The economics of hedge funds ☆

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A B S T R A C T

Hedge fund managers trade off the benefits of leveraging on the alpha-generating strategy against the costs of inefficient fund liquidation. In contrast to the standard risk-seeking intuition, even with a constant-return-to-scale alpha-generating strategy, a risk-neutral manager becomes endogenously risk-averse and decreases leverage following poor performance to increase the fund's survival likelihood. Our calibration suggests that management fees are the majority of the total compensation. Money flows, managerial restart options, and management ownership increase the importance of high-water-mark-based incentive fees but management fees remain the majority. Investors' valuation of fees are highly sensitive to their assessments of the manager's skill.

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1. Introduction

Hedge fund assets under management (AUM) reached $2.13 trillion, a new record, in the first quarter of 2012, according to Hedge Fund Research. One salient feature of this multi-trillion dollar industry is its complicated management compensation structure. Hedge fund management compensation contracts typically feature both management fees and performance-based incentive fees. The management fee is charged as a fraction, e.g., 2%, of AUM. The

incentive fee, a key characteristic that differentiates hedge funds from mutual funds, is calculated as a fraction, e.g., 20%, of the fund's profits. The cost base for the profit calculation is often the investors' high-water mark (HWM), which effectively keeps track of the maximum value of the invested capital and thus depends on history and the manager's dynamic investment strategies. While "two-twenty" is often observed and viewed as the industry norm, compensation contracts vary with fund managers' track records. For example, James Simons' Renaissance Technologies Medallion Fund, one of the most successful hedge funds, charges 5% of the AUM via management fees and a 44% incentive fee.[1]

For investors to pay these fees and break even (net of fees) in present value (PV), managers need to generate risk-adjusted excess returns, known as alpha. Because of returns to scale, managers have incentives to leverage their alpha-generating strategies. Indeed, an important feature of the hedge fund industry is the sophisticated and prevalent use of leverage. Hedge funds may borrow through the repo markets or from prime brokers, as well as use various forms of implicit leverage, such as options and other derivatives. However, leverage also increases fund volatility and hence the likelihood of poor performance. In practice, a fund that performs poorly often faces money outflow, withdrawal/redemption, or liquidation.

We develop an analytically tractable dynamic model to analyze hedge fund leverage policy and to value hedge fund management compensation contracts. The manager dynamically allocates the fund's AUM between the alpha-generating strategy and the risk-free asset. By leveraging the alpha strategy, the manager creates value for investors (in expectation) and hence benefits via performance-linked compensation. However, leveraging also increases the fund's volatility and hence the likelihood of liquidation, resulting in the loss of fees in the future. The manager dynamically trades off the benefit and the downside (e.g., liquidation and money outflow) risk of leverage to maximize the present value (PV) of fees not only from current but also future managed funds. Outside investors rationally participate in the fund given their beliefs about the managerial skills and leverage strategies.

Specifically, our analytically tractable model contains the following important features: (1) an alpha-generating strategy; (2) poor performance-triggered drawdown and liquidation; (3) management fees as a fraction of the AUM; (4) incentive/performance fees linked to the HWM; (5) leverage constraint and margin requirement; (6) managerial ownership, which is often motivated as an incentive alignment mechanism; (7) performance-induced new money inflow; and (8) the manager's option to restart a fund (endogenous managerial outside option) at a cost. To simplify the exposition, in our baseline model, we incorporate the first five features and focus on the manager's key tradeoff between the value creation benefit and the liquidation risk induced by leverage. We then introduce additional features (6), (7), and (8) individually into our baseline model and analyze the economic and quantitative implications.

We exploit our model's homogeneity property and show that the ratio between the fund's AUM and its HWM, denoted by w, determines leverage choice. We analytically characterize the solution for the manager's value and the optimal leverage policy via an ordinary differential equation (ODE) in w with the right boundary condition at $w=1$ reflecting the manager's incentive fee collection and the left boundary condition reflecting the consequence of the fund's liquidation.

In a dynamic framework with downside (drawdown/ liquidation) risks, the risk-neutral manager has incentives to preserve the fund's going-concern value so as to collect fees in the future. The risk-neutral manager is averse to liquidation and this precautionary motive induces risk-averse managerial behavior. The conventional wisdom suggests that managers tend to take excessive risk when they are compensated via incentive fees. Intuitively, embedded options in these convex payoff structures induce risk-shifting behavior noted by Jensen and Meckling (1976). Importantly, we show that this cost of risk shifting can be overstated and the reasoning can be misleading as managers have long horizons and failing to deliver sufficiently good performance is likely to cause the fund to be liquidated. Thus, managerial career concerns induce the manager to behave in a risk-averse manner. Put differently, the downside risk for the manager can be quite significant in contrast to the implications in *static* settings.

Our model predicts that optimal leverage increases with alpha and decreases with volatility despite constant alpha-generating investment opportunities. More interestingly, optimal leverage decreases with the manager's endogenously determined risk aversion. Unlike the Merton-type investor, both the manager's endogenous risk attitude and optimal leverage depend on w, as the manager's moneyness (the long position in incentive fees and the short position in investors' liquidation option) varies. The higher the value of w (i.e. the more distant the fund is from liquidation), the closer the manager is to collecting the incentive fees, the less risk aversely the manager behaves, and consequently the higher the leverage. When the downside liquidation likelihood is very low, the risk-neutral manager may even behave in a risk-seeking way. In this case, the leverage constraint becomes binding and the manager's value is convex in w.

Our baseline model parsimoniously captures the key tradeoff between value creation via leverage on an alpha strategy and the costly liquidation triggered by sufficiently poor performance. Importantly, the manager's aversion to the fund's inefficient liquidation leads to conservative leverage choices. This under-leveraging result is analogous to the arbitrageur's limited ability to exploit mispricing as in Shleifer and Vishny (1997) and related limits-of-arbitrage models.

Additional key institutional features of hedge funds, such as (6)–(8) listed above, may have first-order effects on the manager's leverage choice and valuation of fees. We introduce each new feature, one at a time into the baseline model to study their implications. First, when liquidation risk is low, the manager can be risk seeking and the

margin requirement becomes binding. Second, managerial ownership within the fund mitigates agency conflicts. Third, we incorporate the empirical finding that money chases performance, and find that it has a significant effect on the manager's leverage choices and the PVs of management fees and incentive fees. Finally, we integrate the manager's options to close the current fund and start up new funds and find that these options are quantitatively valuable.[2]

We also conduct quantitative analysis by calibrating our model to empirical leverage moments reported in Ang, Gorovyy, and van Inwegen (2011). Our calibration implies that investors will liquidate the fund if the manager loses 31.5% of the fund's AUM from its HWM. Interestingly, our calibrated drawdown limit of 31.5% is comparable to the drawdown level of 25% quoted by Grossman and Zhou (1993) in their study of investment strategy for institutional (hedge fund) investors. We find that the manager creates about 20% value on the fund's AUM in PV and captures all the surplus via their compensation. Out of the manager's total value creation of 20 cents on a dollar, 75% is attributed to management fees (15 cents) and the remaining 25% goes to incentive fees (5 cents).

By incorporating managerial risk-seeking incentives/leverage constraints, managerial ownership, new money inflow, and fund restart options, the manager has additional incentives to leverage on the alpha strategy, which in turn increases the value of incentive fees, *ceteris paribus*. Overall, we find that quantitatively both management and incentive fees are important contributors to the manager's total value.

Metrick and Yasuda (2010) report similar quantitative results for private equity (PE) funds whose managers also charge management and incentive fees via two-twenty-type compensation contracts. While compensation structures are similar for hedge funds and private equity funds, institutional details such as how management fees and performance fees are calculated differ significantly. Sorensen, Wang, and Yang (2013), henceforth SWY, develop a dynamic portfolio-choice model for an institutional investor who chooses among public equity, bonds, and illiquid private equity investments. Importantly, the PE investment in SWY has to be delegated to the PE manager. SWY also find that both management and incentive fees are important contributors to the PE manager's total compensation in PV.

We also show that investors' net payoffs (from their investments in the fund) critically depend on their ability to correctly assess the manager's alpha. For example, in our baseline calculation, compensating an unskilled manager with a two-twenty-type compensation is very expensive, as investors lose about 15% of their invested capital in PV. Our calibration results provide quantitative support to Swensen (2005) who writes, "Hedge fund investing belongs in the domain of sophisticated investors who commit significant resources to the manager evaluation process. While the promise of hedge funds proves attractive to many market participants, those investors who fail to identify truly superior active managers face a dismal reality. In the absence of superior security-selection, investment strategies that avoid market exposure deliver money-market-like expected returns. The hefty fee arrangements typical of hedge funds erode the already low cash-like return to an unacceptable level, especially after adjusting for risk." Therefore, it is critically important for investors to choose a skilled manager with sufficiently high alpha so that investors can make some profits or at least do not lose money in PV (netting of fees and agency costs).

Related literature: There are only a few theoretical papers on hedge fund valuation and leverage decisions. Goetzmann, Ingersoll, and Ross (2003), henceforth GIR, provide the first quantitative intertemporal valuation framework for management and incentive fees in the presence of the HWM. They derive closed-form valuation formulas for both investors' payoff and the manager's compensation under the assumption of a constant alpha. GIR focuses solely on valuation and does not allow for endogenous leverage or any other decisions such as fund closure/restart.

Panageas and Westerfield (2009), henceforth PW, obtain explicit formulas for leverage and the manager's value in a setting with only HWM-indexed incentive fees and no liquidation boundary for w. The main predictions of PW are (1) leverage is constant at all times, (2) managers are *worse* off if incentive fees *increase* (e.g., from 20% to 30%), and (3) managers are *worse* off if the HWM *decreases*, ceteris paribus. Our calibrated model predicts that (1) leverage is stochastic and tends to increase with w, (2) managers are better off if incentive fees increase, and (3) managers are better off if the HWM decreases, ceteris paribus. Our results are the *opposite* of PW's because the manager in our model is averse to *downside* liquidation risk and tries to stay away from the liquidation boundary for survival, while the manager in PW is averse to crossing the HWM too soon, because incentive fees leave the fund and do not earn excess returns. Moreover, our model allows for leverage constraints and hence can generate risk-loving behavior while PW's does not. We also incorporate realistic features including management fees, managerial ownership, managerial restart options, and money inflows and outflows, in addition to performance-linked downside liquidation risk. Note that the manager trades off the short-term benefits against future payoffs in both our and PW's dynamic models. Conceptually and quantitatively, management fees, downside liquidation risks, managerial restart options, and performance-triggered new money flows are critical and new features of our model.

Hodder and Jackwerth (2007) numerically solve a risk-averse hedge fund manager's investment strategy in a discrete-time finite-horizon model. They argue the importance of endogenous fund restart options for the manager. Buraschi, Kosowski, and Sritakul (2012) study the manager's leverage choice in a finite-horizon setting with one-time compensation at the terminal date. For tractability, they assume that the HWM is fixed and predetermined at the beginning of the contracts' evaluation period. Dai and Sundaresan (2010) show that the hedge fund's short positions in investors' redemption

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[2] The closure/restart option is similar to the manager's option to negotiate with investors to reset the fund's HWM once the incentive fee is sufficiently out of the money. The cost of negotiating with investors to reset the HWM is that some investors may leave the fund and hence the AUM may decrease.

options and funding options (from prime brokers and short-term debt markets) influence the fund's risk management policies, but do not study the effects of management compensation on leverage and valuation. Drechsler (2013) solves the manager's optimization problem by assuming that management fees are proportional to the HWM as opposed to the AUM in order to obtain closed-form solutions.

More broadly, our paper relates to the literature on how compensation contracts influence fund investment strategies.[3] Carpenter (2000) and Ross (2004) show that convex compensation contracts may not increase risk seeking for risk-averse managers. Basak, Pavlova, and Shapiro (2007) and Hugonnier and Kaniel (2010) study mutual fund managers' risk-taking induced by an increasing and convex relationship of fund flows to relative performance.[4]

Few papers have attempted to quantify the effects of management compensation on leverage and valuation of fees. We provide a simple calibration and take a first-step to quantitatively value both management and incentive fees in a model with endogenous leverage choice. Additionally, we show that managerial ownership, performance-dependent new money flows, and a manager's voluntary closure/restart options are also important for hedge fund leverage and valuation.

There has been much recent and continuing interest in empirical research on hedge funds. Fung and Hsieh (1997), Ackermann, McEnally, and Ravenscraft (1999), Agarwal and Naik (2004), and Getmansky, Lo, and Makarov (2004), among others, study the nonlinear feature of hedge fund risk and return.[5] Aragon and Nanda (2012) and Agarwal, Daniel, and Naik (2009, 2011) empirically study the effect of managerial incentives on hedge fund performance. Lo (2008) provides a treatment of hedge funds for their potential contribution to systemic risk in the economy.

2. Baseline model

We now develop a parsimonious model of dynamic leverage with the following essential building blocks.

The fund's investment opportunity: The manager can always invest in the risk-free asset which pays interest at a constant rate r. Additionally, the manager can also invest in an alpha-generating strategy which earns a risk-adjusted expected excess return. Without leverage, the incremental return for the skilled manager's alpha strategy, dR_t, is given by

$$dR_t = (r + \alpha)\,dt + \sigma\,dB_t, \qquad (1)$$

where B is a standard Brownian motion, α denotes the expected return in excess of the risk-free rate r, and σ is the return volatility. Managers often conceal the details of their trading strategies to make it hard for investors and competitors to infer and mimic their strategies.[6] Alpha measures scarce managerial talents, which earn rents in equilibrium (Berk and Green, 2004). As we will show later, even with time-invariant investment opportunity, the optimal leverage will change over time due to managerial incentives.

Let W be the fund's AUM and D denote the borrowing amount in the risk-free asset which earns interest at the risk-free rate r. Therefore, the amount invested in the alpha strategy (1) is given by $A = W + D$. Let π denote the (risky) asset-capital ratio, $\pi = A/W = (W + D)/W$. Hedge funds often borrow via short-term debt and obtain leverage from the fund's prime brokers, repo markets, and the use of derivatives.[7] For a levered fund, $D \geq 0$ and $\pi \geq 1$. For a fund hoarding cash, $D < 0$ and $0 < \pi < 1$.

Management compensation contracts: Managers are paid via both management and incentive fees. The management fee is specified as a constant fraction c of the AUM W, $\{cW_t : t \geq 0\}$. The incentive fee often directly links compensation to the fund's performance via the so-called high-water mark (HWM). In this paper, we take compensation contracts (both management and incentive fees) as given and then analyze optimal leverage and value management compensation.[8]

In the region when $W < H$, the HWM H evolves deterministically. Let g denote the growth rate of H without money outflow. This growth rate g may be zero, the risk-free rate r, or other values. The contractual growth rate g may reflect investors' opportunity costs of not investing elsewhere (e.g., earning the risk-free rate of return). Additionally, the fund's HWM should be adjusted downward when money flows out of the fund. As in GIR, we also allow investors to continuously redeem capital at the rate, δW_t, where $\delta \geq 0$ is a constant. To sum up, when $W < H$, the HWM H grows exponentially at the rate $(g - \delta)$,

$$dH_t = (g - \delta)H_t\,dt \quad \text{if } W_t < H_t. \qquad (2)$$

When $g = \delta$, (2) implies that the HWM H is the running maximum of W, $H_t = \max_{s \leq t} W_s$, in that the HWM is the highest level that the AUM has attained.

When $W \geq H$, the fund's profit is $dH_t - (g - \delta)H_t\,dt > 0$. The manager collects a fraction k of that profit, given by $k[dH_t - (g - \delta)H_t\,dt]$, and then HWM H is reset.

Fund liquidation: As in GIR, the fund can be exogenously liquidated with probability λ per unit of time. By assumption, the manager can do nothing to influence this liquidation likelihood. Let τ_1 denote the stochastic moment at which the exogenous liquidation occurs.

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[3] The other widely used approach studies the design of optimal compensation contracts given agency and/or informational frictions between investors and fund managers. The two approaches are complementary.

[4] Cuoco and Kaniel (2011) analyze equilibrium asset pricing with delegated portfolio management.

[5] For the presence of survivorship bias, selection bias, and back-filling bias in hedge fund databases, see Brown, Goetzmann, Ibbotson, and Ross (1992), among others. For careers and survival, see Brown, Goetzmann, and Park (2001).

[6] Additionally, for key employees (portfolio managers) who are informed about details of the strategies, the fund manager often pays them with long-term contracts (e.g. inside equity) with various provisions discouraging them from exiting (e.g., non-competition clause) partly to keep strategies secretive.

[7] Few hedge funds are able to directly issue long-term debt or secure long-term borrowing.

[8] The optimality of hedge fund management compensation contracts is not the focus of this paper but is an important topic for future research.

Alternatively, if the fund's performance is sufficiently poor, investors may liquidate the fund. For example, large losses may cause investors to lose confidence in the manager, triggering liquidation. Specifically, we assume that when the AUM W falls to a fraction b of its HWM H, the fund is liquidated. GIR make a similar liquidation assumption in their valuation model. Unlike GIR, the AUM dynamics in our model depend on leverage. Let τ_2 denote this endogenous performance-triggered stochastic liquidation moment,

$$W_{\tau_2} = bH_{\tau_2}. \tag{3}$$

The above liquidation condition has been used by Grossman and Zhou (1993) in their study of investment strategies by institutional investors facing what they refer to as "drawdown" constraints. In their terminology, $1-b$ is the maximum "drawdown" that investors allow the fund manager before liquidating the fund in our model. Grossman and Zhou (1993) state that "it is not unusual for managers to be fired subsequent to achieving a large drawdown, nor is it unusual for the managers to be told to avoid drawdowns larger than 25%." The drawdown limit of 25% in the above quote maps to $b=0.75$ in our model. The fund will be liquidated if the manager loses 25% of the AUM from its HWM.

In reality, investors may increase the withdrawal of capital as the manager's performance deteriorates. We may model this performance-dependent withdrawal by allowing δ, the rate at which investors withdraw, to depend on w, a measure of the fund's performance. By either specifying δ as a decreasing function of w or using the lower liquidation boundary as in (3), we incorporate the downside risk into the model, which induces the manager to behave in an endogenously risk-averse manner, which we will show is the key mechanism of our model. For space considerations, we do not include the details for this extension with continuous performance-triggered withdrawal in the paper. The main results remain effectively the same as in our baseline model.

The fund is liquidated either exogenously at stochastic time τ_1 or endogenously at τ_2. At liquidation time $\tau = \min\{\tau_1, \tau_2\}$, the manager receives nothing and investors collect the fund's AUM W_τ. While leveraging on an alpha strategy creates value, the manager is averse to losing future fees upon liquidation. In an effectively (stationary) infinite-horizon framework such as ours, liquidation can be quite costly and hence the manager faces significant downside risk (e.g., losses of all future compensation), unlike *limited* downside risk in typical finite-horizon option-based compensation models. It is thus often optimal for the manager to choose prudent time-varying leverage.

Leverage constraint: The fund may also face institutional and contractual restrictions on leverage. We impose the following leverage constraint at all times t:

$$\pi_t \leq \bar{\pi}, \tag{4}$$

where $\bar{\pi} \geq 1$ is the exogenously specified maximally allowed leverage. Grossman and Vila (1992) study the effects of leverage constraints on portfolio allocations. For assets with different liquidity and risk profiles, $\bar{\pi}$ may

differ. For example, individual stocks have higher margin requirements than Treasury securities do. See Ang, Gorovyy, and van Inwegen (2011) for a summary of various margin requirements for different assets. Investors also contractually impose bounds on leverage for the fund. With a sufficiently tight leverage constraint, the manager's value and leverage will be finite and the optimization problem is well defined even for a risk-neutral manager.

Dynamics of AUM: Prior to liquidation ($t < \tau$), the fund's AUM W_t evolves as follows:

$$dW_t = rW_t\,dt + \pi_t W_t(\alpha\,dt + \sigma\,dB_t) - \delta W_t\,dt - cW_t\,dt \\ - k[dH_t - (g-\delta)H_t\,dt] - W_t\,dJ_t. \tag{5}$$

The first and second terms in (5) describe the change of AUM W given the manager's leverage strategy π. The third term gives the continuous payout to investors, i.e., money outflow. The fourth term gives the flow of management fees (e.g., $c=2\%$), and the fifth term gives the incentive/performance fees which are paid if and only if the AUM exceeds the HWM (e.g., $k=20\%$).[9] Here, J is a jump process with a mean arrival rate λ. If the jump occurs, the fund is exogenously liquidated and hence its AUM W falls to zero. We can further generalize this jump-induced liquidation by specifying the intensity λ as a function of w, the ratio between the fund's AUM W and its HWM H. By specifying a higher jump likelihood λ for a worse performance (a lower w), we introduce performance-triggered stochastic liquidation, which causes the risk-neutral manager to behave in a risk-averse manner.[10]

Various value functions for investors and the manager: We now introduce various present values (PVs) for the manager and investors. Let β denote the manager's discount rate. For a given dynamic leverage strategy π, the PV of total fees, denoted by $F(W,H;\pi)$, is given by

$$F(W,H;\pi) = M(W,H;\pi) + N(W,H;\pi), \tag{6}$$

where $M(W,H;\pi)$ and $N(W,H;\pi)$ are the PVs of management and incentive fees, respectively,

$$M(W,H;\pi) = \mathbb{E}_t\left[\int_t^\tau e^{-\beta(s-t)}cW_s\,ds\right], \tag{7}$$

$$N(W,H;\pi) = \mathbb{E}_t\left[\int_t^\tau e^{-\beta(s-t)}k[dH_s - (g-\delta)H_s\,ds]\right]. \tag{8}$$

The manager collects neither management nor incentive fees after stochastic liquidation.[11]

Similarly, we define investors' value $P(W,H)$ as follows:

$$P(W,H;\pi) = \mathbb{E}_t\left[\int_t^\tau e^{-r(s-t)}\delta W_s\,ds + e^{-r(\tau-t)}W_\tau\right]. \tag{9}$$

In general, investors' value $P(W,H)$ differs from the AUM W because of managerial skills and fees. The total fund value

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[9] The manager collects the incentive fees if and only if $dH_t > (g-\delta)H_t\,dt$, which can only possibly happen at the boundary $W_t = H_t$. In the interior region ($W_t < H_t$), incentive fees are zero since $dH_t = (g-\delta)H_t\,dt$. Hence, $dH_t - (g-\delta)H_t\,dt \geq 0$ and incentive fees are non-negative at all times.

[10] As we will show later, our model solution only depends on the sum of $\lambda + \delta$. See Sections 3 and 4.

[11] In Section 7, we extend the baseline model to allow the manager to close the fund and start a new one. The manager then maximizes the PV of fees from both current and future funds.

$V(W, H)$ is given by the sum of $F(W, H)$ and $P(W, H)$:

$$V(W, H; \pi) = F(W, H; \pi) + P(W, H; \pi). \tag{10}$$

The manager's optimization and investors' participation: Anticipating that the manager behaves in self interest, investors rationally demand that investors' value, $P(W, H)$, is at least as large as their time-0 investment W_0 in order to break even in PV. At time 0, by definition, we have $H_0 = W_0$. Thus, at time 0, we require

$$P(W_0, W_0; \pi) \geq W_0. \tag{11}$$

Note that the constraint (11) is only required at the inception of the fund.[12]

The surplus, $P(W_0, W_0) - W_0$, is divided between investors and the manager depending on their bargaining powers. In competitive markets, the skilled manager collects all the surplus, and the participation constraint (11) binds. However, in periods such as a financial crisis, investors may earn some rents by providing scarce capital and the constraint (11) holds with slack.

When the manager only faces the time-0 investors' voluntary participation constraint (11), we may write the manager optimization problem as follows:

$$\max_{\pi} F(W, H; \pi), \tag{12}$$

subject to the liquidation boundary (3), the leverage constraint (4), the investors' voluntary participation (11), and technical regularity conditions.

3. Solution

We solve the manager's optimization problem using dynamic programming. In the interior region ($W < H$), we have the following Hamilton-Jacobi-Bellman (HJB) equation:

$$(\beta + \lambda)F(W, H) = \max_{\pi \leq \bar{\pi}} cW + [\pi\alpha + (r - \delta - c)]WF_W(W, H)$$
$$+ \frac{1}{2}\pi^2\sigma^2 W^2 F_{WW}(W, H) + (g - \delta)HF_H(W, H), \tag{13}$$

subject to the leverage constraint (4). The discount rate on the left side changes from β to $(\beta + \lambda)$ to reflect the exogenous stochastic liquidation likelihood. For expositional simplicity, we leave the discussions for the boundary conditions to the Appendix.

As we show, the homogeneity property proves valuable in simplifying our analysis. That is, if we double the AUM W and the HWM H, the PV of total fees $F(W, H)$ will correspondingly double. The effective state variable is therefore the ratio between the AUM W and the HWM H, $w = W/H$. We use the lower case to denote the corresponding variable in the upper case scaled by the contemporaneous HWM H. For example, $f(w) = F(W, H)/H$.

Summary of main results: The manager's leverage policy critically depends on the manager's endogenously determined risk attitude. To measure a risk-neutral manager's endogenous risk attitude, motivated by the coefficient of

relative risk attitude for a consumer, we define a risk-neutral manager's risk attitude as follows:

$$-\frac{WF_{WW}(W, H)}{F_W(W, H)} = -\frac{wf''(w)}{f'(w)} \equiv \psi(w), \tag{14}$$

where the equality follows from the homogeneity property. When $\psi(w) > 0$, we refer to the risk-neutral manager as being endogenously risk averse. When $\psi(w) \leq 0$, we refer to the risk-neutral manager as being endogenously risk seeking. Note that risk attitude $\psi(w)$ is endogenous and stochastic. We next solve the manager's optimal leverage policy.

Optimal leverage policy:

- First, we use the following first-order condition (FOC) for leverage π:

  $$\alpha WF_W(W, H) + \pi\sigma^2 W^2 F_{WW}(W, H) = 0. \tag{15}$$

  With the homogeneity property, we further simplify the leverage policy as,

  $$\pi = \frac{\alpha F_W(W, H)}{-\sigma^2 WF_{WW}(W, H)} = \frac{\alpha}{\sigma^2 \psi(w)}, \tag{16}$$

  where $\psi(w)$ is the manager's endogenous risk attitude defined in (14).
- Second, we note that the FOC is not sufficient for the optimality. We also need to check the second-order condition (SOC) for leverage, which is given by $\sigma^2 W^2 F_{WW} < 0$. The homogeneity property of our model simplifies the SOC as,

  $$f''(w) < 0, \tag{17}$$

  which can be equivalently written as $\psi(w) > 0$.
- Third, we incorporate the leverage constraint (4) which may sometimes bind by writing the optimal leverage policy as,

  $$\pi(w) = \min\left\{\frac{\alpha}{\sigma^2 \psi(w)}, \ \bar{\pi}\right\}, \tag{18}$$

  in the region where the risk-neutral manager behaves in a risk-averse manner, $\psi(w) > 0$.
- Now, we turn to the region where the risk-neutral manager behaves in a risk-seeking manner, $\psi(w) \leq 0$. In this case, the FOC (15) no longer characterizes the manager's optimality. Indeed, the SOC is violated. The manager chooses the maximally allowed leverage $\bar{\pi}$, and hence the leverage constraint (4) binds.

Combining the optimal leverage policy solutions in both the risk-averse and the risk-seeking regions, we obtain the following optimal leverage policy:

$$\pi(w) = \begin{cases} \min\left\{\dfrac{\alpha}{\sigma^2 \psi(w)}, \ \bar{\pi}\right\}, & \psi(w) > 0, \\ \bar{\pi}, & \psi(w) \leq 0, \end{cases} \tag{19}$$

where the endogenous risk attitude $\psi(w)$ is defined by (14). When $\psi(w) > 0$ and equivalently $f''(w) < 0$ implying that the second-order condition (SOC) is satisfied, the manager is endogenously risk averse to downside liquidation risk. If $\psi(w)$ is sufficiently large, the leverage constraint given in (4) does not bind and the optimal leverage $\pi(w)$ is then given by the ratio between (1) the excess return α and (2) the

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[12] If investors can liquidate the fund at any time, we then need to require investors' voluntary participation constraints to hold at all times, i.e., $P(W_t, H_t; \pi) \geq W_t$ for all t. In this case, investors never lose money. In Section 8, we analyze this case where investors' voluntary participation constraints need to hold at all times, i.e., $P(W_t, H_t; \pi) \geq W_t$ for all t.

product of variance, σ^2, and endogenous risk attitude $\psi(w)$. Unlike in the standard portfolio model (e.g., Merton, 1971), the manager's motive for fund survival is strong enough to cause the risk-neutral manager's leverage and value function to be well defined and finite despite the option to infinitely leverage on the alpha strategy. When $\psi(w)$ is low enough, the leverage constraint binds and $\pi(w) = \overline{\pi}$.

When $\psi(w) \leq 0$ and equivalently $f''(w) \geq 0$ implying that the SOC is not satisfied, the manager then behaves in a risk-seeking manner by choosing the maximally allowed leverage, causing the leverage constraint given in (4) to bind, $\pi(w) = \overline{\pi}$. In the risk-seeking region, the standard FOC-based analysis of leverage choices is no longer valid. A sufficiently tight leverage constraint given in (4) is necessary to ensure that the manager's optimization problem is well defined. We note that the constraint can bind either when $\psi(w) < 0$ or when $\psi(w) \geq 0$. The mechanisms that cause the constraint given in (4) to bind differ for the two cases.

Importantly, the risk-neutral manager's risk-taking incentives vary significantly with w. Let $\overline{w}$ denote the threshold value of w where the leverage constraint (4) just becomes binding. Because the manager optimally chooses $\overline{w}$, the manager's value $f(w)$ is twice continuously differentiable at $w = \overline{w}$,

$$f(\overline{w}_-) = f(\overline{w}_+), \quad f'(\overline{w}_-) = f'(\overline{w}_+), \quad f''(\overline{w}_-) = f''(\overline{w}_+). \quad (20)$$

Here, $\overline{w}_+$ and $\overline{w}_-$ denote the right and left limits of the endogenously chosen $\overline{w}$.

Both the manager's endogenous risk attitude $\psi(w)$ and the leverage constraint given in (4) help to ensure that the risk-neutral manager's optimization problem is well defined.

Using Ito's formula and the optimal leverage policy (19), we write the dynamics for w as,

$$dw_t = [\pi(w_t)\alpha + r - g - c]w_t\, dt + \sigma\pi(w_t)w_t\, dB_t - w_t\, dJ_t, \quad (21)$$

where J is the pure jump process leading to liquidation as we have previously described.

The manager's value $f(w)$ solves the following ordinary differential equation (ODE):

$$(\beta - g + \delta + \lambda)f(w) = cw + [\pi(w)\alpha + r - g - c]wf'(w) + \tfrac{1}{2}\pi(w)^2\sigma^2 w^2 f''(w), \quad (22)$$

subject to the following boundary conditions:

$$f(b) = 0, \quad (23)$$

$$f(1) = (k + 1)f'(1) - k. \quad (24)$$

Eq. (23) states that the manager's value is zero at the liquidation boundary b. This assumption is the same as the one in GIR. However, unlike GIR, the manager in our model influences the liquidation likelihood via dynamic leverage. Importantly, performance-based liquidation risk, such as the liquidation boundary (23), plays a critical role in our analysis.[13] Eq. (24) gives the manager's value at the right boundary $w = 1$. Our reasoning for this boundary condition follows GIR and PW. Finally, ODE (22) gives the manager's

total value $f(w)$ in the interior region. Note that the withdrawal rate δ and the exogenous liquidation intensity λ appear additively in (22), and thus they have the same effects on valuation.

Investors' voluntary participation condition (11) can be simplified to

$$p(1) \geq 1. \quad (25)$$

Applying the standard method to $P(W, H)$ defined in (9), we have the following ODE for $p(w)$:

$$(r - g + \delta + \lambda)p(w) = (\delta + \lambda)w + [\pi(w)\alpha + r - g - c]p'(w) + \tfrac{1}{2}\pi(w)^2\sigma^2 w^2 p''(w), \quad (26)$$

where $\pi(w)$ is given in (19). If a positive return shock increases the AUM from $W = H$ to $H + \Delta H$, we could use the same argument as the one for $F(W, H)$ and obtain the continuity of value function before and after the adjustment of the HWM,

$$P(H + \Delta H, H) = P(H + \Delta H - k\Delta H, H + \Delta H). \quad (27)$$

By taking the limit as ΔH approaches zero and using Taylor's expansion rule, we have $kP_W = P_H$, which implies

$$p(1) = (k + 1)p'(1). \quad (28)$$

Recall that the investors collect the AUM at the liquidation boundary ($W = bH$), and hence, the following lower condition is held:

$$p(b) = b. \quad (29)$$

The scaled total PV of the fund $v(w)$ is given by $v(w) = f(w) + p(w)$.

If we further assume perfectly competitive capital markets, managers collect all the rents from their skills at time 0 and investors' value satisfies $p(1) = 1$, as in Berk and Green (2004) in the context of mutual funds.[14]

4. Results

We now analyze our baseline model's implications on leverage, valuation of fees, and investors' payoff. We first choose the parameter values and calibrate our model.

4.1. Parameter choices and calibration

We choose the commonly used two-twenty compensation contract, $c = 2\%$ and $k = 20\%$. In reality, part of the management fees cover operating expenses. For small funds, it is quite plausible that management fees simply cover the operating costs. For large funds, it is very likely that management fees at the margin are pure profits due to the scalability of the investment technology. Without loss of generality, we equate the manager's and the investors' discount rates, $\beta = r$. All rates are annualized and continuously compounded when applicable. As in GIR, our model identifies $\delta + \lambda$, the sum of payout rate δ and the fund's exogenous liquidation

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[13] In reality, managers often have options to close the current fund and start a new one. In Section 7, we extend our model to allow the manager to start a new fund, enriching the baseline model by providing the manager with flexible exit and restart options.

[14] Here, if we impose (11) at all times as we discussed earlier, i.e., $p(w) \geq w$ for all w, we will have a tighter constraint but the model remains tractable. See Section 8 for our analysis of the case with the investors' participation constraint $p(w) \geq w$.

intensity λ. We refer to $\delta + \lambda$ as the total withdrawal rate. Similarly, our model identifies $r-g$, which we refer to as the net growth rate of w. We thus only need to choose the following parameter values: (1) the unlevered α; (2) the unlevered volatility σ; (3) the total withdrawal rate, $\delta + \lambda$; (4) the net growth rate of w, $r-g$; and (5) the liquidation boundary b. We also set the leverage constraint at $\bar{\pi} = 4$.

We set the net growth rate of w to zero, i.e., $r-g = 0$. Otherwise, even unskilled managers collect incentive fees by simply holding a 100% position in the risk-free asset. We set the exogenous liquidation probability $\lambda = 10\%$ so that the implied average fund life (with only exogenous liquidation risk) is ten years. Few hedge funds have regular payouts to investors, so we choose $\delta = 0$. The total expected withdrawal rate is thus $\delta + \lambda = 10\%$.

Next, we calibrate the remaining three parameters: the expected unlevered excess return α, the unlevered volatility σ, and the liquidation boundary b. We use two moments from Ang, Gorovyy, and van Inwegen (2011), who report that the average long-only leverage is 2.13 and the standard deviation for cross-sectional leverage is 0.616.

Calibrating to the two leverage moments and the equilibrium condition, $p(1) = 1$, we identify $\alpha = 1.22\%$, $\sigma = 4.26\%$, and $b = 0.685$. The implied Sharpe ratio for the alpha strategy is $\alpha / \sigma = 29\%$. Our calibration implies that the maximum drawdown before investors liquidate the fund (or equivalently fire the manager) is $1-b = 31.5\%$. Interestingly, this calibrated value 31.5% is comparable to the drawdown level of 25% that is quoted by Grossman and Zhou (1993) in their study of investment strategy with drawdown constraints. Our calibration implies a levered (annual) alpha to be about 2.6% and levered volatility to be 9.1%, both of which lie within various estimates in the literature. (Naturally, different styles of hedge funds have different targets for alpha and volatility.) Note that empirical estimates of a fund's alpha and volatility correspond to the levered ones in our model.[15] Table 1 summarizes all the key variables and parameters in the model. Unless otherwise noted, we use these parameter values for our quantitative analysis.

4.2. Leverage π, the manager's value $f(w)$, and risk attitude $\psi(w)$

Dynamic leverage: Fig. 1 plots leverage $\pi(w)$. Despite the fund's constant investment opportunity, leverage π_t depends on w_t, the manager's moneyness in the fund. At the liquidation boundary $b = 0.685$, the fund is barely levered, $\pi(b) = 1.03$. As w increases, the manager increases leverage, and reaches $\pi(1) = 3.18$ at $w = 1$. The higher the manager's moneyness w, the closer the manager is to collecting incentive fees and the more distant the fund is

from liquidation, the higher the leverage $\pi(w)$. The implied annual *levered alpha* $\pi(w)\alpha$ varies from 1.26% at the liquidation boundary $b = 0.685$ to 3.88% at $w = 1$ with an average value of 2.60%. The annual volatility of the levered return $\pi(w)\sigma$ ranges from 4.4% at the liquidation boundary b to 13.4% at $w = 1$.

Common wisdom suggests that managers may gamble for resurrection when their option-based compensation contracts are deep out of the money. In our model, this standard risk-shifting argument is not applicable. First, debt is fully collateralized, short-term (continuously rolled over), and hence, debt is risk-free in our model. There is thus no risk-shifting incentive against creditors.[16]

In our infinite-horizon setting, upon liquidation, the manager loses all future management and incentive fees, which is quite costly, as opposed to the limited downside risk assumed in standard finite-horizon settings with option-based convex payoffs. Therefore, the cost of taking excessive risks today is large in present value because persistent poor performance may lead to liquidation and losses of all future fees. Indeed, the risk-neutral manager in our model behaves in an endogenously risk-averse manner and hence underleverages on the alpha strategy. That is, the agency conflict in our model features excessively prudent leveraging, a form of underinvestment, as opposed to excessive risk taking, a form of overinvestment.

These results are analogous to the underinvestment result for financially constrained firms in Corporate Finance. For example, Bolton, Chen, and Wang (2011) develop a dynamic corporate investment model for a financially constrained firm facing costly external financing. They show that a risk-neutral firm behaves in an effectively risk-averse manner; it underinvests, and moreover the degree of underinvestment increases as corporate liquidity decreases. Additionally, while our model takes the management compensation contract as exogenously given, some key features of our model may also hold in an optimal contracting setting. For example, liquidity-dependent under-investment can be optimal in dynamic contracting models. In our model the AUM-HWM ratio w is the natural measure of liquidity/moneyness for the manager.

We note that GIR also show that the manager becomes more prudent as the fund gets closer to the liquidation boundary as opposed to seeking risk, consistent with our findings.[17]

The manager's value $f(w)$ and endogenous risk attitude $\psi(w)$: Panel A of Fig. 2 plots $f(w)$. At the inception of the fund, for each unit of AUM, the manager creates a 20% surplus in PV, $f(1) = 0.20$, and collects all the surplus via

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Table 1
Summary of key variables and parameters.
    This table summarizes the symbols for the key variables used in the model and the parameter values for the calibration and quantitative exercises. For each upper-case variable in the left column (except D, R, H, J, S), we use its lower case to denote the ratio of this variable to the HWM H. When applicable, parameter values are continuously compounded and annualized.

| Variable	Symbol	Parameters	Symbol	Value
Risk-free borrowing amount	D	Risk-free rate	r	5%
Assets under management (AUM)	W	Unlevered alpha	α	1.22%
The alpha strategy's cumulative returns	R	Unlevered volatility	σ	4.26%
High-water mark (HWM)	H	Indexed growth rate of H	g	5%
Leverage	π	Manager's discount rate	β	5%
Exogenous fund liquidation jump process	J	Investors' withdrawal rate	δ	0
Present value of management fees	M	Probability of liquidation	λ	10%
Present value of incentive fees	N	Management fee parameter	c	2%
Present value of investors' payoffs	P	Incentive fee parameter	k	20%
Present value of total fees	F	Lower liquidation boundary	b	0.685
Total value of the fund	V	Manager's fund ownership	ϕ	0.2
Manager's total value with ownership	Q	New money inflow rate	i	0.8
The manager's endogenous risk attitude	ψ	Restart option parameter	θ_0	−24.75
New fund size	S	Restart option parameter	θ_1	61.47
Present value of new money inflow	X	Restart option parameter	θ_2	−74
		Leverage constraint	$\bar{\pi}$	4
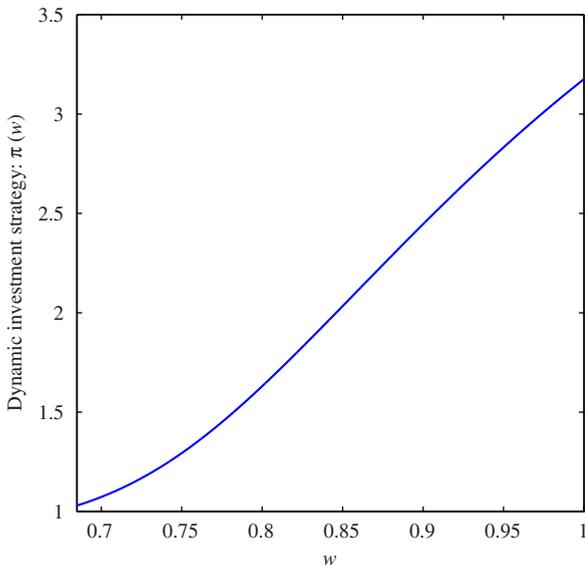
Fig. 1. Hedge fund's dynamic leverage $\pi(w)$. We plot the hedge fund manager's leverage policy $\pi(w)$ as a function of w, the manager's moneyness in the fund. Even with a time-invariant alpha-generating strategy, the manager's optimal leverage is time-varying and depends on w. For our calibrated specification, the manager increases the fund's leverage $\pi(w)$ as the fund's performance increases (i.e., w increases) and decreases $\pi(w)$ as w decreases. Leverage allows the manager to generate a higher expected payoff but increases the downside liquidation risk. The (annualized) parameter values are: the unlevered alpha, $\alpha = 1.22\%$, the unlevered alpha-strategy volatility $\sigma = 4.26\%$, management fee $c = 2\%$, incentive fee $k = 20\%$, the growth rate of the HWM is indexed to the risk-free rate, $g = r$, the total expected payout rate, $\delta + \lambda = 10\%$, and the lower liquidation boundary $b = 0.685$.

management and incentive fees. Panel B of Fig. 2 plots the manager's endogenous risk attitude, $\psi(w)$. The risk-neutral manager is averse to costly fund liquidation. As w increases, liquidation risk decreases and the manager's endogenous risk attitude $\psi(w)$ falls from $\psi(b) = 6.50$ to $\psi(1) = 2.11$.

The marginal value of the AUM W, $F_W(W, H)$: Using the homogeneity property, we have $F_W(W, H) = f'(w)$. Panel A of Fig. 3 plots $f'(w)$. As w increases, $f'(w)$ decreases from $f'(b) = 1.46$ at the liquidation boundary $b = 0.685$ to $f'(1) = 0.33$, which is much lower than $f'(b) = 1.46$. The higher the value of w, the lower the liquidation risk and thus the lower the marginal value of AUM $f'(w)$. A dollar increase of the AUM near the liquidation boundary b is much more valuable than a dollar increase near $w = 1$ because the former decreases the risk of fund liquidation and can potentially save the fund from liquidation.

The marginal impact of the HWM H, $F_H(W, H)$: Again, using the homogeneity property, we have $F_H(W, H) = f(w) - wf'(w)$. Panel B of Fig. 3 plots $F_H(W, H)$ as a function of w. Increasing H mechanically lowers $w = W/H$, which increases the likelihood of investors' liquidation. Because the manager is averse to fund liquidation, increasing H lowers $F(W, H)$, $F_H(W, H) < 0$. Importantly, downside liquidation risk (a sufficiently high b) is critical to generating this result.

Quantitatively, the impact of the HWM H on $F(W, H)$ is significant. Even when the manager is very close to collecting incentive fees ($w = 1$), a unit increase of the HWM H lowers the manager's value $F(H, H)$ by 0.13, which follows from $F_H(H, H) = f(1) - f'(1) = -0.13$. The impact of H on $F(W, H)$ is even greater for lower values of w. At the liquidation boundary $b = 0.685$, the impact of HWM on the manager's value is about one to one in our calibration, $F_H(bH, H) = f(b) - bf'(b) = -1.00$.

Intuitively, increasing H lowers the manager's total value $F(W, H)$ as the manager's option to collect incentive fees is further out of the money and the manager is closer to the liquidation boundary, both of which make the manager worse off, ceteris paribus. In a model with incentive fees only, PW show that the manager's value function increases with the HWM H, which is the opposite of our result. This is because the first-order tradeoffs in the two models are different. In PW, there is no liquidation boundary ($b = 0$) and the manager is averse
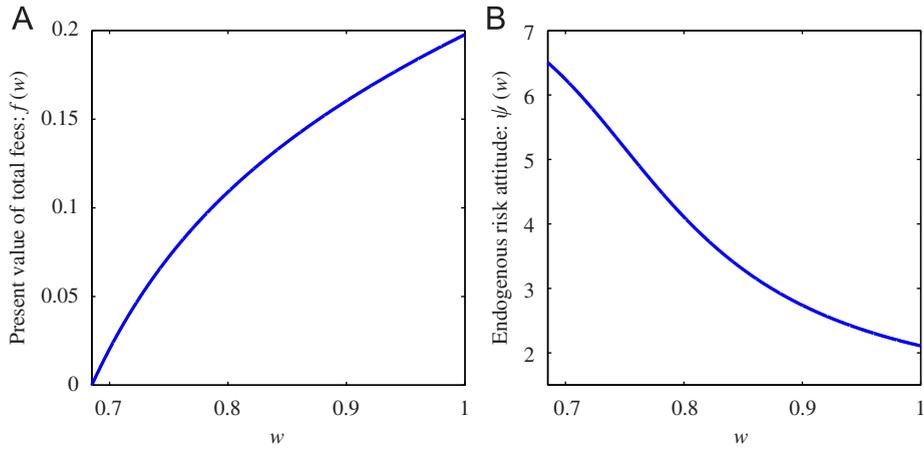
Fig. 2. The hedge fund manager's scaled value function $f(w)$ and the endogenous risk attitude, $\psi(w) = -wf''(w)/f'(w)$. Panel A plots the manager's scaled value $f(w)$ as a function of the manager's moneyness w, and shows that the manager collects 20 cents for each unit of AUM in present value at $w=1$. Panel B shows that the risk-neutral manager's endogenous risk aversion decreases from 6.5 to 2.1 as w increases from $w=b=0.685$ to $w=1$. Intuitively, the manager becomes more risk-averse as the manager's moneyness w decreases and the fund is getting closer to liquidation. (A) The manager's total (scaled) value $f(w)$. (B) The manager's endogenous risk attitude $\psi(w) = -wf''(w)/f'(w)$.
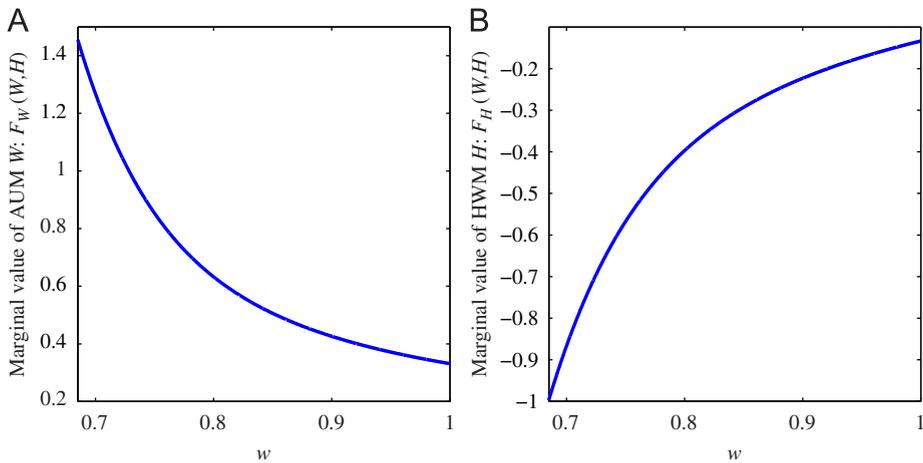

Fig. 3. The sensitivities of the hedge fund manager's value function $F(W,H)$ with respect to the AUM W and the HWM H; $F_W(W,H)$ and $F_H(W,H)$. Panel A shows that the manager's marginal value of the AUM, $F_W(W,H)$, decreases from 1.46 to 0.33 as w increases from $b=0.685$ to $w=1$, which shows that the risk-neutral manager's value function $F(W,H)$ is concave in W. Panel B shows that the manager is worse off as the HWM H increases, i.e., $F_H(W,H) < 0$. (A) The manager's marginal value of AUM W, $F_W(W,H)$. (B) The manager's marginal value of HWM H, $F_H(W,H)$.

to crossing the HWM too soon.[18] In our model, the manager is primarily concerned about the downside liquidation risk.[19]

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[18] In PW, $F_H(W,H) = f(w) - wf'(w) > 0$, which follows from the homogeneity property and no liquidation boundary, $b=0$. In PW, $F_H > 0$ is implied by a concave $f(w)$ and $b=0$. Therefore, with $b=0$, $F_H > 0$ is necessary and sufficient to ensure that the manager's optimization problem is well defined in PW.

[19] It is worth noting that the manager also earns a lower rate of return (no alpha) from collected fees in our model as in PW. However, in our model, the lower rate of return that a manager earns on fees only has a second-order effect in deterring the manager from leveraging and does not influence the main tradeoff between leveraging on the alpha strategy to generate excess returns and being prudent to reduce liquidation risk. Our model and PW focus on very different parameter regions.

4.3. Understanding leverage and valuation via a sample path

Fig. 4 illustrates a sample path for the fund's performance from its inception to its stochastic liquidation. For the simulation, we set the time step $\Delta = 10^{-3}$. Panel A plots the fund's AUM W_t and HWM H_t. Panel B plots $w_t = W_t/H_t$ and leverage π_t. Finally, Panels C and D plot the management fee and incentive fee (both in cash flows), respectively.

At the fund's inception, $W_0 = H_0 = 1$, and $w_0 = 1$ by definition. The manager chooses leverage $\pi_0 = \pi(1) = 3.18$. At the end of the first time step, $t = \Delta = 10^{-3}$, the manager collects the management fee $c \times W_0 \times \Delta = 2 \times 10^{-5}$, as expected. Given that the realized shock at $t=\Delta$, $dR_\Delta = -0.001$, is negative, the AUM $W_\Delta = 0.9967$ and $H_\Delta = 1.00005$ implying
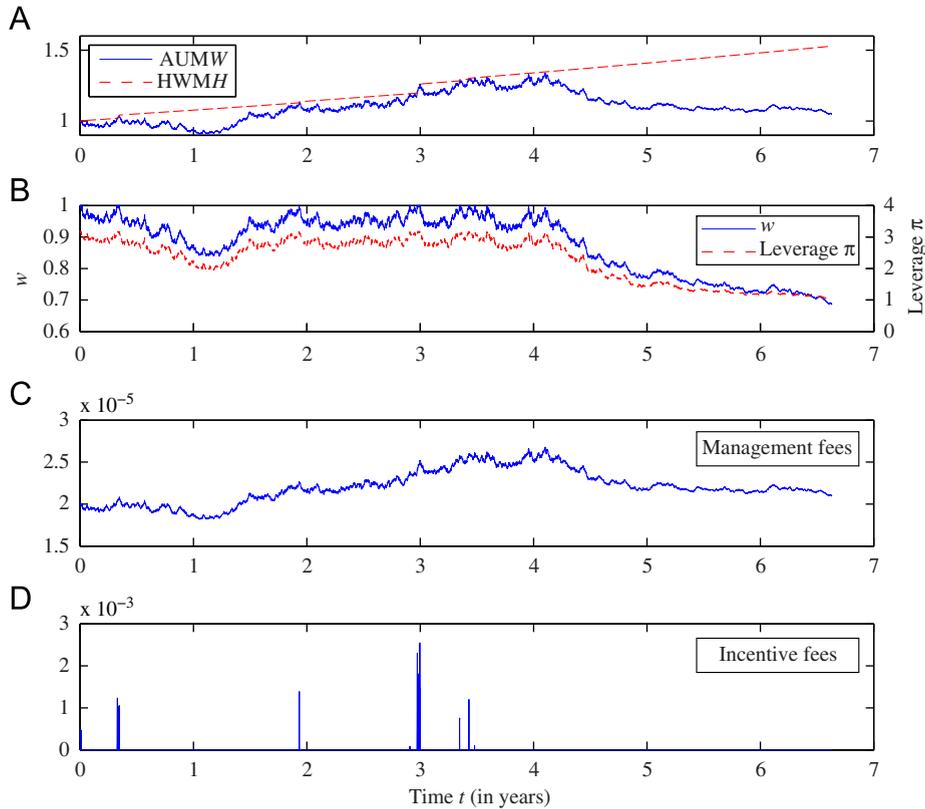
Fig. 4. A sample path of AUM W_t, HWM H_t, w_t, leverage π_t, management fees (in cash flows), and incentive fees (in cash flows) over the calendar time t. This figure plots a simulated sample path of our baseline model solution. Panel A shows that the fund's AUM W_t is always weakly lower than its contemporaneous HWM H_t. This is simply due to the fact that the fund pays out 20% of the profits to the manager whenever the AUM W exceeds the fund's HWM H and the fund resets the HWM H after paying the manager incentive fees. Panel B shows that the fund's w and the manager's optimal leverage π_t is time-varying. Note that leverage $\pi(w)$ increases with the manager's moneyness w. Panel C plots the sample path for management fees, which is simply 2% of the AUM W_t and hence perfectly tracks the sample path of W_t in Panel A. Finally, Panel D plots the incentive fee payments, which is stochastic and lumpy, as the fund only pays out 20% of profits $dH_t-(g-\delta)H_t dt$ to the manager if and only if W_t exceeds the contemporaneous HWM H_t. (A) The sample path of the fund's AUM W and HWM H. (B) The sample path of the fund's w_t and the manager's optimal leverage π_t. (C) The sample path of the management fees cW_t. (D) The sample path of the incentive fees $0.2 \times [dH_t-(g-\delta)H_t \, dt]{\geq}0$.

$w_\Delta = 0.9966 < 1$ and no incentive fee cash flow. Leverage drops to $\pi_\Delta = 3.15$.

At $t=2\Delta=2 \times 10^{-3}$, the fund draws another shock $dR_{2\Delta}$ and the process repeats. As long as $W_t < H_t$, the HWM H_t grows exponentially at a deterministic rate $g-\delta = 5\%$, as we see from the smooth curve in the region $W_t < H_t$.

In contrast, when $W_t=H_t$, a positive shock causes the manager to collect incentive fee cash flow. For example, at $t=8\Delta=0.008$, $w_t{\approx}1$, the realized shock at $t=9\Delta$, $dR_{9\Delta} = 0.0015$, is positive, and the manager collects $0.0023 \times 20\% = 0.00046$ in incentive fees, and then HWM is reset so that $w_{9\Delta}=1$.

This sample path shows that as w_t decreases, leverage π_t decreases in a nonlinear way. For this simulation, the fund is liquidated at time $\tau=6.63$, the first moment that w_τ reaches the lower liquidation boundary $b=0.685$. Panel C shows that the management fee cash flow $cW_t\Delta$ linearly tracks the evolution of AUM W_t. Incentive fees are paid occasionally and the size of the payment is lumpy, as the fund's AUM occasionally exceeds its HWM at random moments. Next, we turn to the valuation of management fees and incentive fees.

4.4. Valuing incentive and management fees

Applying the standard differential equation pricing method to $M(W,H)$ defined in (7) and $N(W,H)$ defined in (8), we obtain the following ODEs for $m(w)$ and $n(w)$, respectively,

$$(\beta-g + \delta + \lambda)m(w) = cw + [\pi(w)\alpha + r-g-c]m'(w)$$
$$+\tfrac{1}{2}\pi(w)^2\sigma^2 w^2 m''(w), \tag{30}$$

$$(\beta-g + \delta + \lambda)n(w) = [\pi(w)\alpha + r-g-c]n'(w)$$
$$+\tfrac{1}{2}\pi(w)^2\sigma^2 w^2 n''(w). \tag{31}$$

Next, using the homogeneity property, we obtain the following boundary conditions:

$$m(1) = (k + 1)m'(1), \tag{32}$$

$$n(1) = (k + 1)n'(1)-k. \tag{33}$$

Notice that the manager loses both management fees and incentive fees forever at the liquidation boundary ($W=bH$), and hence the following lower conditions are held:

$$m(b) = n(b) = 0. \tag{34}$$

The value of incentive fees n(w): Panel A of Fig. 5 plots $n(w)$. As w increases, the manager is closer to collecting incentive fees and $n(w)$ increases. At $w=1$, $n(1)=0.05$, which is one quarter of the manager's total value $f(1) = 0.20$. Incentive fees are a sequence of embedded call options. Intuitively, the delta of $n(w)$, $n'(w)$, increases with w, opposite to the result in PW. Again, the manager is primarily concerned about downside liquidation risk in our model but is averse to collecting incentive fees too soon in PW.

The value of management fees m(w): Panel B of Fig. 5 plots $m(w)$ which increases with w from $m(b)=0$ to $m(1)=0.15$, about 75% of the manager's total value $f(1)=0.20$. Quantitatively, management fees contribute significantly to total compensation. Intuitively, the management fee is effectively a wealth tax on the AUM. The PV of the AUM tax at 2% over a fund's lifespan is thus significant as a fraction of its AUM. In our case, $m(1)$ is 15% of the AUM. For the private equity industry, Metrick and Yasuda (2010) also find that management fees contribute to the majority of total management compensation.

Unlike the value of incentive fees $n(w)$, $m(w)$ is concave in w. The manager collects management fees as long as the fund survives, but only receives incentive fees when the AUM exceeds the HWM. Management fees, as a fraction c of the fund's AUM, effectively give the manager an *unlevered* equity-type cash flow claim from the fund while it is alive. Upon the fund's liquidation, the manager receives nothing and loses all future fees. Therefore, fund liquidation is quite costly for the manager. The manager thus optimally chooses a prudent level of leverage for survival so as to collect fees in the future.
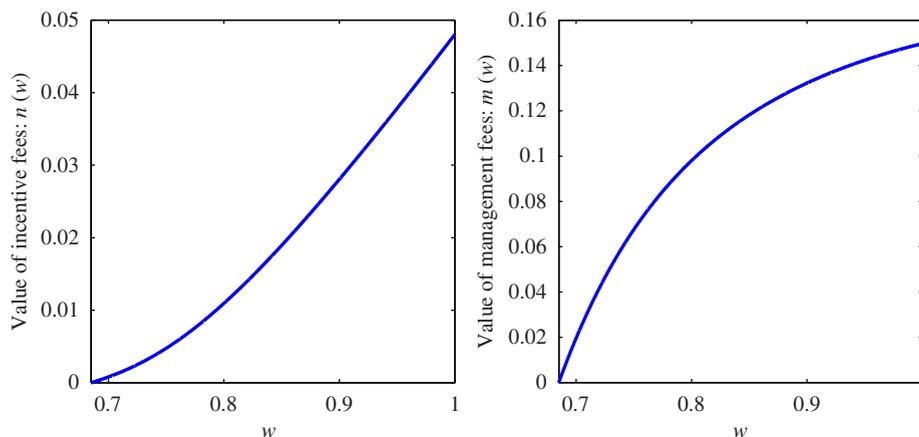
4.5. Investors' payoff p(w) and the fund's total value v(w)

Panels A and B of Fig. 6 plot the scaled investors' value $p(w)$ and its sensitivity $p'(w)$ with respect to AUM. Note that $p(w)$ is convex for $w \leq 0.80$ and concave for $w \geq 0.80$. Panels C and D of Fig. 6 plot the scaled total fund value $v(w)$, and the sensitivity $v'(w)$ with respect to AUM. Total fund value $V(W,H)$ is increasing and concave in AUM W.

4.6. How important is the manager's alpha?

Table 2 reports the comparative static effects of changing the unlevered α on leverage $\pi(1)$, endogenous risk attitude $\psi(1)$, and various values. Recall that in the baseline case (the highlighted row), investors break even, $p(1)=1$, and managers collect $f(1)=0.2$ in PV via fees for their skills. Other than the baseline case, investors either make losses ($p(1)<1$) or collect surplus ($p(1)>1$) in our comparative static exercises, as we expect. The bottom-line of this comparative static exercise is that it is important for both investors and the manager to correctly assess the manager's skill, alpha, from a quantitative perspective.

As we decrease α from 1.22% to 0.61%, investors lose 10.7% by investing with a manager whose unlevered alpha is only half of the baseline value 1.22%. The manager still collects about 16.3% in total fees where the vast majority of the fees, 14.1% out of the total 16.3%, come from management fees, as we expect. Intuitively, a less skillful manager milks investors by overcharging fees, primarily the management fees. This calculation shows that a modern misassessment of the manager's skill leads to significant value losses to investors.

By doubling the manager's skill (increasing the unlevered α from 1.22% to 2.44%), investors' value $p(1)$ increases by about 50% from one to 1.49. And the manager's total value $f(1)$ almost doubles from 19.8% to 39.0%, with the majority of the increase coming from incentive fees $n(1)$, as we expect. This calculation suggests that if investors can

Fig. 5. The scaled PV of incentive fees $n(w)$ and the scaled PV of management fees $m(w)$. We plot the scaled value of management fees $m(w)$ and the scaled value of incentive fees $n(w)$ for our baseline model developed in Sections 2–4 in Panels A and B, respectively. At the inception of a fund ($w=1$), for each unit of the fund's AUM, the manager earns 15 cents in PV via management fees, $m(1)=0.15$, and earns 5 cents in PV via incentive fees, $n(1)=0.05$, out of the total manager's value $f(1)=0.2$. Management fees contribute to the majority of total management compensation. (A) The scaled value of incentive fees $n(w)$. (B) The scaled value of management fees $m(w)$.
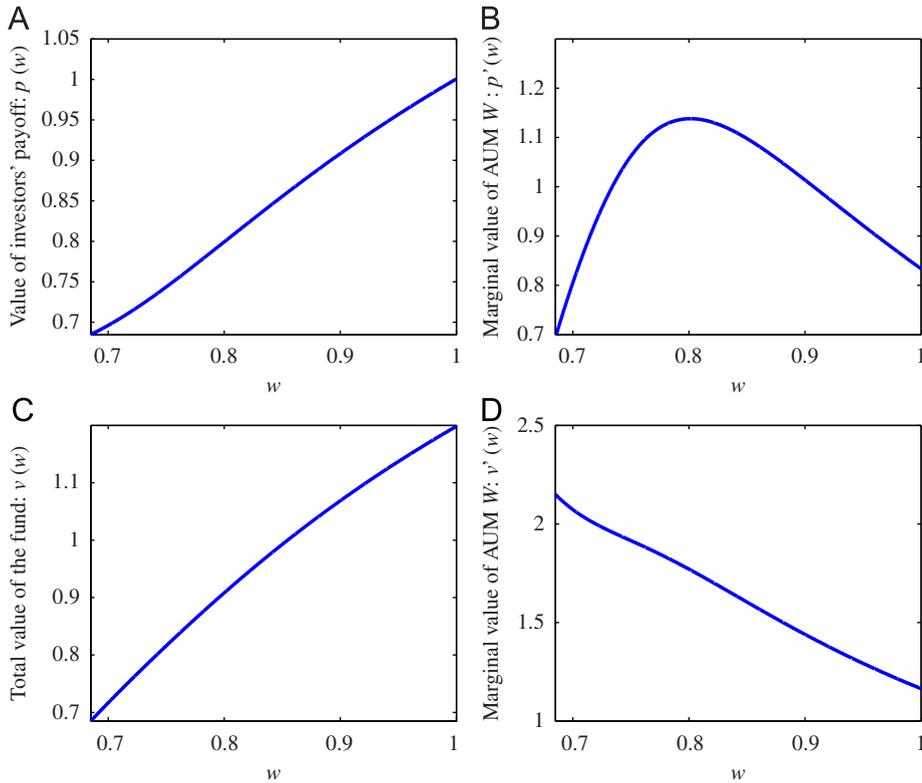
Fig. 6. The scaled investors' value $p(w)$, the scaled total fund value $v(w)$, and their sensitivities, $p'(w)$ and $v'(w)$, with respect to AUM W. This figure plots the investors' value in the baseline case where we require investors break even at time 0, i.e. $p(1) = 1$. (A) The investors' scaled value $p(w)$. (B) The investors' marginal value of the AUM $p'(w)$. (C) The fund's total scaled value $v(w)$. (D) The fund's marginal value of the AUM W, $v'(w)$.

Table 2
Comparative static effects of the unlevered α on leverage $\pi(1)$, the manager's endogenous risk attitude $\psi(1)$, the manager's value $f(1)$, the value of management fees $m(1)$, the value of incentive fees $n(1)$, investors' payoff $p(1)$, and the total fund's value $v(1)$.

For all cases, we vary the manager's unlevered α but keep all remaining parameter values the same as those given in Table 1. For the baseline case (in bold), investors break even at the inception of the fund, $p(1) = 1$. Managers make 20 cents for each dollar under management in PV, i.e. $f(1) = 0.2$. Out of the total management compensation $f(1) = 0.2$, three quarters are management fees, $m(1) = 0.15$, and the remaining one quarter are incentive fees, $n(1) = 0.05$. If investors hired a manager with $\alpha = 2.44\%$, twice as large as the break-even alpha 1.22%, and used the same compensation contract (2–20), they would gain 49% in PV from their hedge fund investments, i.e. $p(1) = 1.49$. However, if investors hired a manager whose alpha is 0.61%, only half of the break-even benchmark value and still compensated the manager with 2–20, investors would lose 11% in PV from their hedge fund investments, $p(1) = 0.89$. Finally, by hiring an unskilled manager with $\alpha = 0$, investors would lose 15% in PV from their investments in the hedge fund, i.e. $p(1) = 0.85$, and the manager would collect 15% of the AUM in PV as transfers from investors via fees, $f(1) = 0.15$. Rational investors will not hire a manager with $\alpha < 1.22\%$ in our example.

α	$\pi(1)$	$\psi(1)$	$m(1)$	$n(1)$	$f(1)$	$p(1)$	$v(1)$
0	0	2.0425	0.1495	0	0.1495	0.8505	1
0.61%	2.5038	1.3381	0.1413	0.0213	0.1627	0.8934	1.0560
1.22%	**3.1753**	**2.1105**	**0.1497**	**0.0480**	**0.1977**	**1**	**1.1977**
2.44%	4.0000	2.6598	0.2248	0.1653	0.3901	1.4899	1.8800

hire a manager who is more skilled than the market expectation, investors can benefit significantly. Correctly assessing the manager's skill is thus essential for investors,

which is consistent with the commonly held belief in the hedge fund industry.

Next, we consider the special case where the manager has no skills, $\alpha = 0$. With no skills, the manager behaves quite prudently by optimally choosing no leverage. Since the growth rate of HWM is indexed to r, the manager thus collects no incentive fees, $n(1) = 0$, but still collects management fees. Table 2 reports that $m(1)$ is effectively unchanged, remaining at 15%. Investors are thus worse off in that $p(1)$ is lowered by 15% from one to 0.85. Management fees transfer wealth from investors to the manager. Hiring an unskilled manager with a two-twenty compensation contract costs investors about 15% of their AUM in PV, which is very costly.

4.7. Leverage constraint

Downside liquidation risk plays a critical role by giving the risk-neutral manager incentives to potentially behave in a risk-averse manner and hence delivers sensible economic predictions in our model. We now conduct the comparative static analysis with respect to the downside risk. As we decrease the liquidation boundary b, the downside risk decreases, leverage increases, and hence the constraint $\pi(w) = \overline{\pi}$ may bind.

Dynamic leverage $\pi(w)$ and endogenous risk attitude $\psi(w)$: Fig. 7 and Table 3 report the comparative static analysis of leverage policy $\pi(w)$ and endogenous risk attitude $\psi(w)$ with respect to the liquidation boundary b.
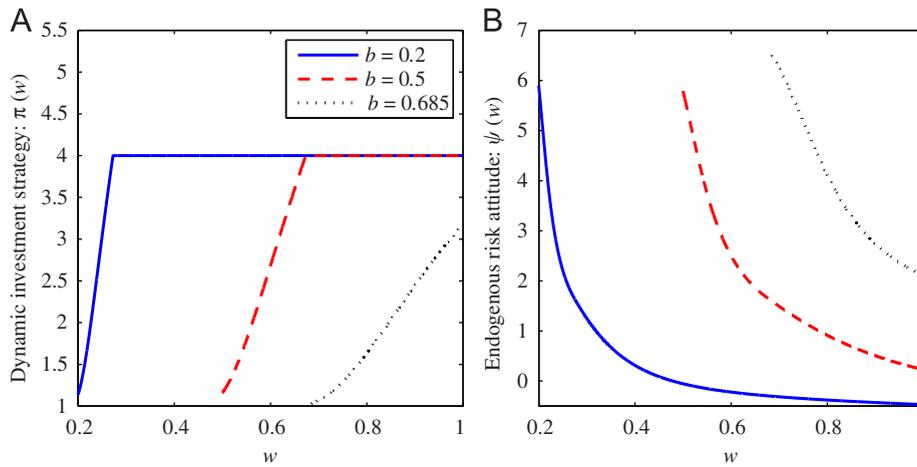
Fig. 7. Dynamic leverage $\pi(w)$ and the manager's endogenous risk attitude $\psi(w)$ for various levels of liquidation boundary $b = 0.2, 0.5, 0.685$. This figures demonstrates the effects of the downside liquidation risk on the manager's leverage choice $\pi(w)$ and endogenous risk attitude $\pi(w)$. As we lower the downside liquidation risk by decreasing b, the manager increases leverage for all values of w. As a result, the leverage constraint becomes binding for sufficiently high values of w for the cases with $b=0.5$ and $b=0.2$. For the case with $b=0.5$, the downside liquidation risk is lower than the baseline case with $b=0.685$, but is still sufficiently high. The leverage constraint $\pi \leq 4$ becomes binding but the manager remains endogenously risk averse for all value of w, i.e., $\psi(w) > 0$ for all w. For the case with $b=0.2$, the leverage constraint $\pi \leq 4$ becomes binding for $w \geq 0.27$. Moreover, for $0.48 \leq w \leq 1$, the manager becomes risk loving, and $\psi(w) \leq 0$. In this case, leverage constraint is critical to ensure that the manager's problem has an economically meaningful finite leverage. (A) Dynamic leverage policy $\pi(w)$. (B) The manager's endogenous risk attitude $\psi(w)$.

Table 3

Comparative static effects of the liquidation boundary b on leverage $\pi(1)$, the manager's endogenous risk attitude $\psi(1)$, the manager's value $f(1)$, the value of management fees $m(1)$, the value of incentive fees $n(1)$, investors' payoff $p(1)$, and the total fund's value $v(1)$.

For all cases, we vary the liquidation boundary b but keep all remaining parameter values the same as those given in Table 1. For the baseline case (in bold) with $b=0.685$, leverage $\pi(1) = 3.17$, the manager's endogenous risk aversion $\psi(1) = 2.11$, the manager's value $f(1) = 0.2$, and investors break even, $p(1) = 1$. As we decrease b to 0.5, the manager increases leverage causing the leverage constraint to bind, $\pi(1) = 4$, the manager's endogenous risk aversion $\psi(1)$ decreases significantly to 0.22, the manager's value of total fees $f(1)$ increases to 0.31, investors are also better off with 15% surplus in PV (netting fees), $p(1) = 1.15$. Further lowering the liquidation boundary b to 0.2 causes the manager to be risk seeking, $\psi(1) = -0.47 < 0$, and the leverage constraint becomes binding, $\pi(1) = 4$. As we lower the liquidation boundary b, both the value of management fees $m(1)$ and the value of incentive fees $n(1)$ increase as the fund has a longer (expected) horizon and the manager's endogenous risk attitude $\psi(1)$ decreases.

b	$\pi(1)$	$\psi(1)$	$m(1)$	$n(1)$	$f(1)$	$p(1)$	$v(1)$
0.2	4	−0.4702	0.2445	0.1235	0.3680	1.2259	1.5939
0.5	4	0.2269	0.1981	0.1091	0.3072	1.1467	1.4539
0.685	**3.1753**	**2.1105**	**0.1497**	**0.0480**	**0.1977**	**1**	**1.1977**

We choose $b = 0.2, 0.5$, and 0.685 and set the leverage constraint $\bar{\pi} = 4$.

First, we restate the results from our baseline case. With sufficiently large liquidation risks ($b=0.685$), the leverage constraint $\pi \leq 4$ does not bind (the maximal leverage is $\pi(1) = 3.18$), and hence the value of relaxing this leverage constraint has no value for the manager.

Second, as we decrease b from 0.685 to 0.5, with more flexibility in managing downside liquidation risk, the manager rationally increases leverage $\pi(w)$. The leverage constraint (4) then binds for $w \geq 0.67$ but remains non-binding otherwise. Importantly, the manager is endogenously risk

averse for all w when $b=0.5$, $\psi(w) > 0$, as seen from Panel B of Fig. 7.

Finally, the manager may be *risk seeking* when liquidation risk is low. When $b = 0.2$, in the region $0.48 \leq w \leq 1$, the leverage constraint binds, $\pi(w) = \bar{\pi} = 4$, as the manager is endogenously risk seeking ($\psi(w) \leq 0$). Table 3 reports $\psi(1) = -0.4702$. In the region $0.27 \leq w < 0.48$, the leverage constraint also binds $\pi(w) = \bar{\pi} = 4$ but the manager behaves in a risk-averse manner, $\psi(w) > 0$. When $w < 0.27$, the manager is sufficiently risk averse and hence the optimal leverage $\pi(w) < \bar{\pi}$ does not bind.

Importantly, we note that the manager may engage in risk seeking for high values of w but still behaves prudently for low w. This is the opposite of the typical risk-seeking intuition in option-based models. We often hear that managers become risk seeking as the firm gets close to liquidation because the manager effectively holds an option with convex payoffs. This intuition does not apply here because debt in our model is risk-free and the manager's loss upon liquidation is quite costly in contrast to limited downside losses. Managerial survival is the dominant consideration in our model when the fund performs poorly.

For simplicity, we have so far intentionally chosen a parsimonious baseline model. In the next three sections, we extend our model along three important dimensions: managerial ownership, new money flows, and managerial voluntary fund closure/restart options.

5. Managerial ownership

Hedge fund managers often have equity positions in funds that they run, which potentially mitigates managerial conflicts with investors. Aragon and Nanda (2012) empirically study the effects of insiders' equity ownership

on the funds' risk taking. We next incorporate inside ownership into our analysis.

Let ϕ denote managerial percentage ownership in the fund. For simplicity, we assume that ϕ remains constant over time. Let $Q(W, H)$ denote the manager's total value including both the value of total fees $F(W, H)$ and the manager's pro rata share of investors' value $\phi P(W, H)$,

$$Q(W, H) = F(W, H) + \phi P(W, H)$$
$$= (1 - \phi)F(W, H) + \phi V(W, H), \tag{35}$$

where the second equality states that the manager's total value $Q(W, H)$ is given by the sum of (1) the PV of fees charged on outside investors $(1 - \phi)F$ and (2) the total fund's value owned by the manager, $\phi V(W, H)$. For simplicity, in this section we assume that the manager's subjective discount rate β equals to r.

The manager dynamically chooses the investment policy π to maximize (35). Using the homogeneity property, we write $Q(W, H) = q(w)H$ where $q(w)$ is the manager's scaled total value. In the Appendix, we show that the optimal investment strategy $\pi(w)$ is given by

$$\pi(w) = \begin{cases} \min\left\{ \dfrac{\alpha}{\sigma^2 \psi_q(w)}, \overline{\pi} \right\}, & \psi_q(w) > 0, \\ \overline{\pi}, & \psi_q(w) \leq 0, \end{cases} \tag{36}$$

where $\psi_q(w)$ is the manager's endogenous risk attitude defined by

$$\psi_q(w) = -\frac{w q''(w)}{q'(w)}. \tag{37}$$

With managerial ownership ϕ, the manager's endogenous risk attitude $\psi_q(w)$ and hence leverage $\pi(w)$ naturally depend on $q(w)$, the sum of the value of fees $f(w)$, and the value of the fund's ownership $\phi p(w)$. The Appendix provides the ODE and boundary conditions for $q(w)$.

Table 4 shows the effects of managerial ownership ϕ on leverage and values, using the parameter values as in the baseline calibration. As we increase ϕ from zero to 20%, leverage $\pi(1)$ increases from 3.18 to 3.30, and endogenous risk attitude $\psi_q(1)$ changes from 2.11 to 2.03, as managerial

ownership improves incentive alignments between the manager and investors by making the manager less concerned about liquidation risk. Managerial ownership increases the value of incentive fees $n(1)$ but lowers the value of management fees $m(1)$, as higher leverage makes liquidation more likely.

Fig. 8 plots the manager's optimal leverage $\pi(w)$ and the risk attitude measure $\psi_q(w)$ for $\phi = 0.2$ and $\phi = 0$ (the baseline case). All other parameter values are the same for the two cases. Leverage $\pi(w)$ is higher and correspondingly the endogenous risk attitude $\psi_q(w)$ is lower, with managerial ownership $\phi = 0.2$ than without ownership, $\phi = 0$, ceteris paribus. The larger the inside equity position ϕ, the more the manager cares about investors' value $p(1)$, encouraging the manager to choose a higher leverage.

6. New money flow

Money chases performances in hedge funds. We next incorporate this feature into the baseline model. Agarwal, Daniel, and Naik (2009) empirically study the effects of money flows on managerial incentives and fund performances.[20] We show that the manager benefits significantly from new money flows.

6.1. Model setup

We model performance-triggered money inflows as follows. Whenever the fund's AUM exceeds its HWM, new money flows into the fund. Let dI_t denote the new money inflows over time increment $(t, t + \Delta t)$. We assume that

$$dI_t = i[dH_t - (g - \delta)H_t \, dt], \tag{38}$$

where the constant parameter $i > 0$ measures the sensitivity of dI_t with respect to the fund's profits measured by $[dH_t - (g - \delta)H_t \, dt] \geq 0$. Because the HWM H grows deterministically at the rate of $(g - \delta)$ in the interior region $W < H$, the fund's AUM only exceeds its HWM, and new money subsequently flows in, when $W_t = H_t$ and $dH_t - (g - \delta)H_t \, dt > 0$. For example, suppose $W_t = H_t = 100$ and the next year's realized AUM is $W_{t+1} = 115$. Then, the manager collects $3 = 20\% \times 15$ in incentive fees. With $i = 0.8$, the new money inflow is $dI_t = 12 = 0.8 \times 15$, which is 12% of the fund's AUM $W_t = 100$.

New money flow increases the fund's AUM which in turn rewards the manager with more future fees. Let $X(W, H)$ denote the present discounted amount of all future money inflows,

$$X(W, H) = \mathbb{E}_t\left[\int_t^{\tau} e^{-r(s - t)} i[dH_s - (g - \delta)H_s \, ds] \right], \tag{39}$$

where τ is stochastic liquidation time. Let $x(w) = X(W, H)/H$.

Current and future investors may attach different values to the fund. Let $P_1(W, H) = p_1(w)H$ and $P_2(W, H) = p_2(w)H$ denote the PV of current investors' value and the

Table 4

Comparative static effects of managerial ownership ϕ on leverage $\pi(1)$, the manager's endogenous risk attitude $\psi_q(1)$, the manager's value $f(1)$, the value of management fees $m(1)$, the value of incentive fees $n(1)$, investors' payoff $p(1)$, and the total fund's value $v(1)$.

For all cases, we vary the manager's fund ownership ϕ but keep all remaining parameter values the same as those given in Table 1. For the baseline case (in bold) with $\phi = 0$, leverage $\pi(1) = 3.17$, the endogenous risk aversion $\psi_q(1) = 2.11$, the manager's value $f(1) = 0.2$, and investors break even, $p(1) = 1$. As we increase managerial ownership ϕ, leverage $\pi(1)$ increases, and the manager's endogenous risk attitude $\psi_q(1)$ decreases. The value of management fees $m(1)$ decreases as the fraction of the AUM raised from outside investors decreases. The value of incentive fees $n(1)$ increases with ϕ as agency costs are reduced by inside equity.

ϕ	$\pi(1)$	$\psi_q(1)$	$m(1)$	$n(1)$	$f(1)$	$p(1)$	$v(1)$
0	**3.1753**	**2.1105**	**0.1497**	**0.0480**	**0.1977**	**1**	**1.1977**
10%	3.2300	2.0749	0.1442	0.0528	0.1970	1.0142	1.2112
20%	3.3038	2.0282	0.1388	0.0568	0.1956	1.0242	1.2197
50%	3.5188	1.9042	0.1255	0.0646	0.1901	1.0408	1.2309

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[20] Chevalier and Ellison (1997) empirically study flow-induced risk-taking by mutual fund managers. Basak, Pavlova, and Shapiro (2007) study mutual fund managers' risk-taking induced by an increasing and convex relationship of fund flows to relative performances.
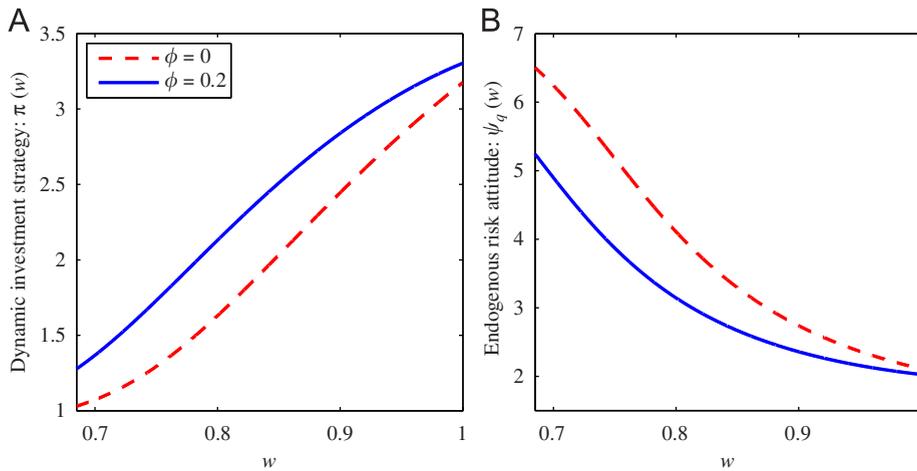
Fig. 8. Dynamic leverage strategy $\pi(w)$ and endogenous risk attitude $\psi_q(w)$ with managerial ownership ($\phi = 0.2$). This figure shows the effects of managerial ownership ϕ on dynamic leverage policy $\pi(w)$ and the manager's endogenous risk aversion $\psi(w)$. Increasing managerial ownership alleviates conflicts of interest between investors and the manager and hence, the manager becomes less concerned about their compensation and rationally increases leverage. As a result, the manager's endogenous risk aversion decreases with managerial ownership ϕ in the fund. Incentives are better aligned with a more concentrated managerial ownership. (A) The manager's dynamic leverage policy $\pi(w)$. (B) The manager's endogenous risk aversion $\psi(w)$.

PV of future investors' contributed capital, respectively. The sum $p_1(w) + p_2(w)$ gives the scaled value of all investors' value, $p(w)$. In general, for future investors, the PV $p_2(w)$ differs from the discounted amount $x(w)$. Because new money only possibly flows into the fund at $w=1$, current and future investors share the same HWM. This property substantially simplifies our analysis because we only need to track a single fund-wide HWM for all investors. See the Appendix for technical details.

6.2. Results

Fig. 9 plots leverage $\pi(w)$ and the manager's value $f(w)$ for the case with new money flow, $i=0.8$, and then compares to the baseline case with $i=0$. Panel A of Fig. 9 plots leverage $\pi(w)$. Leverage is substantially higher with new money inflow, $i=0.8$, than with $i=0$. The leverage constraint (4) binds for $w \geq 0.909$ with $i=0.8$. Panel B of Fig. 9 plots $f(w)$. Intuitively, $f(w)$ is higher with $i=0.8$ than with $i=0$. The impact of new money flow on $f(w)$ is much bigger for larger values of w. For example, at $w=1$, $f(1)$ increases significantly from 0.198 to 0.261. The new money inflow rewards the manager by increasing the AUM size on which the manager collects future management and incentive fees. With more rewards at the upside, the manager behaves in a less risk-averse manner.

Table 5 reports the comparative static effect of new money flow i. Quantitatively, the effect of new money flow is significant. As we increase i from zero to one, leverage $\pi(1)$ increases from 3.18 to 4.00 (where the leverage constraint binds), and the manager's total value $f(1)$ increases by 46% from 0.2 to 0.29. Mostly, the new money flow effect operates through the value of incentive fees $n(1)$, which increases by 2.3 times from 4.8% to 11.2%. New money flow rewards the manager when the fund is doing well and thus strongly influences the value of incentive fees $n(1)$.

What is the fund's expected discounted amount of new money flow (scaled by H), $x(1)$? For $i=1$, we have $x(1) = 0.557$, which states that the new money flow

increases the effective AUM by 56% in the PV sense. While most benefits of the new money flow accrue to the manager, investors are also better off in our comparative analysis as the manager becomes less risk averse. The current investors' value $p_1(1)$ increases by 8% from 1 to 1.079 as the new money flow i increases from zero to one. Interestingly, future investors are also better off by 8% per unit of AUM. This is due to the property that all investors, current and future, in our model share the same HWM, which substantially simplifies our analysis.

7. Restart options

In reality, hedge fund managers often have outside options. It may sometimes be optimal for managers to voluntarily close the current fund as its AUM is sufficiently down from its HWM and incentive fees become sufficiently out of the money. An *Economist* magazine article titled "Hedge-fund closures: Quitting while they're behind" (published on February 18, 2012) provides one such topical discussion.[21] We next model the manager's optimal fund closure and restart options.

7.1. Model setup and solution

We analyze a stationary framework with infinite restart options. In reality, the manager has finite restart options. We later provide a sensitivity analysis with respect to the number of restart options.

A stationary model with infinite restart options: At any moment when the current fund's AUM is W and its HWM is H, the manager has an option to start a new fund with a new AUM, which we denote as $S(W,H)$. Let ν denote the ratio between the new fund's AUM $S(W,H)$ and the previous fund's closing AUM W, i.e., $\nu = S(W,H)/W$. To illustrate the effects of restart options, we assume that the

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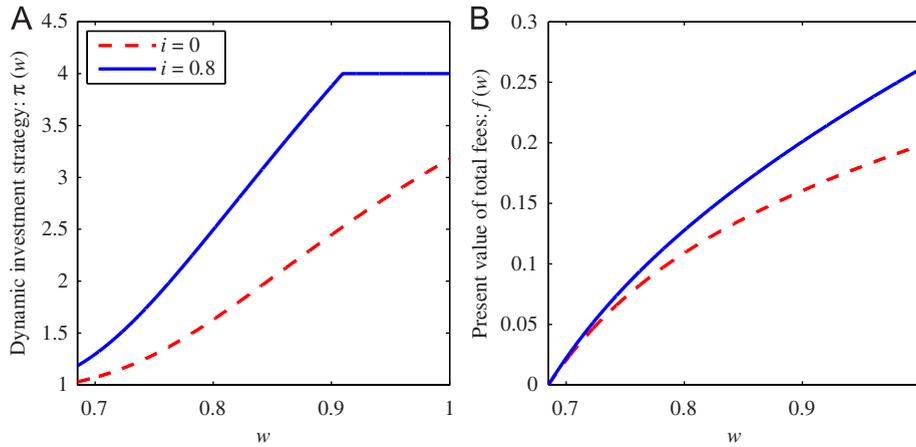[21] See http://www.economist.com/node/21547807.

Fig. 9. Hedge fund leverage $\pi(w)$ and the manager's scaled value function $f(w)$ with new money flows. This figure demonstrates the effects of new money flows on the manager's optimal leverage $\pi(w)$ and the manager's scaled value $f(w)$. Specifically, new money flows in whenever the fund's AUM W exceeds its HWM H, and the inflow amount is given by $i \times [dH_t - (g-\delta) dH_t] \geq 0$. The case with $i=0$ is the baseline case with no money flow. For the case with $i=0.8$, if the profit is 10, the money inflow is 8. By incorporating the positive effect of performance on money flow, we find that the manager increases the fund's leverage as the firm's performance delivers more rewards to the manager via future management and incentive fees. Anticipating this positive feedback effect between performance and money inflows, the manager rationally increases leverage. As a result, the manager's value $f(w)$ increases especially for high values of w. (A) Dynamic leverage policy $\pi(w)$. (B) The manager's scaled value $f(w)$.

Table 5

Comparative static effects of new money flow i on leverage $\pi(1)$, the manager's endogenous risk attitude $\psi(1)$, the manager's value $f(1)$, the value of management fees $m(1)$, the value of incentive fees $n(1)$, the current investors' payoff $p(1)$, and the discounted amount of new money flow $x(1)$.

In our model, new money flows in when the fund's AUM W exceeds its HWM H, and the inflow amount is given by $i \times [dH_t - (g-\delta) dH_t] \geq 0$. Here, the parameter i measures the sensitivity of new money inflow with respect to the performance. For all cases, we vary the new money inflow i but keep all remaining parameter values the same as those given in Table 1. For the baseline case (in bold) with no money inflow, $i=0$, leverage $\pi(1)=3.17$, the endogenous risk aversion $\psi(1)=2.11$, the manager's value $f(1)=0.2$, and investors break even, $p(1)=1$. By increasing i, we see that leverage $\pi(1)$ increases, and the manager's endogenous risk attitude $\psi(1)$ decreases. Both the value of management fees $m(1)$ and the value of incentive fees $n(1)$ increase. The quantitative effects of new money flow are very large. The case with $i=0.8$ implies that the money inflow is 8 if the realized profit $dH_t - (g-\delta)H_t dt$ is 10. Note that the discounted amount of total new money inflow, $x(1)$, is 0.38. That is, in expectation, the discounted total money inflow is 38% of the fund's current AUM. Moreover, with $i=0.8$, the current investors are also better off by 6.7%, i.e. $p_1(1)=1.067$ due to better aligned incentives and less severe under-leveraging result.

i	$\pi(1)$	$\psi(1)$	$m(1)$	$n(1)$	$f(1)$	$p_1(1)$	$x(1)$
0	**3.1753**	**2.1105**	**0.1497**	**0.0480**	**0.1977**	**1**	**0**
0.2	3.6565	1.8325	0.1511	0.0574	0.2085	1.0186	0.0573
0.5	4.0000	1.5189	0.1556	0.0744	0.2300	1.0441	0.1859
0.8	4.0000	1.2212	0.1661	0.0948	0.2609	1.0665	0.3790
1.0	4.0000	1.0260	0.1777	0.1115	0.2892	1.0790	0.5572

ratio ν satisfies

$$\nu(w) = \theta_0 + \theta_1 w + \frac{\theta_2}{2} w^2, \qquad (40)$$

where $\theta_0, \theta_1 > 0$, and $\theta_2 < 0$ are all constant. Intuitively, the better the fund's performance, the larger ν. Additionally, the impact of w on ν diminishes as we increase w. By closing the existing fund and starting a new one, the manager benefits by resetting the fund's HWM and hence being much closer to collecting incentive fees but forgoes

the fees on the closed fund. Additionally, the new fund's AUM $S(W, H)$ may be smaller than the closed fund's AUM W. Finally, it is costly to close the existing fund and start a new fund. The manager optimally chooses the closure/restart timing as well as leverage policies.

Let $f_\infty(w)$ denote the manager's scaled value with infinite restart options. Let $\overline{w}_\infty$ denote the optimal threshold for the restart option. The manager chooses $\overline{w}_\infty$ so that

$$f_\infty(\overline{w}_\infty) = \overline{w}_\infty \nu(\overline{w}_\infty) f_\infty(1), \qquad (41)$$

$$f'_\infty(\overline{w}_\infty) = (\nu(\overline{w}_\infty) + \overline{w}_\infty \nu'(\overline{w}_\infty)) f_\infty(1). \qquad (42)$$

The value-matching condition (41) requires that the manager's value $f_\infty(w)$ is continuous at the moment of abandoning the existing fund and starting a new fund. The smooth-pasting condition (42) ensures that $\overline{w}_\infty$ is optimally chosen. Finally, the ODE for $f_\infty(w)$ is the same as (22), the one for the baseline case.

We may equivalently interpret restart options in our model as options to reset the fund's HWM. Resetting the HWM allows the manager to collect incentive fees much sooner, but also causes some investors to withdraw their capital or leave the fund. We see that both fund restart and HWM reset interpretations are consistent with our model.

7.2. Model results

Parameter choice and calibration: We now calibrate the three new parameters, θ_0, θ_1, and θ_2, in (40). We target (1) the restart boundary W to be 80% of the fund's AUM H, $\overline{w}_\infty = 0.8$; (2) the subsequent fund's AUM to be 75% of the previous fund's AUM, $\nu(\overline{w}_\infty) = 0.75$; and (3) the new fund's size is zero when the manager is forced to liquidate at $w=b$, $\nu(b)=0$. Using these three conditions, we obtain $\theta_0 = -24.75$, $\theta_1 = 61.47$, and $\theta_2 = -74$. Other parameter values are the same as in the baseline calibration.

Leverage $\pi(w)$ and the PV of total fees $f(w)$: Panel A of Fig. 10 plots the optimal leverage $\pi(w)$. Restart options
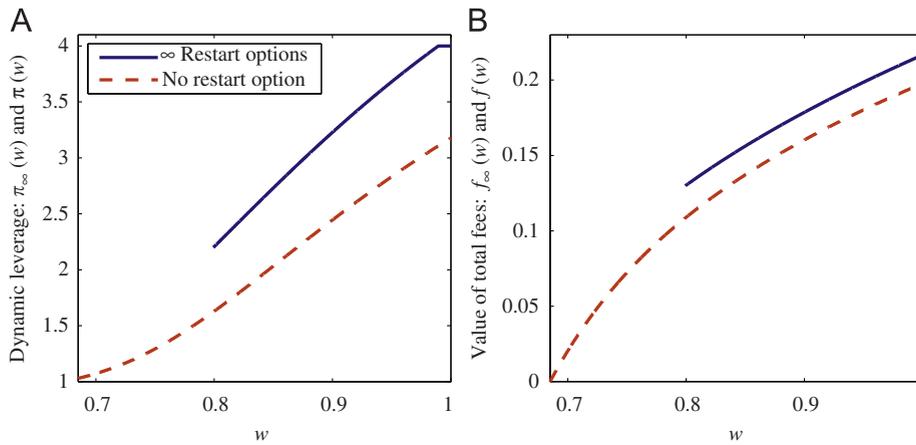
Fig. 10. Hedge fund leverage strategy $\pi(w)$ and the PV of total fees $f(w)$ when fund managers can close the current fund and restart new funds. This figure demonstrates the effects of the manager's endogenous restart options on optimal leverage $\pi(w)$ for the current fund and the manager's scaled value $f(w)$. The manager's value $f(w)$ now includes both the fees from the current fund and also the fees from the future funds. The manager's main benefit of starting a new fund is to reset the HWM, but the new fund is likely to be smaller if the fund manager's performance is not strong and w is not high. With options to close the current fund and to start a new fund, the manager increases the current fund's leverage as the manager now has a higher continuation value with restart options than without. These restart options effectively decrease the relative importance of the current fund and hence cause the manager to increase leverage as the manager is less concerned about the current fund's survival. As a result, the manager's value $f(w)$ increases especially for high values of w. However, quantitatively, our calculation shows that restart options only increase the manager's value $f(w)$ by 10% due to the following reasons. First, future funds are smaller if the fund manager starts a new fund after poor performance. Second, fees in the future from the new funds are discounted in PV. (A) Dynamic leverage policy $\pi(w)$. (B) The manager's value $f(w)$.

make the manager less risk averse, which causes the leverage constraint to bind, $\pi(1) = 4.00$. In our calibration, at the moment of starting up the new fund, the fund is 20% down from its HWM, i.e., $\overline{w}_\infty = 0.8$. The corresponding leverage $\pi(0.8) = 2.20$, which is larger than $\pi(0.8) = 1.63$, the leverage in the baseline case with no restart option. Intuitively, restart options cause the manager to be less risk averse and hence leverage increases. Panel B of Fig. 10 plots $f(w)$. At the optimally chosen restart option boundary $\overline{w}_\infty = 0.8$, $f_\infty(0.8) = 0.130$, which is about 20% higher than the manager's value $f(0.8) = 0.109$ when the fund's AUM is 20% down from its HWM in the baseline case with no restart option. Even at $w = 1$, when the restart option becomes least valuable, $f_\infty(1) = 0.217$, which is 10% higher than $f(1) = 0.198$ in the baseline case.

The value of first, second, and remaining restart options: Table 6 reports the results for the sensitivity analysis with respect to the number of restart options.[22] As we increase the number of restart options, the manager exits the fund sooner (higher $\overline{w}$), chooses a more aggressive investment strategy (higher $\pi(w)$), and becomes more risk tolerant (lower $\psi(w)$). As a result, the manager values incentive fees more, but management fees less. For example, the value of incentive fees $n(1)$ increase from 4.8% for the case with no restart option to 10.4% with infinite restart options, while $m(1)$ decreases from 15% to 11.3%, resulting in an increase of total fees $f(1)$ from 20% to 21.7%. Quantitatively, restart options have much stronger effects on the value of incentive fees than on the value of management fees. Our calibrated exercise also suggests that the first few restart options carry

Table 6

Comparative static effects of increasing the number of restart options on the fund's restart boundary $\overline{w}$, leverage $\pi(1)$, the manager's endogenous risk attitude $\psi(1)$, the manager's value $f(1)$, the value of management fees $m(1)$, the value of incentive fees $n(1)$, and the investors' payoff $p(1)$.

In this generalized version, the manager can close the current fund and start a new fund. This table reports the effects of the manager's voluntary fund closure/restart options on leverage and valuation. We increase the number of restart options but keep all remaining parameter values the same as those given in Table 1. For the baseline case (in bold) with no restart options, the manager runs the fund until it is liquidated by investors at $w = b = 0.685$. The manager chooses leverage $\pi(1) = 3.17$, the endogenous risk aversion is $\psi(1) = 2.11$, the manager's value is $f(1) = 0.2$, and investors break even, $p(1) = 1$. With voluntary fund closure/restart options, the manager behaves in a less risk-averse manner. With restart options, the manager cares less about the downside risk, and hence increases leverage, which in turn lowers the value of management fees and raises the value of incentive fees. The incremental value of each restart option decreases quickly because (1) the time value of fees from future funds are small in PV and (2) future fund size are smaller because the manager will only exercise the restart option if the current fund performance is sufficiently poor.

Option #	$\overline{w}$	$\pi(1)$	$\psi(1)$	$m(1)$	$n(1)$	$f(1)$	$p(1)$
0	**0.685**	**3.1753**	**2.1105**	**0.1497**	**0.0480**	**0.1977**	**1**
1	0.788	3.6051	1.8587	0.1390	0.0675	0.2065	1.0070
2	0.794	3.8204	1.7539	0.1308	0.0802	0.2110	1.0072
∞	0.800	4.0000	1.6420	0.1125	0.1041	0.2166	1.0067

the most value for the manager. If the manager restarts twice, that probably indicates the end of the manager's career. Therefore, the manager may still deploy leverage prudently even with restart options as our analysis indicates.

8. Investors' participation constraints

In our preceding analysis, we only require that investors are willing to participate at time 0, which does not

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[22] See the Appendix for the case with one restart option. For cases with multiple restart options, we have more complicated notations, but the analysis is essentially the same.

imply that investors are always willing to participate in the fund. Indeed, in reality, investors may find it optimal to liquidate their fund positions at any chosen time.[23]

Suppose that investors can liquidate the fund at any time. Anticipating investors' liquidation, the manager effectively faces the investors' participation constraint,

$$P(W_t, H_t) \geq W_t, \quad t \leq \tau. \tag{43}$$

This condition states that the investors are always better off by keeping their capital in the fund. We will show that (43) fundamentally influences leverage. First, we summarize the model solution.

Model solution: Using dynamic programming and the homogeneity property, we jointly solve for the manager's optimal leverage policy $\pi(w)$, the manager's value $f(w)$, and the investors' value $p(w)$ via the following two interrelated ordinary differential equations (ODEs):

$$(\beta - g + \delta + \lambda)f(w) = \max_{\pi(w)} cw + [\pi(w)\alpha + r - g - c]wf'(w)$$
$$+ \tfrac{1}{2}\pi(w)^2\sigma^2 w^2 f''(w), \tag{44}$$

$$(r - g + \delta + \lambda)p(w) = (\delta + \lambda)w + [\pi(w)\alpha + r - g - c]p'(w)$$
$$+ \tfrac{1}{2}\pi(w)^2\sigma^2 w^2 p''(w), \tag{45}$$

subject to the following boundary conditions:

$$f(b) = 0, \quad f(1) = (k+1)f'(1) - k, \tag{46}$$

$$p(b) = b, \quad p(1) = (k+1)p'(1), \tag{47}$$

as well as the following investors' participation constraint, implied by (43):

$$p(w) \geq w, \quad b \leq w \leq 1. \tag{48}$$

Importantly, the investors' participation constraint (48) causes the manager's problem and the investors' valuation problem to be intertwined.[24]

Optimal leverage $\pi(w)$ and the manager's endogenous risk attitude $\psi(w)$: Fig. 11 illustrates the effects of the investors' participation constraint $p(w) \geq w$ on optimal leverage $\pi(w)$ and the manager's endogenous risk attitude $\psi(w)$.

Incorporating the investors' participation constraint $p(w) \geq w$ increases leverage $\pi(w)$ and lowers the manager's endogenous risk attitude $\psi(w)$. For sufficiently high w, the quantitative effect of the investors' participation constraint $p(w) \geq w$ is small. The investors' threat of liquidating the fund is relatively low and the manager's leverage tradeoff is essentially the same as in our baseline case, and therefore, $\pi(w)$ increases with w.

Importantly, the investors' participation constraint $p(w) \geq w$ has much greater effects on the manager's endogenous risk attitude $\psi(w)$ and leverage $\pi(w)$ when the fund does poorly. In the low-w region, the manager becomes less prudent and increases leverage $\pi(w)$, as the fund loses

money and w approaches the liquidation boundary b. The intuition is as follows. Without the investors' participation constraint $p(w) \geq w$, the manager tends to behave more prudently by lowering leverage as w decreases for the purpose of survival as in the baseline case (see the dashed line). However, deleveraging when doing poorly is not in the interest of investors. In order to keep investors at bay (i.e., $p(w) \geq w$), the manager has to create enough value for investors by *increasing* leverage even when doing poorly.

We next demonstrate the effects of $p(w) \geq w$ on the investors' value $p(w)$.

The investors' value $p(w)$: Fig. 12 plots the investors' net value in the fund, $p(w) - w$ and clearly illustrates the effect of the investors' participation constraint on investors' net value $p(w) - w$. Without imposing the investors' voluntary participation constraint $p(w) \geq w$, investors lose money in present value by not liquidating the fund in the low-w region, $w \leq 0.806$ (see the dashed line). Investors are thus better off by exercising the liquidation option if feasible in this low-w region. Anticipating being liquidated by investors before w reaches b, a rational manager optimally chooses leverage to ensure that the constraint $p(w) \geq w$ is satisfied. We note that $p(w) \geq w$ almost never binds. Moreover, $p(w)$ is convex in the low-w region and concave in the high-w region.

Finally, we quantify the effects of the investors' participation constraint $p(w) \geq w$. In Table 7, we show that the quantitative effects of the constraint $p(w) \geq w$ are small. The investors' liquidation option is valuable when this option is in the money, i.e., in the low-w region. For high values of w, the investors' voluntary participation constraint $p(w) \geq w$ is effectively not binding. Therefore, the investors' voluntary participation constraint has little value at the fund's inception where $w = 1$.

9. Two special cases

To highlight our model's mechanism and to better relate to the literature, we next summarize Goetzmann, Ingersoll, and Ross (2003) and Panageas and Westerfield (2009), the two papers most closely related to ours, and compare our main results with these two models.

9.1. *Goetzmann, Ingersoll, and Ross (2003) (GIR)*

GIR value hedge fund compensation contracts when managers are paid via both management fees and HWM-indexed incentive fees. GIR derive closed-form solutions for the values of management fees, incentive fees, and investors' value but do not model the manager's decisions such as leverage and fund closure/restart. We summarize their results below.

Proposition 1. By setting $\pi(w) = 1$ at all times and $\beta = r$, we obtain GIR results via the following explicitly solved value functions:

$$n(w) = \frac{k(w^\gamma - b^{\gamma-\zeta}w^\zeta)}{\gamma(k+1) - 1 - b^{\gamma-\zeta}(\zeta(1+k) - 1)}, \tag{49}$$

$$f(w) = \frac{c}{c + \delta + \lambda - \alpha}w$$

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[23] We are grateful to the anonymous referee for suggesting us to include this section to illustrate the significance of investors' voluntary participation constraints.

[24] In contrast, for the baseline case where we only require investors' time-0 voluntary participation, $p(1) \geq 1$, we can proceed in two steps. First, we solve the manager's problem defined by (44) and (46). Second, we use (45) and (47) to obtain the investors' value $p(w)$. This two-step solution procedure for the baseline case is invalid when the manager faces the investors' participation constraint (48).
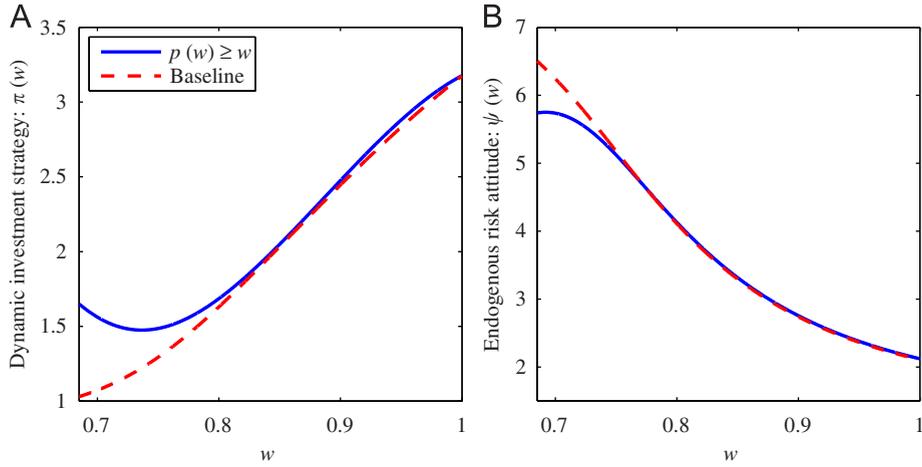
Fig. 11. Hedge fund leverage $\pi(w)$ and the manager's endogenous risk attitude $\psi(w)$: The effects of the investors' participation constraint $p(w) \geq w$. This figure demonstrates the effects of the investors' voluntary participation constraint $p(w) \geq w$ on the manager's leverage choice $\pi(w)$ and the manager's endogenous risk attitude $\psi(w)$. Unlike in the baseline model where we only require $p(1) \geq 1$, $p(w) \geq w$ ensures that investors can exercise the walk-away option from the manager at any time. Importantly, we note that near the liquidation boundary b, the voluntary participation constraint $p(w) \geq w$ induces the manager to increase leverage in order to generate sufficient value for investors so that investors do not want to walk away. As a result, $p(w) \geq w$ lowers the manager's endogenous risk aversion $\psi(w)$ especially near the liquidation boundary b and causes the leverage to be non-monotonic in w. (A) Dynamic leverage policy $\pi(w)$. (B) The manager's endogenous risk attitude $\psi(w)$.
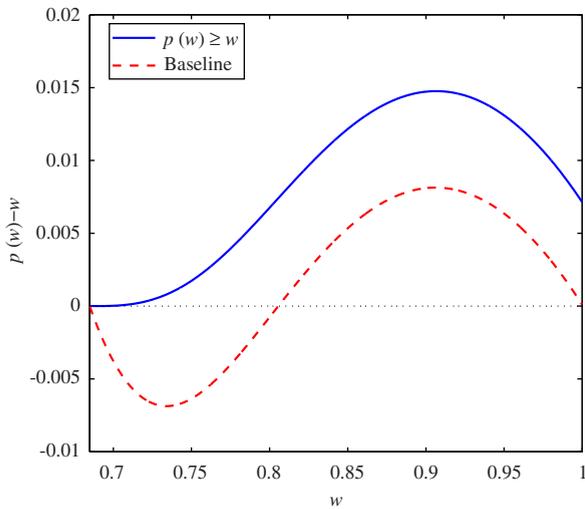

Fig. 12. The PV of investors' net payoff $p(w) - w$ when investors have voluntary participation constraint at all times, i.e. $p(w) \geq w$. This figure demonstrates the effects of the investors' participation constraint $p(w) \geq w$ on the investors' value $p(w)$. Without imposing $p(w) \geq w$, we see that for sufficiently low values of w, $p(w) < w$. See the "Baseline" case in the figure. Intuitively, under this scenario where we only require $p(1) = 1$, the investors are worse off by not walking away from the manager for sufficiently low values of w. Imposing $p(w) \geq w$ at all times improves the investors' payoff $p(1)$. But we note that, ex ante, doing so lowers the manager's scaled value $f(w)$, as the manager faces tighter constraints.

$$+ \frac{(\delta + \lambda - \alpha)k + (\zeta(1+k)-1)cb^{1-\zeta}}{(c + \delta + \lambda - \alpha)(\gamma(k+1)-1-b^{\gamma-\zeta}(\zeta(1+k)-1))} w^{\gamma}$$

$$- \frac{b^{\gamma-\zeta}(\delta + \lambda - \alpha)k + (\gamma(1+k)-1)cb^{1-\zeta}}{(c + \delta + \lambda - \alpha)(\gamma(k+1)-1-b^{\gamma-\zeta}(\zeta(1+k)-1))} w^{\zeta}, \quad (50)$$

$$p(w) = \frac{\delta + \lambda}{c + \delta + \lambda - \alpha} w$$

$$- \frac{(\delta + \lambda)k + (\zeta(1+k)-1)(c-\alpha)b^{1-\zeta}}{(c + \delta + \lambda - \alpha)(\gamma(k+1)-1-b^{\gamma-\zeta}(\zeta(1+k)-1))} w^{\gamma}$$

$$+ \frac{b^{\gamma-\zeta}(\delta + \lambda)k + (\gamma(1+k)-1)(c-\alpha)b^{1-\zeta}}{(c + \delta + \lambda - \alpha)(\gamma(k+1)-1-b^{\gamma-\zeta}(\zeta(1+k)-1))} w^{\zeta}, \quad (51)$$

where γ and ζ are two constants given by

$$\gamma = \frac{1}{2} - \frac{\alpha + r - g - c}{\sigma^2}$$
$$+ \sqrt{\left(\frac{1}{2} - \frac{\alpha + r - g - c}{\sigma^2}\right)^2 + \frac{2(r + \lambda - g + \delta)}{\sigma^2}}, \quad (52)$$

and

$$\zeta = \frac{1}{2} - \frac{\alpha + r - g - c}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\alpha + r - g - c}{\sigma^2}\right)^2 + \frac{2(r + \lambda - g + \delta)}{\sigma^2}}. \quad (53)$$

Finally, $m(w) = f(w) - n(w)$ and the total fund's value is $v(w) = p(w) + f(w)$.

9.2. *Panageas and Westerfield (2009) (PW)*

Building on the valuation framework of GIR, PW derive explicit formulas for leverage and the manager's value, by making two key simplifying assumptions: (1) no liquidation boundary, $b = 0$, and (2) no management fees, $c = 0$. We next summarize the main results in PW.

Proposition 2. With no liquidation boundary ($b = 0$), no management fees ($c = 0$), a non-binding leverage constraint, and $g = \delta = 0$, PW derive the following optimal leverage:

$$\pi = \frac{\alpha}{(1-\eta)\sigma^2}. \quad (54)$$

Table 7

The effects of the investors' participation constraint $p(w) \geq w$ on leverage $\pi(1)$, the manager's endogenous risk attitude $\psi(1)$, the manager's value $f(1)$, the value of management fees $m(1)$, the value of incentive fees $n(1)$, the investors' payoff $p(1)$, and the total value $v(1)$.

In the baseline case, we only require investors to break even at the inception of the fund, i.e. $p(1) = 1$. We now consider the case where investors at least break even at each point in time. That is, at any point in time, investors can walk away. Quantitatively, this walk-away option only makes investors slightly better off, $p(1) = 1.007$. Not surprisingly, the effects on leverage and various values are insignificant at the inception of the fund. Importantly, the investors' participation constraint $p(w) \geq w$ has much greater effects on the manager's endogenous risk attitude $\psi(w)$ and leverage $\pi(w)$ when the fund does poorly. For example, the participation constraint increases the leverage $\pi(b)$ from 1.0301 to 1.6246 at the liquidation boundary $b = 0.685$. Correspondingly, the manager's endogenous risk attitude $\psi(b)$ decreases from 6.5050 in the baseline case to 5.7874.

	$\pi(b)$	$\psi(b)$	$\pi(1)$	$\psi(1)$	$m(1)$	$n(1)$	$f(1)$	$p(1)$
Baseline	**1.0301**	**6.5050**	3.1753	2.1105	0.1497	0.0480	0.1977	1
$p(w) \geq w$	**1.6246**	**5.7874**	3.1765	2.1232	0.1479	0.0494	0.1973	1.0071

The manager's scaled value f(w) is given by

$$f(w) = n(w) = \frac{k}{\eta(k+1)-1} w^\eta, \tag{55}$$

where η is a constant and given by

$$\eta = r + \beta + \lambda + \frac{\alpha^2}{2\sigma^2} - \frac{\sqrt{\left(r + \beta + \lambda + \frac{\alpha^2}{2\sigma^2}\right)^2 - 4r(\beta + \lambda)}}{2r}. \tag{56}$$

To ensure that f(w) is increasing, concave, and finite, PW impose the following condition:

$$\eta(1 + k) > 1. \tag{57}$$

The main results of PW are as follows: (1) leverage $\pi(w)$ is constant at all times; (2) the higher the incentive fee k, the *lower* the manager's value $F(W,H)$; (3) the higher the HWM H, the *higher* the manager's value $F(W,H)$; and (4) the value of incentive fees $N(W,H)$ is concave in W. In PW, the risk-neutral manager is averse to collecting incentive fees too soon, because incentive fees are consumed by the manager rather than being reinvested in the fund to earn a (levered) excess return. The manager may delay the timing of collecting incentive fees by reducing leverage because of aversion to the *upside* risk caused by the incentive payout at $w = 1$. The condition (57), which is Eq. (15) in PW, is critical and ensures that the mechanism discussed above works in PW. However, PW's predictions (2)–(3)–(4) listed above are hard to reconcile with commonly held beliefs.

In contrast, we find that (1) leverage $\pi(w)$ is time-varying and tends to increase following good performances; (2) the higher the incentive fee k, the *higher* the manager's value $F(W,H)$; (3) the higher the HWM, the *lower* the manager's value $F(W,H)$; and (4) the value of incentive fees is convex in W. The two models generate opposite predictions as the mechanisms are different. The manager in our model is mostly averse to the *downside* liquidation risk, while the manager in PW is averse to collecting incentive fees too soon.[25] By incorporating management fees as we do, the manager has more to

lose upon liquidation, which further makes the manager behave prudently.

10. Conclusion

We develop a dynamic model of hedge fund leverage and compensation valuation where the manager dynamically chooses leverage to maximize the PV of the sum of the AUM-based management fees and the HWM-indexed incentive fees from the current and future managed funds (with money inflow/outflow, fund closure/restart options, and investors liquidation options). The risk-neutral manager trades off the value creation by leveraging on the alpha strategy with the cost of inefficient liquidation and redemption/drawdown. The manager has incentives to preserve the fund's going-concern value so as to collect fees in the future. This survival/precautionary motive causes the manager to behave in an endogenously risk-averse manner. The greater the liquidation risk/costs, the more prudently the manager behaves. In our calibrated setting, leverage increases with alpha and decreases with both volatility and the manager's endogenously determined risk attitude. The ratio between the fund's AUM and its HWM, w, measures the manager's moneyness in the fund. The higher the value of w, the less likely the fund is liquidated and the more likely the manager collects the incentive fees, and, consequently, the higher the manager's leverage.

Quantitatively, our calibration suggests that the manager needs to create significant value to justify their compensation contracts. Both management fees and incentive fees are quantitatively important contributors. Our baseline calibration suggests that the manager needs to create 20% surplus in PV on the AUM to justify a two-twenty contract. Out of the manager's total surplus, 75% is attributed to management fees (15 cents on a dollar) and the remaining 25% is due to incentive fees (5 cents on a dollar). By incorporating features such as managerial ownership, new money flows, and fund closure/restart options, the relative contributions of management fees and incentive fees to the manager's total value become more balanced. In private equity, Metrick and Yasuda (2010) also find that management and incentive fees are significant contributors to the manager's value. We show that the investors' investment performance highly depends on their ability to assess the manager's true skills. As Swensen (2000, 2005) do, we emphasize that only sophisticated investors with expertise of and resources to evaluate the manager's investment and value-creating skills

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[25] In our calibration, the "upside" risk of crossing the HWM too soon (the key mechanism in PW) still exists but plays a minor role. This should be expected since our model includes theirs as a special case. However, the key tradeoff and intuition in our model are different from those in PW.

should invest in hedge funds or other sophisticated actively managed investment funds (e.g. private equity and venture capital). Otherwise, the hefty fees and the misalignment of incentives can create significant value losses to investors.

For sophisticated investors, managerial skills may be unknown and time-varying. Even managers may not know their own skill levels. How learning about managerial skills influences investors' asset allocation and managerial decision making is a topic for future research. Moreover, managers with no skills may pretend to be skilled, which further complicates investors' inference and learning about the manager's skill. To highlight the effect of managerial incentives on leverage and valuation, we have intentionally assumed that the managerial skill features constant alpha and Sharpe ratio. It is also important to incorporate managerial risk aversion (Hodder and Jackwerth, 2007) because managers are often poorly diversified and incomes from fund management compensation thus carry additional idiosyncratic risk premiums.

Additionally, the recent crisis reveals that market liquidity and funding liquidity are first-order issues (Brunnermeier and Pedersen, 2009), and that funding costs can increase significantly in crisis periods. Anticipating these liquidity problems, the manager rationally chooses the fund's leverage and prudently manages risk in a state-contingent way (Dai and Sundaresan, 2010 and Liu and Mello, 2011). In future work, we plan to integrate market and funding illiquidity.

Finally, it is likely that the manager's skill (alpha) decreases with fund size. This decreasing-returns-to-scale feature may explain why some funds with strong performance records voluntarily close to new investors. Goetzmann, Ingersoll, and Ross (2003) find that large hedge funds avoid taking new money and argue that hedge funds may not have a constant-returns-to-scale technology. Anticipating the negative impact of fund size on the alpha-generating strategy, the manager chooses leverage even more prudently. Interestingly, the key results that we emphasize in this paper tend to remain valid even with decreasing returns to scale. As the richer specification does not have the homogeneity property in the AUM W and the HWM H, incorporating this feature will significantly complicate our analyses and substantially lengthen our paper. Admittedly, by assumption, the constant-returns-to-scale model inevitably misses the impact of fund size on the alpha-generating strategy, leverage choices, and management compensation valuation.

Appendix A. Details for Section 3

The upper boundary ($W=H$): Our reasoning for the boundary behavior essentially follows GIR and PW. A positive return shock increases the AUM from $W=H$ to $H+\Delta H$. The PV of the manager's total fees is then given by $F(H+\Delta H, H)$ before the HWM adjusts. Immediately after the positive shock, the HWM adjusts to $H+\Delta H$. The manager collects an incentive fee $k\Delta H$, and thus the AUM is lowered from $H+\Delta H$ to $H+\Delta H-k\Delta H$. The PV of total fees is $F(H+\Delta H-k\Delta H, H+\Delta H)$. Using the continuity of $F(W,H)$ before and after the adjustment of the HWM, we have

$$F(H+\Delta H, H) = k\Delta H + F(H+\Delta H-k\Delta H, H+\Delta H). \quad (A.1)$$

By taking the limit as ΔH approaches zero and using Taylor's expansion rule, we obtain

$$kF_W(H,H) = k + F_H(H,H). \quad (A.2)$$

The above is the value-matching condition for the manager on the boundary $W=H$.

The lower liquidation boundary ($W=bH$): At the liquidation boundary $W=bH$, the manager loses all future fees in our baseline model, in that

$$F(bH,H) = 0. \quad (A.3)$$

This assumption is the same as the one in GIR. Unlike GIR, the manager in our model influences the liquidation likelihood via dynamic leverage.

Homogeneous value functions: We conjecture that the value function $F(W,H)$ takes the following homogeneous form in W and H:

$$F(W,H) = f(w)H, \quad (A.4)$$

where $w=W/H$. When the manager's value is given by (A.4), we have

$$F_W(W,H) = f'(w), \quad F_{WW}(W,H) = f''(w)/H,$$
$$F_H(W,H) = f(w)-wf'(w). \quad (A.5)$$

Substituting them into the HJB Eq. (13) and the boundary conditions (A.2)–(A.3), we obtain ODE (22) with boundary conditions (23)–(24).

Appendix B. Details for Section 4

For the boundary behavior at $W_t=H_t$, we use the same argument as the one for $F(W,H)$. Consider a positive return shock that increases the AUM from $W=H$ to $H+\Delta H$. Using the continuity of value function before and after the adjustment of the HWM,

$$M(H+\Delta H, H) = M(H+\Delta H-k\Delta H, H+\Delta H), \quad (B.1)$$

$$N(H+\Delta H, H) = k\Delta H + N(H+\Delta H-k\Delta H, H+\Delta H). \quad (B.2)$$

By taking the limit as ΔH approaches zero and using Taylor's expansion rule, we obtain

$$kM_W = M_H, \quad kN_W = k + N_H. \quad (B.3)$$

Using the homogeneity properties for $M(W,H)$ and $N(W,H)$, we obtain the valuation formulas for $m(w)$ and $n(w)$ reported in Section 4.

Appendix C. Details for Section 5

With managerial ownership, in the region $W < H$, the manager's total value $Q(W,H)$ solves

$$(\beta+\lambda)Q(W,H) = \max_{\pi \leq \bar{\pi}} [c+\phi(\delta+\lambda)]W$$
$$+[\pi\alpha+(r-\delta-c)]WQ_W(W,H)$$
$$+\tfrac{1}{2}\pi^2\sigma^2W^2Q_{WW}(W,H)+(g-\delta)HQ_H(W,H). \quad (C.1)$$

When $\pi < \bar{\pi}$, the FOC for leverage π is

$$\alpha WQ_W(W,H) + \pi\sigma^2 W^2 Q_{WW}(W,H) = 0. \quad (C.2)$$

Using the homogeneity property, $Q(W, H) = q(w)H$, we simplify (C.1) and obtain the following ODE for $q(w)$:

$$(\beta - g + \delta + \lambda)q(w) = [c + \phi(\delta + \lambda)]w$$
$$+ [\pi(w)\alpha + r - g - c]wq'(w) + \tfrac{1}{2}\pi(w)^2\sigma^2 w^2 q''(w). \quad \text{(C.3)}$$

The optimal leverage $\pi(w)$ is given by (36)–(37). Because $q(w) = f(w) + \phi p(w)$, the lower boundary condition becomes $q(b) = \phi b$. Using the same analysis as the one for $F(W, H)$, we obtain the upper boundary condition $q(1) = (k + 1)q'(1) - k$.

Appendix D. Details for Section 6

We solve various value functions in four steps: (1) $f(w)$, $m(w)$, and $n(w)$; (2) the current investors' value $p_1(w)$; (3) the total expected discounted amount of new capital $x(w)$; and (4) the PV of future investors' contributed capital $p_2(w)$.

The PVs of total fees, management, and incentive fees: $f(w)$, $m(w)$, and $n(w)$: The continuity of the manager's value before and after hitting the HWM implies $F(H + \Delta H, H) = k\Delta H + F(H + \Delta H - k\Delta H + i\Delta H, H + \Delta H + i\Delta H)$. By taking the limit as ΔH approaches zero and using Taylor's expansion rule, we obtain

$$(k - i)F_W(H, H) = k + (1 + i)F_H(H, H). \quad \text{(D.1)}$$

Using the homogeneity property, we simplify the boundary condition (D.1) as

$$f(1) = \frac{(k + 1)f'(1) - k}{1 + i}. \quad \text{(D.2)}$$

Similarly, we may also obtain the following boundary conditions for $m(w)$ and $n(w)$:

$$m(1) = \frac{(k + 1)m'(1)}{1 + i}, \quad \text{(D.3)}$$

$$n(1) = \frac{(k + 1)n'(1) - k}{1 + i}. \quad \text{(D.4)}$$

At the liquidation boundary $w = b$, the manager collects no fees, which implies

$$f(b) = m(b) = n(b) = 0. \quad \text{(D.5)}$$

Current investors' value $p_1(w)$: The amount of the new money inflow upon the adjustment of the HWM is $i\Delta H$, which has a PV of $i\Delta H \partial P_1(H + \Delta H, H)/\partial W$. Before the new money inflow, the existing current investors' value (i.e., before HWM adjustment) equals the existing current investors' value after the HWM adjustment minus $i\Delta H \partial P_1(H + \Delta H, H)/\partial W$, the PV of the new money inflow. The continuity of the value function P_1 implies

$$P_1(H + \Delta H, H) = -i\Delta H \frac{\partial P_1(H + \Delta H, H)}{\partial W}$$
$$+ P_1(H + \Delta H(1 - k + i), H + \Delta H + i\Delta H). \quad \text{(D.6)}$$

By taking the limit as ΔH approaches zero and using Taylor's expansion rule, we obtain

$$k\frac{\partial P_1(H, H)}{\partial W} = (1 + i)\frac{\partial P_1(H, H)}{\partial H}. \quad \text{(D.7)}$$

Simplifying the above condition yields

$$p_1(1) = \frac{(k + 1 + i)p_1'(1)}{1 + i}. \quad \text{(D.8)}$$

Using the standard pricing method, the current investors' value $p_1(w)$ solves

$$(r - g + \delta + \lambda)p_1(w) = (\delta + \lambda)w + [\pi(w)\alpha + r - g - c]wp_1'(w)$$
$$+ \tfrac{1}{2}\pi(w)^2\sigma^2 w^2 p_1''(w), \quad \text{(D.9)}$$

with the boundary conditions (D.8) and

$$p_1(b) = b. \quad \text{(D.10)}$$

The current investors' voluntary participation condition is

$$p_1(1) \geq 1. \quad \text{(D.11)}$$

The total expected discounted amount of new money flow $x(w)$: At the moment when new money flows in, $X(W, H)$ satisfies the value-matching condition,

$$X(H + \Delta H, H) = i\Delta H$$
$$+ X(H + \Delta H - k\Delta H + i\Delta H, H + \Delta H + i\Delta H). \quad \text{(D.12)}$$

Intuitively, the first term on the right side gives the instant new money flow and the second term gives the total expected discounted amount of future new money inflows. By taking the limit $\Delta H \to 0$ and using Taylor's expansion rule, we obtain

$$(k - i)X_W(H, H) = i + (1 + i)X_H(H, H). \quad \text{(D.13)}$$

Using the homogeneity property, we simplify (D.13) as follows:

$$x(1) = \frac{(k + 1)x'(1) - i}{1 + i}. \quad \text{(D.14)}$$

The homogeneity property also implies that $x(w)$ satisfies the following ODE:

$$(r - g + \delta + \lambda)x(w) = [\pi(w)\alpha + r - g - c]wx'(w)$$
$$+ \tfrac{1}{2}\pi(w)^2\sigma^2 w^2 x''(w). \quad \text{(D.15)}$$

From ODE (D.15), there is no money inflow when $w < 1$. In sum, $x(w)$ solves (D.15) subject to boundary condition (D.14) and $x(b) = 0$, the condition at the liquidation boundary b.

The PV of future investors' contributed capital $p_2(w)$: Following the same argument as the one for our analysis of $p_1(w)$, the continuity of value function P_2 implies

$$P_2(H + \Delta H, H) = i\Delta H \frac{\partial P_1(H + \Delta H, H)}{\partial W}$$
$$+ P_2(H + \Delta H - k\Delta H + i\Delta H, H + \Delta H + i\Delta H). \quad \text{(D.16)}$$

By taking the limit as ΔH approaches zero and using Taylor's expansion rule, we obtain

$$(k-i)\frac{\partial P_2(H,H)}{\partial W} = i\frac{\partial P_1(H,H)}{\partial W} + (1+i)\frac{\partial P_2(H,H)}{\partial H}. \quad (D.17)$$

Simplifying the above condition yields

$$p_2(1) = \frac{(k+1)p'_2(1) - ip'_1(1)}{1+i}. \quad (D.18)$$

The future investors' scaled value $p_2(w)$ satisfies the following ODE:

$$(r-g+\delta+\lambda)p_2(w) = [\pi(w)\alpha + r - g - c]wp'_2(w)$$
$$+\frac{1}{2}\pi(w)^2\sigma^2 w^2 p''_2(w), \quad (D.19)$$

with the boundary conditions (D.18) and

$$p_2(b) = 0. \quad (D.20)$$

The fund's total net surplus $z(w)$: The scaled fund's value $v(w)$ equals the sum of all investors' value $p(w)$ and the manager's value $f(w)$. Summing the existing capital w and discounted amount of future money inflows $x(w)$ gives the total capital, $w + x(w)$. Let $z(w)$ denote the scaled total net surplus, which equals the difference between $v(w)$ and $w + x(w)$,

$$z(w) = v(w) - (w + x(w)) = p_1(w) + p_2(w)$$
$$+ f(w) - (w + x(w)). \quad (D.21)$$

Appendix E. Details for Section 7

Consider the case where the manager has one restart option. Let W_1 and H_1 denote the first fund's AUM and its HWM at the moment of closure. And the manager has an option to start up a new fund whose size is denoted as W_2, equals $S(W_1, H_1) = v(w_1)W_1$, where $w_1 = W_1/H_1$ and the function $v(w)$ is given in (40).

At the moment of closing the current fund and starting a new one, the manager's value is continuous, in that

$$F_1(W_1, H_1) = F_2(v(w)W_1, v(w)W_1). \quad (E.1)$$

Note that the HWM is reset when the manager restarts the fund as seen from the right side of (E.1). Let $\overline{w}_1$ be the optimal boundary to restart the new fund. We thus have

$$f_1(\overline{w}_1) = \overline{w}_1 v(\overline{w}_1)f_2(1), \quad (E.2)$$

$$f'_1(\overline{w}_1) = (v(\overline{w}_1) + \overline{w}_1 v'(\overline{w}_1))f_2(1), \quad (E.3)$$

where (E.2) is the value-matching condition and the smooth-pasting condition (E.3) describes the manager's optimal exercising of the exit/restart option.

References

Ackermann, C., McEnally, R., Ravenscraft, D., 1999. The performance of hedge funds: risk, return, and incentives. Journal of Finance 54, 833–874.

Agarwal, V., Daniel, N.D., Naik, N.Y., 2009. Role of managerial incentives and discretion in hedge fund performance. Journal of Finance 64, 2221–2256.

Agarwal, V., Daniel, N.D., Naik, N.Y., 2011. Do hedge funds manage their reported returns? Review of Financial Studies 24, 3281–3320.

Agarwal, V., Naik, N.Y., 2004. Risks and portfolio decisions involving hedge funds. Review of Financial Studies 17, 63–98.

Ang, A., Gorovyy, S., van Inwegen, G.B., 2011. Hedge fund leverage. Journal of Financial Economics 102, 102–126.

Aragon, G.O., Nanda, V., 2012. Tournament behavior in hedge funds: high-water marks, fund liquidation, and managerial stake. Review of Financial Studies 25, 937–974.

Basak, S., Pavlova, A., Shapiro, A., 2007. Optimal asset allocation and risk shifting in money management. Review of Financial Studies 20, 1583–1621.

Berk, J.B., Green, R.C., 2004. Mutual fund flows and performance in rational markets. Journal of Political Economy 112, 1269–1295.

Bolton, P., Chen, H., Wang, N., 2011. A unified theory of Tobin's q, corporate investment, financing, and risk management. Journal of Finance 66, 1545–1578.

Brown, S.J., Goetzmann, W.N., Ibbotson, R.G., 1999. Offshore hedge funds: survival and performance, 1989–1995. Journal of Business 72, 91–119.

Brown, S.J., Goetzmann, W.N., Ibbotson, R.G., Ross, S.A., 1992. Survivorship bias in performance studies. Review of Financial Studies 5, 553–580.

Brown, S.J., Goetzmann, W.N., Park, J., 2001. Careers and survival: competition and risk in the hedge fund and CTA industry. Journal of Finance 56, 1869–1886.

Brunnermeier, M., Pedersen, L., 2009. Market liquidity and funding liquidity. Review of Financial Studies 22, 2201–2238.

Buraschi, A., Kosowski, R., Sritrakul, W., 2012. Incentives and endogenous risk taking: implications for hedge fund alphas. Unpublished working paper. Imperial College Business School.

Carpenter, J., 2000. Does option compensation increase managerial risk appetite? Journal of Finance 15, 2311–2331.

Chevalier, J., Ellison, G., 1997. Risk taking by mutual funds as a response to incentives. Journal of Political Economy 105, 1167–1200.

Cuoco, D., Kaniel, R., 2011. Equilibrium prices in the presence of delegated portfolio management. Journal of Financial Economics 101, 264–296.

Dai, J., Sundaresan, S., 2010. Risk management framework for hedge funds: role of funding and redemption options on leverage. Unpublished working paper. Columbia University.

Drechsler, I., 2013. Risk Choice under High-Water Marks. Unpublished working paper. New York University.

Fung, W., Hsieh, D., 1997. Empirical characteristics of dynamic trading strategies: the case of hedge funds. Review of Financial Studies 10, 275–302.

Getmansky, M., Lo, A.W., Makarov, I., 2004. An econometric model of serial correlation and illiquidity in hedge fund returns. Journal of Financial Economics 74, 529–609.

Goetzmann, W.N., Ingersoll, J., Ross, S.A., 2003. High-water marks and hedge fund management contracts. Journal of Finance 58, 1685–1717.

Grossman, S.J., Vila, J-L., 1992. Optimal dynamic trading with leverage constraints. Journal of Financial and Quantitative Analysis 27, 151–168.

Grossman, S.J., Zhou, Z., 1993. Optimal investment strategies for controlling drawdowns. Mathematical Finance 3, 241–276.

Hodder, J.E., Jackwerth, J.C., 2007. Incentive contracts and hedge fund management. Journal of Financial and Quantitative Analysis 42, 811–826.

Hugonnier, J., Kaniel, R., 2010. Mutual fund portfolio choice in the presence of dynamic flows. Mathematical Finance 20, 187–227.

Ibbotson, R.G., Chen, P., Zhu, K.X., 2011. The ABCs of hedge funds: alphas, betas, and costs. Financial Analysts Journal 67, 15–25.

Jensen, M.C., Meckling, W., 1976. Theory of the firm: managerial behavior, agency costs and ownership structure. Journal of Financial Economics 3, 305–360.

Kosowski, R., Naik, N.Y., Teo, M., 2007. Do hedge funds deliver alpha? A Bayesian and bootstrap analysis. Journal of Financial Economics 84, 229–264.

Liu, X., Mello, A., 2011. The fragile capital structure of hedge funds and the limits to arbitrage. Journal of Financial Economics 102, 491–506.

Lo, A., 2008. Hedge Funds: An Analytic Perspective. Princeton University Press, Princeton, NJ.

Merton, R.C., 1971. Consumption and portfolio rules in a continuous-time model. Journal of Economic Theory 3, 373–413.

Metrick, A., Yasuda, A., 2010. The economics of private equity funds. Review of Financial Studies 23, 2303–2341.

Myers, S., 1977. Determinants of corporate borrowing. Journal of Financial Economics 5, 147–175.

Panageas, S., Westerfield, M.M., 2009. High-water marks: high risk appetities? Convex compensation, long horizons, and portfolio choice. Journal of Finance 64, 1–36.

Ross, S.A., 2004. Compensation, incentives, and the duality of risk aversion and riskiness. Journal of Finance 59, 207–225.

Shleifer, A., Vishny, R.W., 1997. The limits of arbitrage. The Journal of Finance 52, 35–55.

Sorensen, M., Wang, N., Yang, J., 2013. Valuing private equity. Conditionally accepted, Review of Financial Studies. Columbia University and Shanghai University of Finance and Economics.

Swensen, D.F., 2000. Pioneering Portfolio Management: An Unconventional Approach to Institutional Investment. Free Press.

Swensen, D.F., 2005. Unconventional success: a fundamental approach to personal investment. Free Press.