’s Lagrange multiplier (Φ_i^*), while positive, in general differs from the Ramsey planner’s Lagrange multiplier Φ_0^* . Thus, we now have a simple Lagrange-multiplier invariance test for implementability and no longer need to check implementability on

a case-by-case basis by computing “constructed” Lagrange multipliers for all continuation Ramsey planners.

6.2 Declining $\{b_{-1,i}\}$ bring increasing $\{\tau_{0,i}\}$

Here government spending unfolds at a constant rate $G_i = G = 0.2$ for all $i \geq 0$ and the initial debt structure $\{b_{-1,i}\}_{i \geq 0}$ starts with $b_{-1,0} = b_0 = 0.2$ in period 0 and decays exponentially at a rate of $\theta = 0.5$ each period. Figure 3 reports the Ramsey outcome and our implementation. As $b_{-1,i}$ declines monotonically as time period i advances (panel A), the tax rate $\tau_{0,i}^*$ increases (panel B.) Since $C_{0,i}^* = 1 - \tau_{0,i}^*$, by setting a higher tax rates when debt $b_{-1,i}$ is higher, the Ramsey planner makes the marginal utility $U_C(C_{0,i}^*)$ higher. The Ramsey planner thereby minimizes the time 0 discounted value of $\{b_{-1,i}\}_{i \geq 1}$. Consumption growth rate $g_{0,i}^* = \log(C_{0,i+1}^*/C_{0,i}^*)$ is negative,¹⁹ but becomes less negative over time and approaches zero. Coincidentally, the equilibrium interest rate $r_{0,i}^* = \rho + g_{0,i}^*$ increases over time and approaches the discount rate $\rho = 0.5$ (see panel C). By keeping the interest rate low in early periods when debt is high, the planner lowers the market value of its liabilities.

The Ramsey planner front loads consumption so much that increases in output $N_{0,i}^* = C_{0,i}^* + G$ overwhelm decreases in the tax rate $\tau_{0,i}^* = 1 - C_{0,i}^*$, making the primary surplus $S_{0,i}^* = \tau_{0,i}^* N_{0,i}^* - G = (1 - C_{0,i}^*)(C_{0,i}^* + G) - G$ increase over time. Since $b_{-1,i}$ decreases over time, $S_{0,i}^* - b_{-1,i}$ is negative at $i = 0$, turns positive at $i = 2$, and converges to 15% (see panel D).²⁰

Government liabilities accumulate according to $\Pi_{i+1}^* = (\Pi_i^* + (b_{-1,i} - S_{0,i}^*)\Delta) e^{r_{0,i}^*\Delta}$, starting from $\Pi_0^* = 0$. Panel E shows that Π_i^* increases over time and converges to 0.38 as $i \rightarrow \infty$.²¹ The government honors its liabilities by issuing one-period debt $b_{i-1,i}\Delta$ at time $i - 1$ that equals the sum of cumulative liability Π_i^* (in panel E) and the initial debt due in period i : $b_{-1,i}\Delta$. As $i \rightarrow \infty$, $b_{-1,i}\Delta \rightarrow 0$, $\lim_{i \rightarrow \infty} b_{i-1,i}\Delta = \lim_{i \rightarrow \infty} \Pi_i^* = 0.38$.

¹⁹In this example, $C_{0,i}^* = 2b_0 e^{-\theta i \Delta} \Phi_0^* / (\sqrt{1 + 4b_0 e^{-\theta i \Delta} \Phi_0^* (1 + \Phi_0^*)} - 1)$. Therefore, the consumption growth rate is negative: $g_{0,t}^* = \log(C_{0,i+1}^*/C_{0,i}^*) < 0$ because $dC_{0,i}^*/di = -b_0 \Phi_0^* \theta \Delta e^{-\theta i \Delta} / \sqrt{1 + 4b_0 e^{-\theta i \Delta} \Phi_0^* (1 + \Phi_0^*)} < 0$.

²⁰As $i \rightarrow \infty$, $b_{-1,i} \rightarrow 0$. We can show that $C_{0,i}^* \rightarrow \frac{1}{1 + \Phi_0^*}$ and $\lim_{i \rightarrow \infty} S_{0,i}^* - b_{-1,i} = \frac{1}{1 + \Phi_0^*} \left(1 - \frac{1}{1 + \Phi_0^*} - G\right) = 15\%$, using $G_i = 0.2$ and $\Phi_0^* = 0.99$.

²¹By letting $i \rightarrow \infty$ in $\Pi_i^* = (\Pi_{i-1}^* + (b_{-1,i-1} - S_{0,i-1}^*)\Delta) e^{r_{0,i-1}^*\Delta}$, we obtain $\lim_{i \rightarrow \infty} \Pi_i^* = (\lim_{i \rightarrow \infty} \Pi_i^* + \lim_{i \rightarrow \infty} (b_{-1,i-1} - S_{0,i-1}^*)\Delta) e^{\rho \Delta}$ and thus $\lim_{i \rightarrow \infty} \Pi_i^* = \frac{\Delta e^{\rho \Delta}}{e^{\rho \Delta} - 1} \lim_{i \rightarrow \infty} (S_{0,i-1}^* - b_{-1,i-1}) = \frac{\Delta e^{\rho \Delta}}{e^{\rho \Delta} - 1} \frac{1}{1 + \Phi_0^*} \left(1 - \frac{1}{1 + \Phi_0^*} - G\right)$.

Figure 3: Ramsey plan and our debt management for exponentially decaying debt structure: $b_{-1,i} = b_0 e^{-\theta i \Delta}$. Parameter values: $b_0 = b^H = 0.2$, $\theta = 0.5$, $\eta = \gamma = 1$, $\rho = 0.5$, $G = 0.2$, and $\Delta = 1$.

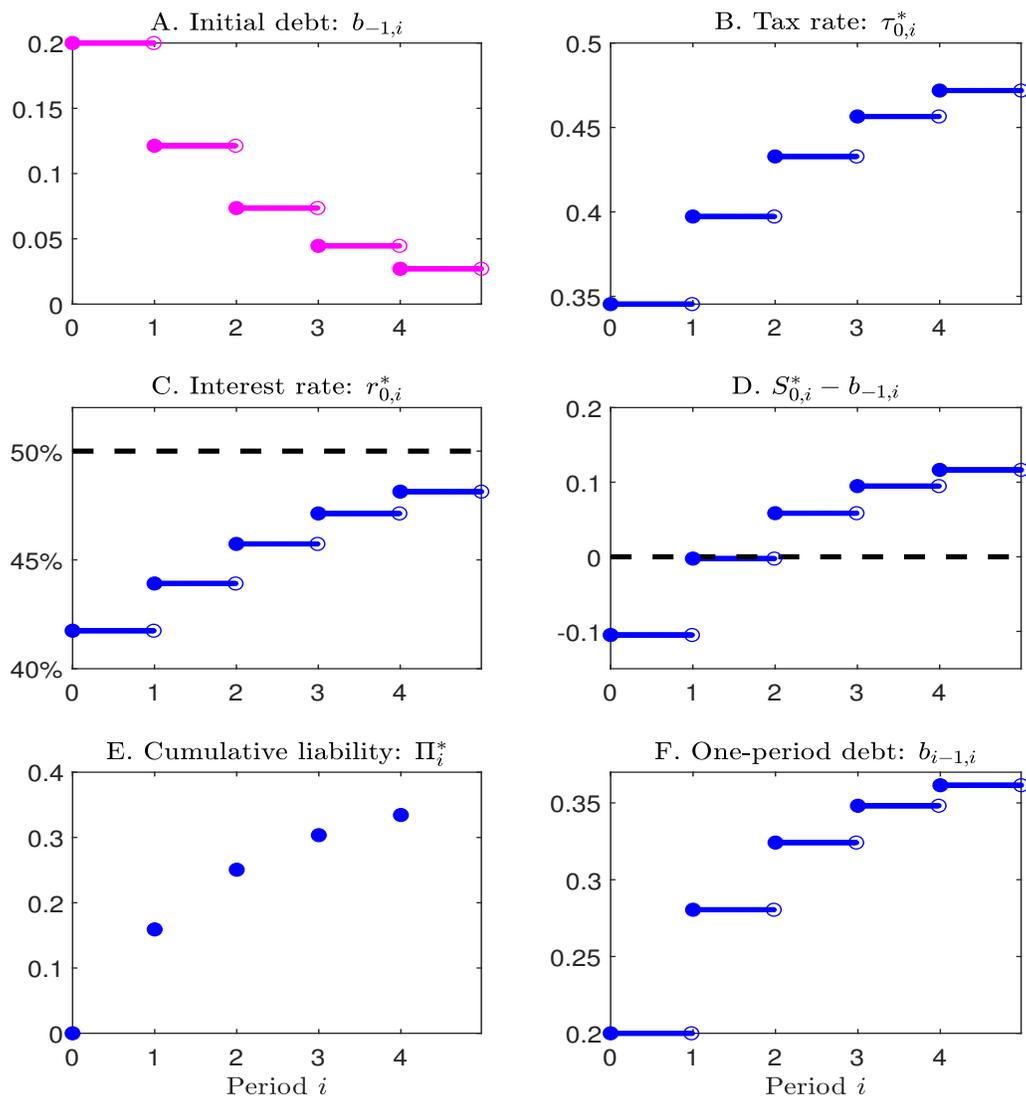
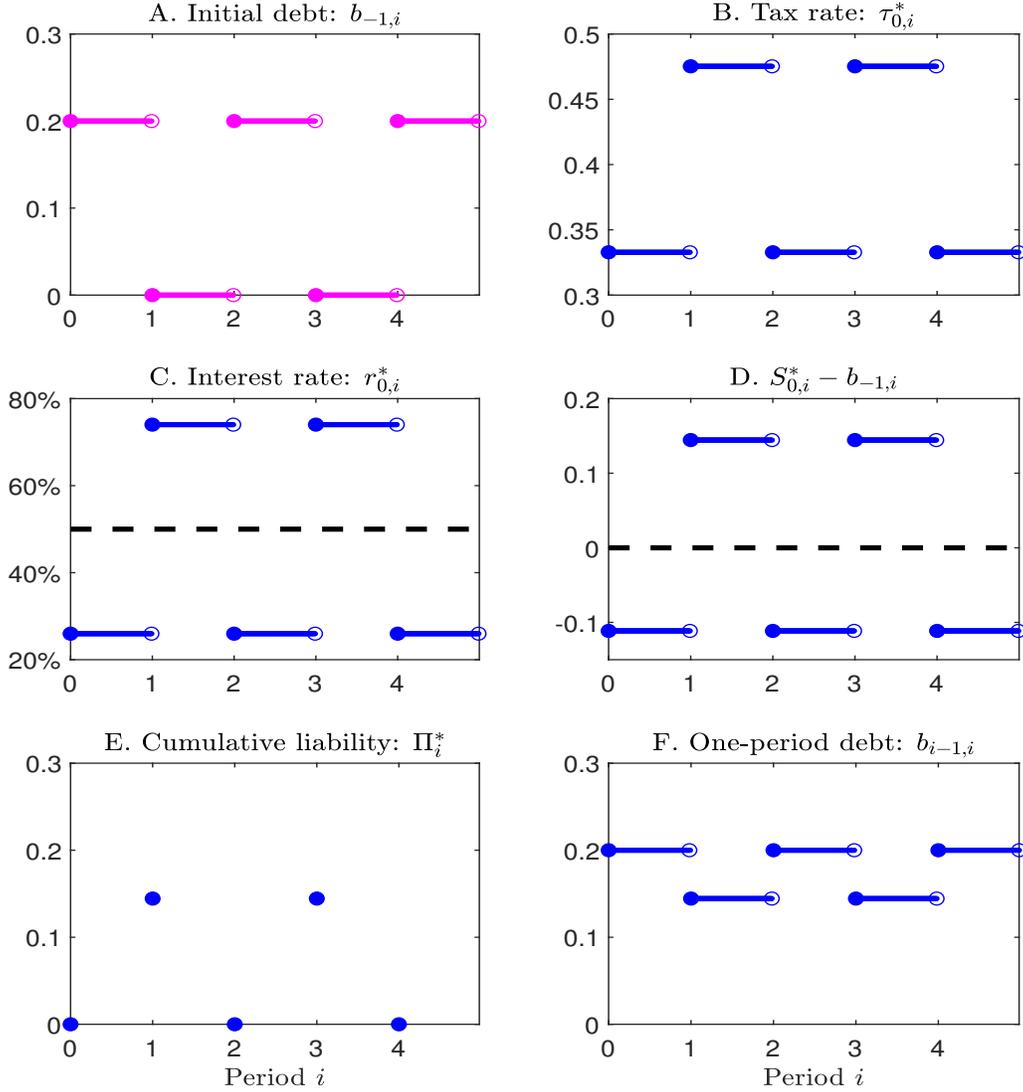


Figure 4: Ramsey plan and debt management in an example where $b_{-1,i}$ is high (b^H) in even periods and is low (b^L) in odd periods. Parameter values: $b^H = 0.2, b^L = 0, \eta = \gamma = 1, \rho = 0.5, G = 0.2$, and $\Delta = 1$.



6.3 Periodic $\{b_{-1,i}\}$ brings periodic $\{\tau_{0,i}\}$

Here initial debts $b_{-1,i}$ oscillate between two levels: $b_{-1,i} = b^H$ in even periods and $b_{-1,i} = b^L < b^H$ in odd periods. We set $b^H = 0.2$ and $b^L = 0$, start with $b_{-1,0} = b^H$. Figure 4 plots

outcomes under the Ramsey plan and our timing protocol and debt management policy.

The tax rate $\tau_{0,i}^*$ takes a high value when $b_{-1,i}$ is low. Panel B of Figure 4 indicates that $\tau_{0,i}^* = \tau^H = 0.47$ in odd periods and $\tau_{0,i}^* = \tau^L = 0.33$ in even periods. Since $C_{0,i}^* = 1 - \tau_{0,i}^*$, the Ramsey planner sets marginal utility $U_C(C_{0,i}^*)$ to be low in high debt periods and high in low debts period. This policy makes the time 0 value of the government's debt payments be low.

The equilibrium interest rate $r_{0,i}^*$ equals the sum of ρ and the consumption growth rate $g_{0,i}^* = \log(C_{0,i+1}^*/C_{0,i}^*)$. This sum moves in lock step with $g_{0,i}^*$. Evidently, $g_{0,i}^*$ moves inversely with $b_{-1,i}$ because $g_{0,i}^* < 0$ when consumption is high and consumption is high when initial debt due is high. Thus, $r_{0,i}^*$ is high in odd periods: $r^H = 0.74$ when $b_{-1,i} = b^L$, but is low ($r^L = 0.26$) in even periods (see panel C).

Panel D shows that $S_{0,i}^* - b_{-1,i}$ is $S_{0,i}^* - b_{-1,i} = -0.11 < 0$ in even periods when debt is high ($b^H = 0.2$) and $S_{0,i}^* - b_{-1,i} = 0.14 > 0$ in odd periods when $b_{-1,i} = b^L = 0$. Panel E reports cumulative liabilities $\Pi_i^* = (\Pi_{i-1}^* + (b_{-1,i-1} - S_{0,i-1}^*)\Delta) e^{r_{0,i-1}^*\Delta}$, starting from $\Pi_0^* = 0$. It shows that Π_i^* is 0.14 in an odd periods and zero in an even periods. A high level of Π_i^* accompanies high debt ($b_{-1,i-1} = b^H = 0.2$) and low surplus ($S_{0,i-1}^* = 0.09$) in period $i - 1$. A zero level of Π_i^* is an outcome of zero debt ($b_{-1,i-1} = b^L = 0$) and a high surplus ($S_{0,i-1}^* = 0.14$) in period $i - 1$. Here the government policy uses the surplus to finance all of its accumulated liabilities $\Pi_{i-1}^* = 0.14$.

Finally, in our implementation, the government honors its liabilities by issuing one-period debt $b_{i-1,i}\Delta$ that equals the sum of its cumulative liabilities Π_i^* (in panel E) and initial debt term debt due in period i : $b_{-1,i}\Delta$. In an even period i , the government simply issues one-period debt to pay for its initial debt $b_{-1,i}$, since $\Pi_i^* = 0$. After that, in odd periods i , the government issues one-period debt to pay for its cumulative liability $\Pi_i^* = 0.14$, since $b_{-1,i} = 0$. Consequently, $b_{i-1,i}$ oscillates between 0.2 in even periods and 0.14 in odd periods (panel F.)

6.4 Periodic $\{G_i\}$ and constant $\{b_{-1,i}\}$ brings constant $\{\tau_{0,i}\}$

This example assumes a cyclical government spending process G_i that oscillates between two levels: $G_i = G^H$ in even periods and $G_i = G^L < G^H$ in odd periods. We set $G_0 = G^H$ and $G^H = 0.3$ and $G^L = 0.1$, fix $b_{-1,i} = b = 0.1$ for all i , and plot Ramsey outcomes and continuation debt structures under our implementation in Figure 5.

The Ramsey planner sets a constant flat tax rate $\tau_{0,i}^* = 61\%$ for all i , since $\tau_{0,i}^* =$

$1 - C_{0,i}^*$ and $C_{0,i}^* = 0.39$ (panel B). Outcomes resemble the tax smoothing in Barro (1979), where the interest rate is an exogenous constant, and example 4 of Lucas and Stokey (1983, sec. 3), where the interest rate is endogenous. In our example the interest rate is endogenous but, because consumption $C_{0,i}^*$ is constant over time, turns out to be constant: $r_{0,i}^* = \rho + \log(C_{0,i+1}^*/C_{0,i}^*) = \rho$ (see panel C).

Since $\tau_{0,i}^*$ and $N_{0,i}^* = G_i + C_{0,i}^*$ are constant, the primary surplus $S_{0,i}^* = \tau_{0,i}^* N_{0,i}^* - G_i = C_{0,i}^* (1 - C_{0,i}^* - G_i)$ moves inversely to G_i . Consequently, $S_{0,i}^* - b_{-1,i}$ is also cyclical, attaining a high value when spending is low ($G_i = G^L$) and a low value when spending is high ($G_i = G^H$). Panel E reports the government's accumulated liabilities Π_i^* (with $\Pi_0^* = 0$), which it shows that Π_i^* oscillate between 0.08 in an odd periods and zero in even periods. A high Π_i^* accompanies high spending (G^H) and a negative $S_{0,i-1}^* - b_{-1,i-1}$. Similarly, if government spending G_{i-1} is low, the government manages its surplus in period $i - 1$ to cover all of its accumulated liabilities Π_{i-1}^* , making $\Pi_i^* = 0$.

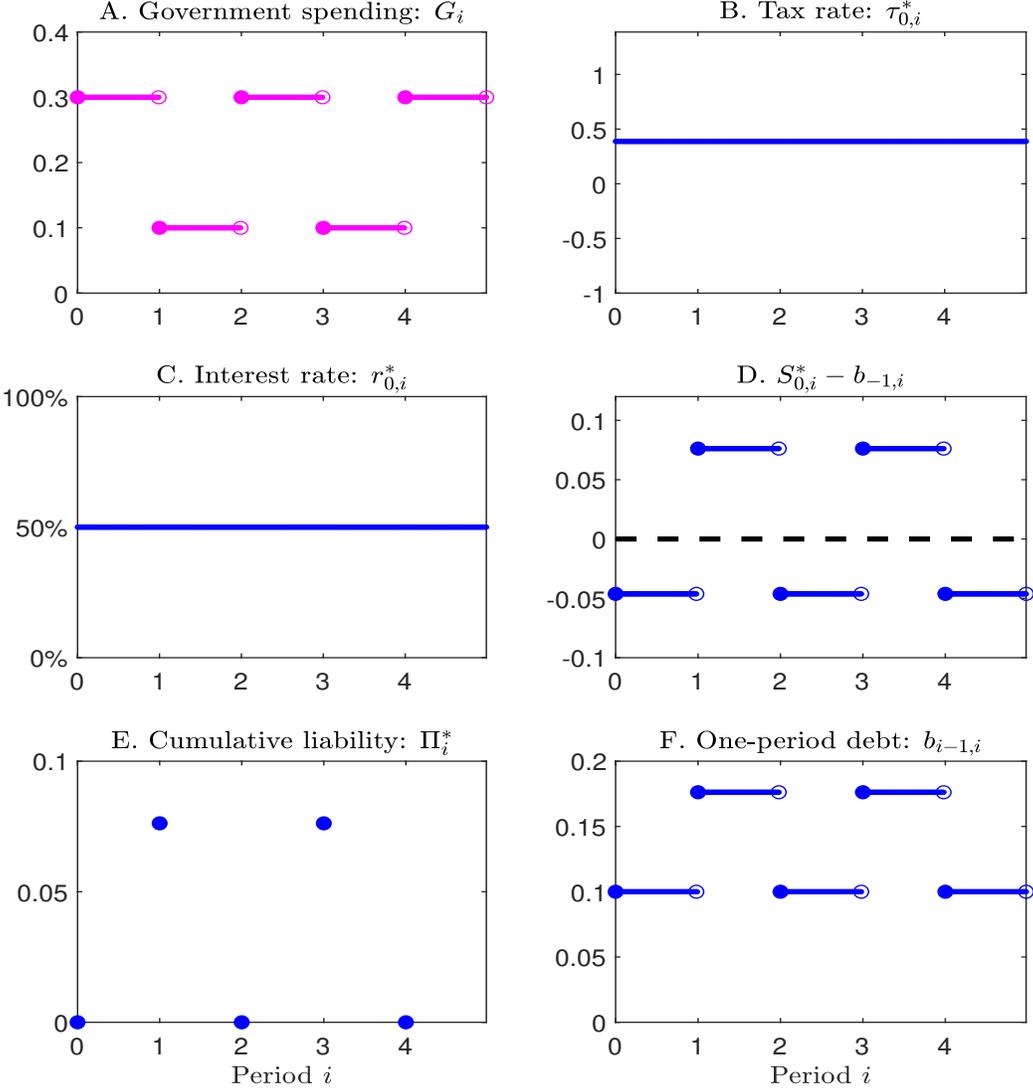
Turning to the government's debt management strategy under our implementation, since $b_{-1,i} = b = 0.1$ for all $i \geq 0$, in even periods i , the government issues one-period debt to finance $b = 0.1$ and $\Pi_i^* = 0$, but in odd periods i , the government issues one-period debt to finance the sum of its accumulated liabilities $\Pi_i^* = 0.08$ and its time i initial debt obligation $b = 0.1$. Consequently, $b_{i-1,i}$ oscillates between 0.1 in even periods and 0.18 in odd periods (panel F).

7 Concluding Remarks

Debortoli, Nunes, and Yared (2021) showed that Lucas and Stokey (1983) way of implementing a Ramsey plan for flat-rate taxes on labor fails for a simple initial government debt structure in which debt is too big. Our paper shows how to implement that Ramsey plan by slightly modifying Lucas and Stokey's timing protocol for setting tax rates and by using a "short route" debt management strategy instead of Lucas and Stokey's debt management plan that uses consols. Our implementation always works, regardless of initial debt levels or whether the Ramsey plan sets tax rates above or below peaks of Laffer curves.

Our innovations involve two complementary modifications to Lucas and Stokey's analysis. First, we expand continuation Ramsey planners' authority by allowing them to pre-set a one period-ahead tax rate, while also shrinking their authority by preventing them from resetting current-period tax rates. Second, we propose a debt management strategy that leaves longer-term debts unchanged while using only short-term debt to finance accumu-

Figure 5: Ramsey plan and debt management in an example where government spending G_i is high (G^H) in even periods and is low (G^L) in odd periods, and $b_{-1,i} = b$ for all $i \geq 0$. Parameter values: $G^H = 0.3, G^L = 0.1, b = 0.1, \eta = \gamma = 1, \rho = 0.5$, and $\Delta = 1$.



lated unpaid government liabilities. This “short route” debt management policy implements all Ramsey plans and equalizes Lagrange multipliers that appear in the constrained optimum problems faced by the Ramsey planner and all continuation planners. We have

explained how equality of those Lagrange multipliers is decisive evidence for implementability.

The leading role of short-term debt in our implementation opens a way to formulate the Ramsey problem recursively. By constructing a “state” variable based on accumulated government liabilities Π_i^* and establishing the invariance of Lagrange multipliers across time, our approach makes it possible to formulate a tractable Bellman equation for continuation planners. Such a recursive representation facilitates quantitative versions of richer variants of our model, potentially bridging gaps between theoretical fiscal policy design and practical implementation.²²

A useful extension would incorporate uncertainty.²³ This would let us study how stochastic shocks to government spending, productivity, or other economic fundamentals affect optimal tax policies and debt management strategies under limited commitment. Invariance of Lagrange multipliers and associated recursive formulations promise to prevail under uncertainty. Further, by computing limits as the time increment Δ approaches zero, we can develop continuous-time formulations of our model that should bring insights and convenient mathematical expressions.²⁴

²²Kydland and Prescott (1980) described a concise way of representing some Ramsey plans recursively. That paper is silent about implementation and did nothing to dampen their skepticism about the plausibility of Ramsey plans that they expressed in Kydland and Prescott (1977).

²³Again, see Lucas and Stokey (1983, fnt. 1).

²⁴Jiang, Sargent, and Wang (2024) provide a continuous-time analysis that interprets the timing protocol restricting the current tax rate as a covenant on government debt. By contrast, we adopt a different interpretation in the present paper. Jiang et al. also present a recursive formulation of the Ramsey problem along with the associated HJB equation.

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Appendices

A Technical Details

We state four conditions that guarantee that a Ramsey plan and a continuation Ramsey plan exist.

Condition 1. *For a given utility function $U(C, C + G)$ that is concave and differentiable, the function $(CU_C + (C + G)U_N - bU_C)$ is also concave and differentiable in C when $G > 0, b \geq 0$.*

Condition 2. *For a given initial debt structure $\{b_{-1,i}\}_{i=0}^\infty$, there exists allocations $\{C_{0,i}\Delta\}_{i=0}^\infty$ that strictly satisfy the IC (8).*

Condition 3. *For period j ($j \geq 1$) continuation planner that inherits a debt structure $\{b_{j-1,i}\}_{i=j}^\infty$ and is subject to new timing protocol given in Definition 2, there exists allocations $\{C_{j,i}\Delta\}_{i=j}^\infty$ that strictly satisfy the CIC (16).*

Condition 4. *For a period j ($j \geq 1$) continuation planner who inherits a debt structure (23)-(24) and is subject to new timing protocol given in Definition 2, there exists allocations $\{C_{j,i}\Delta\}_{j=i}^\infty$ that strictly satisfy the CIC (25).*

Condition 1 ensures both Ramsey problem and continuation Ramsey problem being concave problems. Conditions 2, 3, and 4 are Slater's conditions for the Ramsey problem and continuation Ramsey problem, respectively. For example, the economy studied in Debortoli et al. (2021) satisfies these four regularity conditions when initial debt is not too high.

Proof of Proposition 2. Note that the first condition states the Ramsey problem is a concave problem, and the second condition is a Slater's condition that ensures the strong duality for this concave problem (i.e., Boyd and Vandenberghe, 2004). Then for the Ramsey plan $\{C_{0,i}^*\}_{i=0}^\infty$, there must exist a Φ_0^* such that the following Karush-Kuhn-Tucker

condition holds:

$$\begin{aligned}
C_{0,i}^* &= \arg \max_{C_{0,i}} [U(C_{0,i}, C_{0,i} + G_i) + \Phi_0^* (C_{0,i} U_{C,i} + (C_{0,i} + G_i) U_{N,i} - b_{-1,i} U_{C,i})] \\
\Phi_0^* \cdot \left(\sum_{i=0}^{\infty} e^{-\rho i \Delta} [C_{0,i}^* U_{C,i} + (C_{0,i}^* + G_i) U_{N,i} - b_{-1,i} U_{C,i}] \cdot \Delta \right) &= 0 \\
\Phi_0^* &\geq 0 \\
\sum_{i=0}^{\infty} e^{-\rho i \Delta} [C_{0,i}^* U_{C,i} + (C_{0,i}^* + G_i) U_{N,i} - b_{-1,i} U_{C,i}] \cdot \Delta &\geq 0
\end{aligned}$$

Hence, optimal consumption satisfies $C_{0,i}^* = C(\Phi_0^*; b_{-1,i}, G_i)$, where $C(\cdot; \cdot, \cdot)$ is defined in (10).

Lastly, we show that the optimal Lagrange multiplier must be positive, $\Phi_0^* > 0$, and thus the implementability condition (11) must hold with equality. We prove it by the method of contradiction. Assume $\Phi_0^* = 0$. Then optimality requires the optimal consumption $C_{0,i}^*$ must satisfy

$$U_C(C_{0,i}^*, C_{0,i}^* + G_i) + U_N(C_{0,i}^*, C_{0,i}^* + G_i) = 0$$

and thus the utility $U(C_{0,i}^*, C_{0,i}^* + G_i)$ achieves its global maximum. Contradiction then arises because the implementability condition (8) does not hold because

$$\sum_{i=0}^{\infty} e^{-\rho i \Delta} [C_{0,i}^* U_{C,i} + (C_{0,i}^* + G_i) U_{N,i} - b_{-1,i} U_{C,i}] \cdot \Delta = - \sum_{i=0}^{\infty} (G_i + b_{-1,i}) U_{C,i} \Delta < 0.$$

□

Proof of Proposition 4. Given the inherited short-term balance $b_{j-1,j} \Delta = \Pi_j^* + b_{-1,j} \Delta$ and the tail of the initial term debt $\{b_{-1,i} \Delta; i \geq j+1\}$, the period i continuation Ramsey planner's Lagrangian is given by (26), and the optimal consumptions for period $i \geq j+1$, given a Lagrange multiplier Φ_j , satisfy (27).

We then show that optimality requires $\Phi_j^* > 0$ and the continuation implementability condition binds, $I_j(\Phi_j^*) = 0$, where $I_j(\cdot)$ is given by (28). We prove this by the method of contradiction. If $\Phi_j^* = 0$, the optimal consumption $C_{j,i}^*$ defined in (27) must satisfy

$$U_C(C_{j,i}^*, C_{j,i}^* + G_i) + U_N(C_{j,i}^*, C_{j,i}^* + G_i) = 0, \quad i \geq j+1, \quad (\text{A-1})$$

and $U(C_{j,i}^*, C_{j,i}^* + G_i)$ achieves the global maximum. On the other hand, using (21), the

continuation implementability condition (25), if holds, implies that

$$\begin{aligned}
& \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} (C_{j,i}^* U_{C,i} + (C_{j,i}^* + G_i) U_{N,i} - b_{-1,i} U_{C,i}) \Delta \\
& \quad + (C_{0,j}^* U_{C,j} + (C_{0,j}^* + G_j) U_{N,j} - b_{-1,j} U_{C,j}) \Delta - \Pi_j^* U_{C,j} \\
& = \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} (C_{j,i}^* U_{C,i} + (C_{j,i}^* + G_i) U_{N,i} - b_{-1,i} U_{C,i}) \Delta \\
& \quad + \sum_{i=0}^j e^{-\rho(i-j)\Delta} (C_{0,i}^* U_{C,i} + (C_{0,i}^* + G_i) U_{N,i} - b_{-1,i} U_{C,i}) \Delta \geq 0.
\end{aligned}$$

We then obtain an allocation $\{C_{0,0}^*, \dots, C_{0,j}^*, C_{j,j+1}^*, C_{j,j+2}^*, \dots, \}$ for the Ramsey planner than satisfies the implementability condition (8). Note that $U(C_{j,i}^*, C_{j,i}^* + G_i) > U(C_{0,i}^*, C_{0,i}^* + G_i); i \geq j + 1$ because $U(C_{j,i}^*, C_{j,i}^* + G_i)$ achieves the global maximum. As a result, the households utility (2) under this constructed allocation is larger than that given in Proposition 2. A contradiction arises. \square

Proof of Lemma 1. Let $\Phi_j = \Phi_0^*$, then $C_{j,i} = C(\Phi_j; b_{-1,i}, G_i) = C_{0,i}^*$ for $i \geq j + 1$. Substituting into (28) and using (21), we obtain

$$\begin{aligned}
I_j(\Phi_0^*) & = \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} (C_{0,i}^* U_{C,i} + (C_{0,i}^* + G_i) U_{N,i} - b_{-1,i} U_{C,i}) \Delta \\
& \quad + (C_{0,j}^* U_{C,j} + (C_{0,j}^* + G_j) U_{N,j} - b_{-1,j} U_{C,j}) \Delta - \Pi_j^* U_{C,j} . \\
& = \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} (C_{0,i}^* U_{C,i} + (C_{0,i}^* + G_i) U_{N,i} - b_{-1,i} U_{C,i}) \Delta \\
& \quad + \sum_{i=0}^j e^{-\rho(i-j)\Delta} (C_{0,i}^* U_{C,i} + (C_{0,i}^* + G_i) U_{N,i} - b_{-1,i} U_{C,i}) \Delta \\
& = e^{j\Delta} I_0(\Phi_0^*) = 0.
\end{aligned}$$

When $I_j(\cdot, \cdot) = 0$ has a unique root, the period j planner chooses $\Phi_j^* = \Phi_0^*$ and thereby confirms the Ramsey plan. \square

B Other Possible Debt-Management Policies

Our section 4 issue-only-short-term-debt strategy implements a Ramsey plan. As discussed in subsection 5.3, for some initial debt structures but not for others, debt management policies that issue both short and some longer term debts also implement the Ramsey plan. To shed more light on this claim, we adopt Debortoli et al.'s period utility function $U(C, N) = \log C - \eta \frac{N^\gamma}{\gamma}$ and an initial debt structure $\{b_{-1,i}\Delta\}_{i=0}^\infty$. Assume that the Ramsey plan implies liabilities accumulated up to period 1 equal to

$$\Pi_1^* = \frac{1}{q_{0,1}^*} (b_{-1,0} - S_{0,0}^*) \Delta = \frac{e^{\rho\Delta} C_{0,1}^*}{C_{0,0}^*} (b_{-1,0} - C_{0,0}^* (1 - \eta(C_{0,0}^* + G_1)^\gamma)) \Delta. \quad (\text{A-2})$$

Assume that at time 0 the Ramsey planner chooses to finance $b_{-1,1}\Delta$ due in period 1 and a fraction $\alpha \in [0, 1]$ of Π_1^* by issuing one-period debt

$$b_{0,1}\Delta = \alpha \Pi_1^* + b_{-1,1}\Delta, \quad (\text{A-3})$$

and to finance remaining liabilities including a fraction $(1 - \alpha)$ of Π_1^* by issuing longer-term debts, $\{b_{0,i}\}_{i \geq 2}$. The period-1 continuation planner's first-order-necessary condition requires that the continuation Lagrange multiplier Φ_1 satisfies

$$\frac{1}{C_{1,i}} - \eta(C_{1,i} + G_i)^{\gamma-1} + \Phi_1 \left(\frac{b_{0,i}}{C_{1,i}^2} - \eta\gamma(C_{1,i} + G_i)^{\gamma-1} \right) = 0, \quad i \geq 2.$$

Confirming the Ramsey plan requires $C_{1,i} = C_{0,i}^*$ for $i \geq 2$, so for $i \geq 2$, $b_{0,i}$ must satisfy

$$b_{0,i}\Delta = \frac{C_{0,i}^*}{\Phi_1} \left[-1 + \eta(C_{0,i}^* + G_i)^{\gamma-1} C_{0,i}^* + \Phi_1 \eta\gamma(C_{0,i}^* + G_i)^{\gamma-1} C_{0,i}^* \right] \Delta. \quad (\text{A-4})$$

Proposition 7. *Suppose that the conditions of Theorem 1 hold. (i) Temporarily assume that $\Pi_1^* \geq 0$. Then for $1 \geq \alpha_1 > \alpha_2 \geq 0$, if a Ramsey plan is implemented at period 1 for debt strategy (A-3)-(A-4) with $\alpha = \alpha_2$, it is also implemented for debt strategy (A-3)-(A-4) with $\alpha = \alpha_1$. (ii) Now assume that $\Pi_1^* > 0$. Then there exists $\alpha^* \in [0, 1)$ such that a Ramsey plan is implemented at period 1 for debt strategy (A-3)-(A-4) with any $\alpha \in (\alpha^*, 1]$.*

Proof. (i) Substituting (A-4) into CIC (38) and solving for Φ_1 , we obtain the constructed

Lagrange multiplier $\widehat{\Phi}_1$:

$$\widehat{\Phi}_1 = \frac{-\sum_{i=2}^{\infty} e^{-\rho(i-1)\Delta} \left[1 - \eta (C_{0,i}^* + G_i)^{\gamma-1} C_{0,i}^* \right]}{\left(1 - \eta(C_{0,1}^* + G)^\gamma - \frac{b_{0,1}}{C_{0,1}^*} \right) + \sum_{i=2}^{\infty} e^{-\rho(i-1)\Delta} \left(1 - \eta(C_{0,i}^* + G)^\gamma - \eta\gamma (C_{0,i}^* + G_i)^{\gamma-1} C_{0,i}^* \right)}$$

(A-5)

As pointed out by Debortoli et al. (2021), if $\widehat{\Phi}_1$ is positive, then a Ramsey plan is implemented. When $\Pi_1^* \geq 0$, the numerator of (A-5) is always positive because

$$\begin{aligned} & -\sum_{i=2}^{\infty} e^{-\rho(i-1)\Delta} \left[1 - \eta (C_{0,i}^* + G_i)^{\gamma-1} C_{0,i}^* \right] \\ < & -\sum_{i=2}^{\infty} e^{-\rho(i-1)\Delta} \left[1 - \eta (C_{0,i}^* + G_i)^\gamma - \frac{b_{-1,i}}{C_{0,i}^*} \right] = -\frac{\Pi_1^*}{C_{0,1}^*} \leq 0, \end{aligned}$$

where the equality holds due to the implementability constraint (8). On the other hand, using (A-3), the denominator $D(\alpha)$ of (A-5) satisfies

$$\begin{aligned} D(\alpha) &= \left(1 - \eta(C_{0,1}^* + G)^\gamma - \frac{\alpha\Pi_1^* + b_{-1,1}\Delta}{C_{0,1}^*} \right) \\ &+ \sum_{i=2}^{\infty} e^{-\rho(i-1)\Delta} \left(1 - \eta(C_{0,i}^* + G)^\gamma - \eta\gamma (C_{0,i}^* + G_i)^{\gamma-1} C_{0,i}^* \right), \end{aligned} \quad (\text{A-6})$$

and $D(\alpha)$ is decreasing in α .

For debt strategy (A-3)-(A-4) with $\alpha = \alpha_2$, if a Ramsey plan is implemented, the associated constructed Lagrange multiplier $\widehat{\Phi}_1$ is positive and we have $D(\alpha_2) < 0$. Because $D(\alpha_1) < D(\alpha_2) < 0$ for $\alpha_1 > \alpha_2$, $\widehat{\Phi}_1$ for $\alpha = \alpha_1$ is positive and the Ramsey plan is also implemented with the debt strategy (A-3)-(A-4) with $\alpha = \alpha_1$.

(ii) As we established in part (i), the constructed Lagrange multiplier $\widehat{\Phi}_1$ satisfies (A-5) and its denominator is described by equation (A-6). Because $\Pi_1^* > 0$, $D(\alpha)$ is strictly decreasing in α . According to Theorem 1, when $\alpha = 1$, $D(1) < 0$ and $\widehat{\Phi}_1 = \Phi_0^* > 0$. Consider two cases: 1) $D(0) \geq 0$, and 2) $D(0) < 0$. In case 1), there exists $1 > \alpha^* \geq 0$ such that $D(\alpha^*) = 0$ and therefore, for any $\alpha > \alpha^*$, $D(\alpha) < 0$ and $\widehat{\Phi}_1 > 0$. In case 2), $\alpha^* = 0$, and for any $\alpha > \alpha^* = 0$, $D(\alpha) < 0$ and $\widehat{\Phi}_1 > 0$. \square

Proposition 8. *For the utility function in Theorem 1 with $\eta > 0, \gamma \geq 1, \eta$, Debortoli et al.'s initial debt given by (36), and a constant government spending process, we have (i)*

for a given $b \in (0, \bar{b})$, a Ramsey plan can be implemented with debt strategy (47)-(48) with any $\alpha \in (\alpha^*(b), 1]$ for some $\alpha^*(b) \in [0, 1)$. (ii) $\alpha^*(b)$ is non-decreasing in b .

Proof. Substituting (48) into the CIC (38) and solving for Φ_1 , we obtain the ‘constructed’ Lagrange multiplier $\hat{\Phi}_1$:

$$\hat{\Phi}_1 = \frac{-(1 - \eta(C_1^* + G)^{\gamma-1}C_1^*)}{-(1 - \alpha)e^{\rho\Delta}(\eta(C_1^* + G)^\gamma - 1) - \eta\gamma(C_1^* + G)^{\gamma-1}C_1^*}. \quad (\text{A-7})$$

As pointed out by Debortoli, Nunes, and Yared (2021), if $\hat{\Phi}_1$ is positive, then a Ramsey plan is implemented.

Let $D(\alpha; b)$ denote the denominator of (A-7). Since $1 - \eta(C_1^* + G)^\gamma > 0$ (see Proposition 2 in Debortoli, Nunes, and Yared (2021)), the numerator of (A-7) is always negative and $D(\alpha; b)$ is a strictly decreasing function of α . If $\alpha = 1$, then $D(1) < 0$ and $\hat{\Phi}_1 = \Phi_0^* > 0$.

Consider two cases: 1) $D(0; b) \geq 0$, and 2) $D(0; b) < 0$. In case 1), there exists $1 > \alpha^*(b) \geq 0$ such that $D(\alpha^*(b); b) = 0$ and therefore, for any $\alpha > \alpha^*(b)$, $D(\alpha; b) < 0$ and $\hat{\Phi}_1 > 0$. In case 2), $\alpha^*(b) = 0$, and for any $\alpha > \alpha^*(b) = 0$, $D(\alpha; b) < 0$ and $\hat{\Phi}_1 > 0$.

(ii) It follows that $D(\alpha; b)$ decreases in C_1^* . Because C_1^* is strictly decreasing in b (see Proposition 1 in Debortoli, Nunes, and Yared (2021)), $D(\alpha; b)$ increases in b . Note that $D(\alpha; b)$ is a strictly decreasing function of α .

For $0 \leq b_1 < b_2 < \bar{b}$, consider three cases: 1) $\alpha^*(b_1) > 0$ and $\alpha^*(b_2) > 0$; 2) $\alpha^*(b_1) > 0$ and $\alpha^*(b_2) = 0$; and 3) $\alpha^*(b_1) = 0$ and $\alpha^*(b_2) = 0$. For cases 1) and 2), if $\alpha^*(b_1) > \alpha^*(b_2)$, a contradiction arises because $0 \geq D(\alpha^*(b_2); b_2) > D(\alpha^*(b_2); b_1) > D(\alpha^*(b_1); b_1) = 0$, and therefore, we have $\alpha^*(b_1) \leq \alpha^*(b_2)$. For case 3), $\alpha^*(b_1) = \alpha^*(b_2) = 0$, so $\alpha^*(b)$ is non-decreasing in b .

□

C Modified Ramsey Plan and its Implementation

We now describe the implications of making the Ramsey planner’s choice set look like a continuation Ramsey planner’s. In particular, we now assume that the Ramsey planner inherits a time-0 tax rate that it cannot alter. We express this situation by imposing $\tau_{0,0} = \tau_{-1,0}$. We describe a plan under this $\tau_{0,0} = \tau_{-1,0}$ constraint that we call a modified Ramsey plan, and show that our short-term debt only strategy implements it.

For convenience, we denote the modified Ramsey plan as $\{\tilde{C}_{0,i}; i \geq 1\}$ and let $\{\tilde{I}_j(\Phi_j), j \geq 1\}$

0} denote the sequence of functions associated with the implementability constraint (IC) and the continuation implementability constraints (CICs). Modifying our Proposition 2 analysis to take into account the $C_{0,0} = C_{-1,0}$ constraint, we can show that there exists a positive $\tilde{\Phi}_0$ such that $\tilde{C}_{0,i} = C(\tilde{\Phi}_0; b_{j-1,i}, G_i)$ and $\tilde{I}_0(\tilde{\Phi}_0) = 0$. Evidently, because $\tilde{\Phi}_0 \neq \Phi_0^*$, this modified Ramsey plan $\{\tilde{C}_{0,i}; i \geq 1\}$ differs from the Ramsey plan in the body of paper, $\{C_{0,i}^*; i \geq 1\}$. Nevertheless, we can show that our short-term-debt-only strategy implements this modified Ramsey plan under the conditions in Theorem 1.

To sketch the key steps, following our proof of Lemma 1, we can verify that $\tilde{\Phi}_0$ is a root of the CIC: $\tilde{I}_j(\tilde{\Phi}_0) = 0$. We then adapt our proof of Theorem 1 and verify that $\tilde{I}_j(\Phi_j) = 0$ has a unique root. Consequently, $\tilde{\Phi}_j = \tilde{\Phi}_0$, so the modified Ramsey plan $\{\tilde{C}_{0,i}; i \geq 1\}$ is implemented.