Option Exercise Games and the q Theory of Investment *

Min Dai† Zhaoli Jiang‡ Neng Wang§

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Abstract

Back and Paulsen (2009) advocate using closed-loop equilibria as the solution concept to characterize firm strategies for real option exercise games analyzed in Grenadier (2002). This approach allows a firm to respond to its competitor's strategy, resulting in a Markov subgame perfect equilibrium. Back and Paulsen (2009) identify a closed-loop equilibrium where firms use the simple net present value (NPV) rule as doing so forms mutually best responses. The resulting outcome is equivalent to a perfectly competitive scenario in which firms ignore the option value of waiting and make zero profits. We define closed-loop equilibria and show that there exist two classes of (infinitely many) closed-loop equilibria where firms invest more quickly than in the open-loop equilibrium of Grenadier (2002) yet more slowly than in the perfectly competitive outcome. In equilibrium, firms earn positive profits, confirming Back and Paulsen (2009)'s conjecture. Furthermore, we find that the highest option value among the closed-loop equilibria corresponds to the lowest investment speed, below which preemption becomes a profitable deviation.

Keywords: closed-loop equilibria, irreversible investment, duopoly, marginal q, real options, singular control

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†The Hong Kong Polytechnic University. E-mail: mindai@polyu.edu.hk.

‡The Hong Kong Polytechnic University. E-mail: zhaoli.jiang@polyu.edu.hk.

§CKGSB and ABFER. E-mail: newang@gmail.com.

1 Introduction

How do firms exercise their growth options and accumulate capital under uncertainty in an oligopolistic competition? Grenadier (2002) derives explicit open-loop strategies and characterizes the symmetric open-loop equilibrium. Back and Paulsen (2009) advocate using closed-loop equilibria as the solution concept to characterize firm strategies, ensuring that the equilibria are Markov subgame perfect. They identify a closed-loop equilibrium in which all firms optimally use the simple net present value (NPV) rule for capital budgeting decisions. This outcome is Markov subgame perfect because using the simple NPV rule is a firm's best response when its competitor uses the simple NPV rule. These closed-loop strategies, which are mutually best responses, yield an oligopolistic equilibrium outcome that is the same as a perfectly competitive equilibrium outcome. In this equilibrium, firms make zero profits and the value of industry growth options is completely eroded.

Having significantly advanced our understanding of option exercising game and investment under uncertainty, Back and Paulsen (2009) raised open questions and suggested directions for future research. In their introduction, they noted: *"There are difficulties in even defining the game in closed-loop form."* They also pointed out that there may be equilibria (other than the zero-profits equilibrium they analyzed) in which firms make positive profits and preserve some option values of waiting. In their conclusion, they wrote: *"It is an open question whether the perfectly competitive boundary is the unique boundary such that closed-loop strategies of the form (13) are impervious to preemption."*

In this paper, we respond to the open questions they raised. First, we propose a definition of closed-loop equilibria for the real option exercise game studied by Grenadier (2002) and Back and Paulsen (2009). The equilibria are Markov subgame perfect, allowing a firm to respond to its competitor's strategy over time. Second, we provide sufficient conditions for the existence of closed-loop equilibria and a procedure to identify these closed-loop equilibria via a verification theorem. We thus confirm Back and Paulsen (2009)'s conjecture in their conclusion: *"It seems likely that there would be other closed-loop equilibria, if the strategy spaces and mapping from strategy n-tuples to outcomes could be specified."*

Third, we apply our solution method to the widely studied duopoly setting in the literature where the market inverse demand function is of a linear (or constant-elasticity) form. For this economy, we derive closed-form equilibrium investment strategies and firms' value functions for a set of (infinitely many) closed-loop equilibria. These equilibria are associated with two classes of investment threshold (trigger) functions: linear and nonlinear. Next, we discuss how to characterize these closed-loop equilibria. To ease exposition, we focus on symmetric equilibria, meaning that the functional form for the two firms' investment strategies is the same, although they may have different levels of capital stock.

We start by conjecturing an equilibrium trigger function $\mathcal{X}$ that depends on the two firms' capital stocks (K_{at}, K_{bt}), which we later verify. If industry demand $X_t < \mathcal{X}(K_{at}, K_{bt})$, neither firm invests. Intuitively, when demand is below the threshold, the existing capital stock is high enough so that neither firm has incentives to invest. If $X_t > \mathcal{X}(K_{at}, K_{bt})$, the larger firm stays inactive while the smaller firm invests until $X_t = \mathcal{X}(K_{at}, K_{bt})$ is reached.[1] However, if the two firms are of equal size, then they both invest to reach the point where the equilibrium threshold $\mathcal{X}(K_{at}, K_{bt})$ intersects with the 45 degree line, $K_{at} = K_{bt}$. Regardless of past shocks, once the two firms have the same size at time t, they will always have the same size going forward, responding to the industry demand shock in lock step along the 45-degree line.

Note that the equilibrium with the lowest option value in the set is the one with zero profits of investment, studied by Back and Paulsen (2009). In addition to identifying this equilibrium with the lowest option value, we can also characterize the closed-loop equilibrium that yields the highest option value. The equilibrium investments in all other closed-loop equilibria are bounded by the investments in the two equilibria with the highest and lowest option values. Any investment lower than that of the equilibrium with the highest option value will lead to a profitable preemption, thereby making the (closed-loop) equilibrium unattainable. We thus have found infinitely many closed-loop equilibria in which investment strategies are Markov subgame perfect. This answers Back and Paulsen (2009)'s *"open question whether the perfectly competitive boundary is the unique boundary"* for closed-loop

[1] If $X_t = \mathcal{X}(K_{at}, K_{bt})$, then firm invests if and only if $dX_t > 0$.

equilibria posed in their conclusion.

Next, we discuss how firms preserve their option values and why profits are positive in these newly identified equilibria. To ease exposition, consider the case where one firm is larger than the other. In this case, the larger firm optimally chooses to be inactive (making no investment) and the smaller firm, its competitor, invests but only to the extent that its marginal q equals the marginal cost of purchasing a unit of capital.[2] Indeed, this is a key optimality condition that we use to pin down the equilibrium industry-demand threshold. Note that the smaller firm has incentives to preserve its own option value of waiting because it incurs losses if investing too much (by going beyond the equilibrium industry-demand threshold). That is, the option value of waiting, although eroded a bit relative to the monopoly's problem, is preserved to a degree, characterized by the equilibrium industry-demand threshold. How much option value is eroded by competition depends on which equilibrium the industry finds itself in. Recall that in the closed-loop equilibrium with the lowest option value, firms earn zero profit (Back and Paulsen, 2009; Leahy, 1993). In contrast, the newly identified equilibria, which have a lower investment speed than the perfect competition equilibrium, preserve greater option values and consequently lead to positive profits.

Our work also contributes to the q theory of investment under irreversibility by extending the standard models (Abel and Eberly, 1996; Hayashi, 1982; Lucas and Prescott, 1971) in a monopoly setting to a duopoly setting. We show that a firm's investment decision under duopoly depends not only on its marginal value of capital, i.e., marginal q, but also the firm's marginal value of its competitor's capital. Because of Markov subgame perfection, a firm's marginal q must be bounded from above by the marginal cost of investing and moreover, the firm's marginal value of its competitor's capital is non-positive in closed-loop equilibria.

For an equilibrium with a linear trigger, the investment trigger function $\mathcal{X}(K_{at}, K_{bt})$ is assumed to grow linearly with the capital stocks of the two firms, K_{at} and K_{bt}. To achieve

[2] Of course, the smaller firm does not want to stop short of reaching the equilibrium industry-demand threshold. Had it done that, the larger firm, which should have been inactive on the equilibrium path, would invest to the point where the equilibrium industry-demand threshold is reached. This makes the smaller firm worse off and hence it is off the equilibrium path.

equilibrium, each firm's marginal q must be lower than the marginal cost of investing, and each firm's marginal value of its competitor's capital must be non-positive. These two constraints result in a class of equilibria, ranging from the perfect competition equilibrium with the lowest option value to the closed-loop equilibrium with the highest option value. In the latter equilibrium, firm value (average q) is close to that in the open-loop equilibrium of Grenadier (2002), as firms form equilibrium beliefs and choose the lowest possible investments that are feasible on the closed-loop equilibrium path.

Regarding this class of equilibria, we find that when the smaller firm invests and the larger firm does not, the larger firms' marginal q is strictly lower than the marginal cost of investing. This finding motivates us to relax the linear trigger assumption and impose an additional constraint that the larger firms' marginal q exactly equals the marginal cost of investing in this scenario. Intuitively, this additional constraint implies that when the smaller firm should invest but chooses to deviate (i.e., not invest), the larger firm responds in its own best interest and preempts the investment. Using this additional constraint, we can endogenously derive a class of nonlinear investment trigger functions and equilibria. The extreme case in this class of equilibria remains the perfect competition equilibrium with the lowest option value. Since the additional constraint narrows the set of admissible strategies, firm value (average q) is significantly smaller than in the previous class of equilibria, yet still higher than in the perfect competition equilibrium.

Technically, our paper contributes to the theory of differential games with singular control. There are two key challenges in the differential games studied in this paper. First, the capital stocks serve as both (singular) control and state variables, making it challenging to define a closed-loop strategy. To tackle this challenge, we decompose the capital stock process into a continuous component and a discontinuous component. A key observation is that, in equilibrium, the continuous component must evolve in tandem with the running maximum of the demand stock. This evolution allows us to adopt the investment speed relative to this running maximum as a new control variable. Each firm's response to its competitor's actions can be effectively integrated into this control variable. Second, in the existing literature on differential games with singular control,[3] the action regions of different players are separate under equilibrium, allowing each player to solve a single-agent singular control given the actions of their competitors. Consequently, equilibrium strategies can be determined using smooth-pasting conditions. In contrast, in our model, different players may take actions simultaneously, which invalidates the use of smooth-pasting conditions and necessitates a distinct characterization of the resulting equilibrium value functions.

2 Model

Consider a duopoly industry consisting of two firms: a and b. Let $\{X_t; t \geq 0\}$ represent the demand shock, evolving according to the following time-homogeneous diffusion process:

$$dX_t = \mu(X_t)dt + \sigma(X_t)d\mathcal{W}_t\,, \quad X_0 = x_0, \tag{2.1}$$

where $\{\mathcal{W}_t; t \geq 0\}$ is a one-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with $\mathcal{W}_0 = 0$.

Let K_{it} denote the capital stock of firm i at time t, where $i \in \{a, b\}$. For notational convenience, we use $-i$ to refer to the competitor of firm i, i.e., $a = -b$ and $b = -a$. Let Q_{it} represent firm i's output at time t. We assume that the profit rate for firm i at time t, F_{it}, is given by

$$F_{it} = (P_t - c_Q)\, Q_{it}, \tag{2.2}$$

where P_t is the market price for a unit of output, and c_Q is the constant marginal cost of selling output (for expositional simplicity, we assume this cost is the same for both firms). Firm i produces its output using the following constant return to scale (i.e., AK) technology:

$$Q_{it} = AK_{it}\,, \tag{2.3}$$

where A is the constant marginal product of capital (assumed to be the same for the two firms for simplicity). By combining (2.2) and (2.3), we express firm i's profit flow as follows:

$$F_{it} = (\Pi_t - c)\, K_{it}\,, \tag{2.4}$$

[3] See, e.g., Kwon and Zhang (2015) and De Angelis and Ferrari (2018).

where $c = c_Q A$ is a constant and $\Pi_t = AP_t$ is the productivity-adjusted inverse demand function.

As in the literature, we assume that Π_t depends on the demand shock X_t and decreases with the total output in the industry: $Q_t = Q_{at} + Q_{bt}$. Due to the AK production technology, we can equivalently write $\Pi_t = \Pi(X_t, K_t)$, where K_t is the total capital stock in the industry. To capture the downward-sloping demand curve in the total capital stock, we assume $\Pi_K(X_t, K_t) < 0$. For brevity, we refer to $\Pi(X_t, K_t)$ as the inverse demand function. By definition, since the inverse demand function must increase with demand X_t, we have $\Pi_X(X_t, K_t) > 0$. Then, firm i's profit at time t is given as follow:

$$F_i(X_t, K_{at}, K_{bt}) = [\Pi(X_t, K_{at} + K_{bt}) - c]\, K_{it}\,. \tag{2.5}$$

In line with the real-options game literature (Back and Paulsen, 2009; Grenadier, 2002), we assume that corporate investment is irreversible and that capital stock does not depreciate, implying that K_{it} is a nondecreasing process. For the sake of regularity, we require the process K_{it} to be left-continuous with right limits (**caglad**) for any $t \geq 0$. Let $\mathcal{A}(k_i)$ denote the set of firm i's admissible capital stock processes with an initial capital level of k_i:

$$\mathcal{A}(k_i) = \{K_i := \{K_{it}; t \geq 0\} \mid K_{it} \text{ is } \mathcal{F}_t\text{-adapted, nondecreaing}, \textbf{caglad}, K_{i0} = k_i\}.$$

In the following, we denote by $t+$ the moment immediately after t, since firms may instantaneously increase their capital stocks.

Firm i selects an admissible investment policy to solve the following problem:

$$\sup_{K_i \in \mathcal{A}(k_i)} \mathbb{E}\left[\int_0^\infty e^{-rt}\Big(F_i(X_t, K_{at}, K_{bt})dt - p\, dK_{it}\Big)\right], \tag{2.6}$$

where $r > 0$ is the constant discount rate and $p > 0$ is the constant marginal cost of purchasing a unit of capital. Next, we introduce our solution concepts. First, we define closed-loop strategies and equilibria.

2.1 Closed-Loop Strategies and Equilibria

Since corporate investment is irreversible and capital stock may increase when the market demand exceeds its historical maximum, we define the running maximum of the market demand as follows:

$$M_t = \max_{s\in[0,t]} X_s, \tag{2.7}$$

which plays a critical role in defining closed-loop equilibrium strategies.

For an admissible capital stock process $\{K_{it}; t \geq 0\}$, let $\{K_{it}^C; t \geq 0\}$ denote its continuous component (where the superscript C refers to continuous) and $\Delta K_{it} := K_{it+} - K_{it}$ denote the discontinuous change of K_i. We can thus decompose $\{K_{it}; t \geq 0\}$ as follows:

$$K_{it+} = K_{it}^C + \sum_{s\in[0,t]} \Delta K_{is}\,, \quad t \geq 0\,. \tag{2.8}$$

We consider a set of closed-loop Markovian (feedback) strategies defined along both the continuous and discontinuous-change components as follows. First, the continuous part of firm i's strategy $\{K_{it}^C; t \geq 0\}$ follows:

$$dK_{it}^C = u_{it} dM_t \tag{2.9}$$

where u_{it} is a process adapted to $\{\mathcal{F}_t; t \geq 0\}$, representing firm i's continuous investment rate related to dM_t, the change in the running maximum of the demand shock. We restrict u_{it} to be a Markov process that depends on the current states (X_t, K_{at}, K_{bt}) as well as the opponent's continuous investment rate u_{-it}, i.e.,

$$u_{it} = \boldsymbol{u}_i(X_t, K_{at}, K_{bt}, u_{-it}), \tag{2.10}$$

where the function $\boldsymbol{u}_i(x, k_a, k_b, u_{-i})$ is deterministic and measurable.

Second, firm i may also instantaneously increase its capital stock by ΔK_{it} in response to its opponent's discontinuous (i.e., lumpy) investment amount ΔK_{-it} as follows:

$$\Delta K_{it} = \boldsymbol{v}_i(X_t, K_{at}, K_{bt}, \Delta K_{-it})\,, \tag{2.11}$$

where $\boldsymbol{v}_i(x, k_a, k_b, v)$ is a deterministic and measurable function.[4] Let $\varphi_i = (\boldsymbol{u}_i, \boldsymbol{v}_i)$ denote firm i's closed-loop strategy, summarizing both its continuous and discontinuous components.

To simplify notation, let $\{Z_t; t \geq 0\}$ denote the joint Markov process for the three state variables, where $Z_t = (X_t, K_{at}, K_{bt})$. Let $\mathbb{E}_t^z[\cdot] = \mathbb{E}_t[\cdot|Z_t = z]$, where Z_t takes the value of

[4]From (2.13), $\boldsymbol{v}_i$ is allowed to depend on the opponent's contemporaneous jump v_{-i}. This is explained in footnote 5 of Back and Paulsen (2009) and Simon and Stinchcombe (1989): in continuous-time games, agents can react instantaneously, so information lags may truly be negligible.

$z = (x, k_a, k_b)$. For $t = 0$, we simply write $\mathbb{E}_0^z[\cdot]$ as $\mathbb{E}^z[\cdot]$. Let $\mathbb{X}$ denote the state space for X_t, and let $\mathbb{Z} = \mathbb{X} \times [0, \infty) \times [0, \infty)$ denote the state space of the game.[5]

We are ready to give a rigorous definition of a feasible (closed-loop) strategy.

Definition 1 A pair of closed-loop Markov strategies by firms a and b, $\varphi := (\varphi_a, \varphi_b)$ with $\varphi_i = (\boldsymbol{u}_i, \boldsymbol{v}_i)$, is feasible if the following conditions hold:

(i) For any $z = (x, k_a, k_b)$, there exists a unique pair $(u_a, u_b) \in \mathbb{R}_+^2$ satisfying

$$u_a = \boldsymbol{u}_a(z, u_b), \qquad u_b = \boldsymbol{u}_b(z, u_a) \tag{2.12}$$

and a unique pair $(v_a, v_b) \in \mathbb{R}_+^2$ satisfying

$$v_a = \boldsymbol{v}_a(z, v_b), \qquad v_b = \boldsymbol{v}_b(z, v_a). \tag{2.13}$$

Moreover, for any initial value $(x_0, k_{a0}, k_{b0}) \in \mathbb{Z}$, there exists a unique pair $K_a \in \mathcal{A}(k_{a0})$ and $K_b \in \mathcal{A}(k_{b0})$ satisfying (2.8)-(2.11).[6]

(ii) For any $z \in \mathbb{Z}$, we have $\mathbb{E}^z\left[|K_{iT}|^n\right] < \infty$ for any finite $T > 0$ and $n > 1$, and

$$\mathbb{E}^z\left[\int_0^\infty e^{-rt}|F_i(X_t, K_{at}, K_{bt})|dt\right] < \infty. \tag{2.14}$$

Let $\mathcal{S}$ denote the set of all feasible (closed-loop) strategies.

For any $z \in \mathbb{Z}$ and any feasible (closed-loop) Markov strategy pair $\varphi = (\varphi_a, \varphi_b) \in \mathcal{S}$, we define

$$J^i(z; \varphi) := J^i(z; \varphi_a, \varphi_b) = \mathbb{E}^z\left[\int_0^\infty e^{-rt}\Big(F_i(X_t, K_{at}, K_{bt})dt - pdK_{it}\Big)\right]. \tag{2.15}$$

Next, we define closed-loop equilibria.

Definition 2 A feasible strategy pair $(\varphi_a, \varphi_b) \in \mathcal{S}$ is a *Markov subgame perfect equilibrium*

[5] Since K_{at} and K_{bt} are nondecreasing, we can also relax the definition to consider a set $\mathbb{Z} \supseteq \mathbb{X} \times [k_{a0}, \infty) \times [k_{b0}, \infty)$, where k_{i0} is firm i's initial capital. See Section 6 for two examples. The state space $\mathbb{X}$ is either $\mathbb{R}$ or $[0, \infty)$. For a geometric Brownian motion process ($\mu(x) = \mu x$ and $\sigma(x) = \sigma x$), $\mathbb{X} = [0, \infty)$. For an arithmetic Brownian motion process ($\mu(x) = \mu$ and $\sigma(x) = \sigma$), $\mathbb{X} = \mathbb{R}$.

[6] By definition, the continuous investment rate (u_{at}, u_{bt}) is the unique solution to (2.12) with $z = (X_t, K_{at}, K_{bt})$, and the lumpy investment $(\Delta K_{at}, \Delta K_{bt})$ is the unique solution to (2.13) with $z = (X_t, K_{at}, K_{bt})$.

strategy if for any $z \in \mathbb{Z}$, the following conditions hold

$$J^a(z; \varphi_a, \varphi_b) \geq J^a(z; \widetilde{\varphi}_a, \varphi_b), \quad \forall\, (\widetilde{\varphi}_a, \varphi_b) \in \mathcal{S}, \tag{2.16}$$

$$J^b(z; \varphi_a, \varphi_b) \geq J^b(z; \varphi_a, \widetilde{\varphi}_b), \quad \forall\, (\varphi_a, \widetilde{\varphi}_b) \in \mathcal{S}. \tag{2.17}$$

We refer to the pair, $J^a(z; \varphi_a, \varphi_b)$ and $J^b(z; \varphi_a, \varphi_b)$, in this equilibrium, as the equilibrium value functions, and the pair, $\{K_{at}, K_{bt}; t \geq 0\}$, as equilibrium capital processes.

In our model, firm i only needs to determine the functions $\varphi_i = (\boldsymbol{u}_i, \boldsymbol{v}_i)$ rather than capital stock process K_i. This enables us to define closed-loop strategies and closed-loop equilibrium as usual. It is evident that mutual responses exist between the two firms in our model, and playing a closed-loop equilibrium is the best response for both firms.

3 Verification Theorem

In this section we propose a verification theorem for a class of closed-loop equilibrium strategies and value functions. The theorem requires some key conditions for a pair of candidate value functions, $V^a(z)$ and $V^b(z)$, to be the equilibrium value functions for firms a and b, $z = (x, k_a, k_b) \in \mathbb{Z}$. While these conditions are sufficient but may not be necessary, they have clear economic implications.

Next we introduce these conditions and provide their economic implications. We first impose a condition for candidate value functions.

Condition 1 *For $V^i(z) \in \mathcal{C}^{0,1,1}(\mathbb{Z})$ where $i = a, b$, we require*

$$\frac{\partial V^i(z)}{\partial k_i} \leq p \quad \textit{and} \quad \frac{\partial V^i(z)}{\partial k_{-i}} \leq 0, \quad \forall z = (x, k_a, k_b) \in \mathbb{Z}. \tag{3.1}$$

The intuition for this condition is as follows. The first inequality in (3.1) indicates that firm i's equilibrium marginal value of capital, also known as marginal q, cannot exceed its marginal cost of investing, p. Otherwise, firm i would benefit by increasing its capital stock at the level where its marginal q exceeds p, thereby disrupting the equilibrium. The second inequality suggests an intuitive relationship: firm i's equilibrium value will not improve as its competitor's capital stock increases.

Next, we define several regions that are useful for our analysis, using the two constraints in Condition 1.

Definition 3 The regions, $\mathbf{I}_i(x)$ and $\mathbf{N}_i(x)$, where $i = a, b$, are defined as follows:

$$\mathbf{I}_i(x) := \{(k_a, k_b) \in \mathbb{R}_+^2 : \frac{\partial V^i(z)}{\partial k_i} = p \ \text{ and } \ \frac{\partial V^{-i}(z)}{\partial k_i} = 0\}, \tag{3.2}$$

$$\mathbf{N}_i(x) := \mathbb{R}_+^2 \setminus \mathbf{I}_i(x). \tag{3.3}$$

Let $\mathbf{I}_{ab}(x)$ denote the intersection of $\mathbf{I}_a(x)$ and $\mathbf{I}_b(x)$ and similarly let $\mathbf{N}_{ab}(x)$ denote the intersection of $\mathbf{N}_a(x)$ and $\mathbf{N}_b(x)$:

$$\mathbf{I}_{ab}(x) := \mathbf{I}_a(x) \bigcap \mathbf{I}_b(x) \quad \mathbf{N}_{ab}(x) := \mathbf{N}_a(x) \bigcap \mathbf{N}_b(x). \tag{3.4}$$

Let $\overline{\mathbf{N}}_{ab}(x)$ denote the closure of $\mathbf{N}_{ab}(x)$, which includes both its interior and boundary.

Intuitively, when firm i invests, its marginal value of investing, i.e., its marginal q, must equal the marginal cost of investing p, and moreover, its competitor's value function must be invariant in equilibrium. It is worth noting that firm i invests only when both $\frac{\partial V^i(z)}{\partial k_i} = p$ and $\frac{\partial V^{-i}(z)}{\partial k_i} = 0$ hold. We later show that $\mathbf{I}_i(x)$ is the region where firm i invests in equilibrium.

The $\mathbf{N}_i(x)$ region is where firm i does not invest, the $\mathbf{I}_{ab}(x)$ region is where both firms invest, and the $\mathbf{N}_{ab}(x)$ region is where neither firm invests. Later, we will demonstrate that our capital stock pair (K_{at}, K_{bt}) lies solely within the $\overline{\mathbf{N}}_{ab}(x)$ region for all $t > 0$ (i.e., after the firms' initial investment). We will verify and discuss these results in detail later.

Next, we impose another condition for candidate equilibrium value functions.

Condition 2 *For $V^i \in \mathcal{C}^{2,1,1}(\{z := (x, k_a, k_b) \in \mathbb{Z} : (k_a, k_b) \in \overline{\mathbf{N}}_{ab}(x)\})$, where $\overline{\mathbf{N}}_{ab}(x)$ is the closure of $\mathbf{N}_{ab}(x)$ given in Definition 3, assume for any $z = (x, k_a, k_b) \in \mathbb{Z}$,*

$$rV^i(z) = \mathcal{L}V^i(z) + F_i(z), \quad \forall\ (k_a, k_b) \in \mathbf{N}_{ab}(x), \tag{3.5}$$

where $F_i(z)$ is given by (2.5), and $\mathcal{L}V^i$ is the infinitesimal generator given by

$$\mathcal{L}V^i(z) := \frac{1}{2}\sigma(x)^2 \frac{\partial^2 V^i(z)}{\partial x^2} + \mu(x) \frac{\partial V^i(z)}{\partial x}\,. \tag{3.6}$$

The intuition for Condition 2 is as follows. Equation (3.5) is the dynamic programming equation for V^i in the region $\mathbf{N}_{ab}(x)$ where neither firm invests. Note that the sum of the profit flow $F_i(z)$ and the expected change of the value function, as given by (3.6), equals $rV^i(z)$. This resembles the standard asset-pricing equation (Duffie, 2001).

The following condition is imposed on candidate equilibrium strategies.

Condition 3 *Consider a candidate equilibrium strategy pair* $(\varphi_a, \varphi_b) \in \mathcal{S}$ *associated with the candidate value function pair* (V^a, V^b) *satisfying Condition 1. For any given initial state* $(X_0, K_{a0}, K_{b0}) = (x, k_a, k_b) \in \mathbb{Z}$*, the capital stock process pair* (K_{at}, K_{bt}) *generated by* (φ_a, φ_b) *has the following properties:*

- *From* $t = 0$ *to* $t = 0+$*, the following value-matching condition holds:*

$$V^i(x, K_{a0+}, K_{b0+}) - p(K_{i0+} - k_i) = V^i(x, k_a, k_b)\,, \tag{3.7}$$

- $(K_{at}, K_{bt}) \in \overline{\mathbf{N}}_{ab}(X_t)$ *for any* $t > 0$*, and* K_{it} *is continuous in* $t > 0$ *and evolves according to:*

$$K_{it} = K_{i0+} + \int_{0+}^{t} \mathbf{1}_{(K_{as}, K_{bs}) \in \mathbf{I}_i(X_s)} dK_{is}\,, \tag{3.8}$$

  where the integrand is an indicator function (equal to one if the event occurs, and zero otherwise), and the $\mathbf{I}_i(x)$ *and* $\overline{\mathbf{N}}_{ab}(x)$ *regions are given in Definition 3.*

When no lumpy investments occur at $t = 0$, equation (3.7) holds automatically. In the presence of lumpy investments at $t = 0$, equation (3.7) aligns with (3.2). To illustrate this, consider scenarios where at least one firm makes a lumpy investment at time 0: (i) $K_{i0+} > k_i$ and $K_{-i0+} = k_{-i}$, or (ii) $K_{a0+} > k_a$ and $K_{b0+} > k_b$. In case (i), that firm i makes a lumpy investment at the initial time to increase its capital stock from k_i to K_{i0+} while firm $-i$ does not invest, i.e., $[k_i, K_{i0+}] \times \{k_{-i}\} \subset \mathbf{I}_i(x)$. By the definition of (3.2), we infer that $\frac{\partial V^i}{\partial k_i} = p$ in the region $[k_i, K_{i0+}] \times \{k_{-i}\}$, which, together with $K_{-i0+} = k_{-i}$, yields (3.7). In case (ii), both firms make a lump sum investment at the initial time, i.e., $[k_a, K_{a0+}] \times [k_b, K_{b0+}] \subset \mathbf{I}_{ab}(x)$. By (3.2) we infer that $\frac{\partial V^i}{\partial k_i} = p$ and $\frac{\partial V^i}{\partial k_{-i}} = 0$ in the region $[k_a, K_{a0+}] \times [k_b, K_{b0+}]$, yielding (3.7).

Condition 3 implies that the capital stock process K_{it} is always continuous except for a possible initial jump. Equation (3.8) indicates that firm i invests only when (K_{at}, K_{bt})

lies within the region $\mathbf{I}_i(X_t)$. Since we require that $(K_{at}, K_{bt}) \in \overline{\mathbf{N}}_{ab}(X_t)$ for any $t > 0$ and $\mathbf{I}_i(x) \cap \mathbf{N}_{ab}(x) = \emptyset$, capital investments occur at the boundary of $\overline{\mathbf{N}}_{ab}(X_t)$ after initial time.[7]

Next, we impose a condition that restricts firms' responses against their competitors' deviation. For exposition, we focus on firm a's deviation.

Condition 4 *Let $(\varphi_a, \varphi_b) \in \mathcal{S}$ be a candidate equilibrium strategy pair associated with the candidate value function pair (V^a, V^b) satisfying Condition 1. Consider any other feasible deviation pair, $(\widetilde{\varphi}_a, \varphi_b) \in \mathcal{S}$, where firm a deviates.*[8] *Under the $(\widetilde{\varphi}_a, \varphi_b) \in \mathcal{S}$ strategy pair, firm b's capital stock process $K_{bt}^{\widetilde{\varphi}_a, \varphi_b}$ is continuous in $t > 0$, and the capital stock process pair satisfies*

$$(K_{at}^{\widetilde{\varphi}_a, \varphi_b}, K_{bt}^{\widetilde{\varphi}_a, \varphi_b}) \in \overline{\mathbf{N}}_{ab}(X_t), \quad \forall\ t > 0\,, \tag{3.9}$$

where $\overline{\mathbf{N}}_{ab}(x)$ is given in Definition 3.

Condition 4 specifies that a candidate equilibrium strategy is limited to those whose off-equilibrium capital stock process pair, after $t > 0$, falls within the closure of the no-investment region associated with the candidate equilibrium value function. Intuitively, when firm a deviates and invests more, (3.9) holds automatically.[9] Conversely, when firm a deviates and invests less, Condition 4 indicates that firm b will preempt the investment and move the capital stock process pair at least to the boundary of the no-action region. Later we will see that in equilibrium, firm i's potential preemption deters its opponent's deviation.

Theorem 1 *Assume there exist functions $V^a(z)$ and $V^b(z)$ that satisfy Conditions 1 and 2. Assume that there exists a feasible strategy $(\varphi_a, \varphi_b) \in \mathcal{S}$, which is associated with $(V^a(z), V^b(z))$, satisfies Conditions 3 and 4. Then (φ_a, φ_b) is a Markov perfect equilibrium strategy and the corresponding equilibrium value functions are $V^a(z)$ and $V^b(z)$ for firms a*

[7] The definition of $\mathbf{N}_{ab}(x)$ implies that $\mathbf{N}_{ab}(x)$ is a relatively open set of $\mathbb{R}_+^2$. Therefore, $\mathbf{I}_i(x) \cap \overline{\mathbf{N}}_{ab}(x)$ lies on the boundary of $\overline{\mathbf{N}}_{ab}(x)$.

[8] By symmetry, we can impose essentially the same condition for any other feasible strategy pair, $(\varphi_a, \widetilde{\varphi}_b) \in \mathcal{S}$, where firm b deviates. For brevity, we leave the details out.

[9] This is because $(K_{at}, K_{bt}) \in \overline{\mathbf{N}}_{ab}(X_t)$ in the equilibrium and higher capital results in lower prices and reduces the incentive to invest.

and b under a regularity condition[10]*:*

$$J^a(z;\varphi_a,\varphi_b) = V^a(z), \qquad J^b(z;\varphi_a,\varphi_b) = V^b(z). \tag{3.10}$$

4 Linear Demand and Monopoly Solution

In this section and onwards, we assume that (i) the inverse demand function $\Pi(x,k)$ is linear in market demand x and total capital stock k, expressed as[11]

$$\Pi(x,k) = x - \eta k\,, \tag{4.1}$$

where $\eta > 0$ measures the slope of the inverse demand function; and (ii) the demand shock X follows the widely-used geometric Brownian motion (e.g., Grenadier (2002); Back and Paulsen (2009)):

$$dX_t = \mu X_t dt + \sigma X_t d\mathcal{W}_t\,, \quad X_0 > 0\,, \tag{4.2}$$

where μ is the drift and $\sigma \neq 0$ is the volatility parameter. We assume $\mu < r$ to ensure the finiteness of value functions.

First, we summarize the solution for the monopoly case. A monopolist's profit flow function is given by $F(x,k) = (x - \eta k)k - ck$. The monopolist maximizes

$$\sup_{K\in\mathcal{A}(k)} \mathbb{E}\left[\int_0^\infty e^{-rt}\Big([X_t - (\eta K_t + c)]K_t dt - p\, dK_t\Big)\right], \tag{4.3}$$

where $(X_0, K_0) = (x,k)$. Let $V^{\mathrm{m}}(x,k)$ denote the monopolist's value function for the optimization problem (4.3).

To ease exposition, we first introduce $\mathcal{X}^{\mathrm{m}}(\,\cdot\,)$, a function of capital stock:

$$\mathcal{X}^{\mathrm{m}}(k) = \rho(2\eta k + rp + c)\,, \tag{4.4}$$

where ρ is a constant given by

$$\rho = \frac{\beta}{\beta - 1}\frac{r-\mu}{r} \tag{4.5}$$

[10]With any initial state $z \in \mathbb{Z}$, for any $\widetilde{\varphi}_i$ s.t. $\widetilde{\varphi} \in \mathcal{S}$, where $\widetilde{\varphi} = (\widetilde{\varphi}_a, \varphi_b)$ when $i = a$ and $\widetilde{\varphi} = (\varphi_a, \widetilde{\varphi}_b)$ when $i = b$, firms' capital processes $K^{\widetilde{\varphi}}_{as}$ and $K^{\widetilde{\varphi}}_{bs}$ under the strategy $\widetilde{\varphi}$ satisfy that $\lim_{n\to+\infty} \mathbb{E}^z[e^{-rt_n} V^i(X_{t_n}, K^{\widetilde{\varphi}}_{at_n}, K^{\widetilde{\varphi}}_{bt_n})] = 0$ for some sequence of real values $\{t_n\}$ converging to infinity as $n \to +\infty$ and that $\mathbb{E}^z\left[\int_0^t |e^{-rs}\frac{\partial V^i(X_s, K^{\widetilde{\varphi}}_{as}, K^{\widetilde{\varphi}}_{bs})}{\partial x}\sigma(X_s)|^2 ds\right] < +\infty,\ \forall\, t \geq 0$.

[11]The constant elasticity model can be handled in a similar manner and is discussed in Appendix A.

with β being the constant describing the optionality of the investment opportunity:[12]

$$\beta = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}\,. \tag{4.6}$$

For convergence, we require $\beta > 2$. As we show later, $\mathcal{X}^{\mathrm{m}}(k)$ is the optimal threshold of the demand shock process X that characterizes the monopolist's optimal investment policy. That is, the monopolist invests only when $X_t \geq \mathcal{X}^{\mathrm{m}}(K_t)$.

Next, we summarize the monopolist's investment strategy and value function.[13]

Proposition 1 *Given the current state $(X_t, K_t) = (x, k)$, the optimal investment policy is as follows.*

(i) If $X_t < \mathcal{X}^{\mathrm{m}}(K_t)$, inaction is optimal ($dK_t = 0$).

(ii) If $X_t > \mathcal{X}^{\mathrm{m}}(K_t)$, the monopolist invests ΔK_t to the level K_t satisfying $x = \mathcal{X}^{\mathrm{m}}(K_t)$, which yields that

$$K_{t+} = K_t + \Delta K_t = \frac{1}{2\eta}\left(\frac{X_t}{\rho} - (rp + c)\right) > K_t\,. \tag{4.7}$$

(iii) If $X_t = \mathcal{X}^{\mathrm{m}}(K_t)$, the instantaneous change of K_t, dK_t, is proportional to dM_t, the instantaneous change of the running maximum M_t given in (2.7):

$$dK_t = \frac{dM_t}{2\eta\rho}\,. \tag{4.8}$$

In sum, the capital stock process under the optimal investment strategy is given by $K_t = \inf\{k \geq k_0 \mid M_t \leq \mathcal{X}^m(k)\},\ t \geq 0$.

The value function in this inaction region where $x < \mathcal{X}^{\mathrm{m}}(k)$ is given by

$$V^{\mathrm{m}}(x, k) = \frac{k}{r - \mu}x - \frac{\eta k^2 + ck}{r} + H^{\mathrm{m}}(k)x^{\beta}\,, \tag{4.9}$$

where $H^{\mathrm{m}}(k)$ is the coefficient of monopolist's option value of investing given by

$$H^{\mathrm{m}}(k) = \frac{1}{2\eta\rho\beta(\beta - 2)(r - \mu)}(\mathcal{X}^{\mathrm{m}}(k))^{2-\beta} \tag{4.10}$$

[12] Here β is the positive root of the following (fundamental) quadratic equation: $\sigma^2 x(x-1)/2 + \mu x - r = 0$.

[13] See Abel and Eberly (1996), Grenadier (2002), and Back and Paulsen (2009).

with $\mathcal{X}^{\mathrm{m}}(k)$ being the threshold function given in (4.4). In the investment region where $x \geq k$, the value function is given by

$$V^{\mathrm{m}}(x,k) = V^{\mathrm{m}}(x, \mathcal{K}^{\mathrm{m}}(x)) - p(\mathcal{K}^{\mathrm{m}}(x) - k), \tag{4.11}$$

where $\mathcal{K}^{\mathrm{m}}(\,\cdot\,)$ denotes the inverse function of $\mathcal{X}^{\mathrm{m}}(\,\cdot\,)$.

The first two terms in (4.9) represent the monopolist's value if it has no future investment options at all, i.e., $K_t = k$ for all t. The $H^{\mathrm{m}}(k)x^{\beta}$ term is convex in x and measures the option value of investing. Equation (4.7) describes lumpy investment when the firm's demand shock X_t exceeds the investment threshold $\mathcal{X}^{\mathrm{m}}(K_t)$. This lumpy adjustment of capital stock can only occur at $t = 0$ since X is continuous.

Finally, when the demand shock X_t reaches its running maximum and a positive shock occurs ($dX_t > 0$), the running maximum increases: $dM_t = dX_t > 0$. In this case, the optimal investment dK_t is proportional to dM_t as given in (4.8). The greater the price impact of production (higher η), the more costly it is to invest. This is why, in response to $dM_t > 0$, the investment sensitivity $dK_t/dM_t = 1/(2\eta\rho)$ decreases with η.

Next, we solve the duopoly problem in Sections 5 and 6.

5 Duopoly Solution: A Linear Trigger Function

In a duopoly setting, firm i's profit is given by

$$F_i(x, k_a, k_b) = [x - \eta(k_a + k_b)]k_i - ck_i\,. \tag{5.1}$$

When neither firm ever invests, i.e., when $dK_{at} = dK_{bt} \equiv 0$ for all t, firm i's value equals

$$\Psi^i(z) = \mathbb{E}^z\left[\int_0^{\infty} e^{-rt} F_i(X_t, k_a, k_b)dt\right] = \frac{k_i}{r-\mu}x - \left(\frac{\eta(k_a+k_b)+c}{r}\right)k_i \tag{5.2}$$

for $z = (x, k_a, k_b)$. The corresponding marginal q is given by

$$\frac{\partial \Psi^i(z)}{\partial k_i} = \frac{x}{r-\mu} - \frac{2\eta k_i + \eta k_{-i} + c}{r}. \tag{5.3}$$

5.1 A Linear Trigger Function

Inspired by the trigger function (4.4) for the monopoly case, we conjecture and verify a set of closed-loop equilibria, characterized by a linear trigger function:

$$\mathcal{X}^i(k_i, k_{-i}) := \rho(\theta^i_i \eta k_i + \theta^i_{-i} \eta k_{-i} + rp + c), \quad (k_i, k_{-i}) \in \mathbb{R}^2_+, \tag{5.4}$$

where the positive pair $(\theta^i_i, \theta^i_{-i})$ is to be determined and ρ is a constant given by (4.5). Intuitively, firm i invests when the demand shock exceeds the trigger $\mathcal{X}^i(k_i, k_{-i})$. The intuition behind the trigger function (5.4) is as follows. First, similar to the monopoly case, the term proportional to k_i captures the negative effect of firm i's own production on the output price. Second, different from the monopoly case, the term proportional to k_{-i} in (5.4) is new and captures the negative effect of the competitor's production on the output price.

To ease exposition, we focus on symmetric equilibria[14] where the two firms use the same investment threshold strategy, i.e., $\theta_+ := \theta^a_a = \theta^b_b$ and $\theta_- := \theta^a_b = \theta^b_a$. We then simplify (5.4) as follows:

$$\mathcal{X}^i(k_i, k_{-i}) = \rho(\theta_+ \eta k_i + \theta_- \eta k_{-i} + rp + c), \quad (k_i, k_{-i}) \in \mathbb{R}^2_+ . \tag{5.5}$$

Note that in a symmetric equilibrium, firms may have different levels of capital stock: $k_i \neq k_{-i}$, even though their strategy functions are symmetric.

In the (k_a, k_b) plane, we work with the following trigger function:

$$\mathcal{X}(k_a, k_b) := \begin{cases} \rho(\theta_+ \eta k_a + \theta_- \eta k_b + rp + c), & \text{if } k_a \leq k_b, \\ \rho(\theta_- \eta k_a + \theta_+ \eta k_b + rp + c), & \text{if } k_a \geq k_b. \end{cases} \tag{5.6}$$

The symmetry property implies that $\mathcal{X}(k_a, k_b) = \mathcal{X}(k_b, k_a)$.

The trigger function induces the following investment boundary, denoted by $\mathbf{\Gamma}(x)$:

$$\mathbf{\Gamma}(x) := \{(k_a, k_b) \in \mathbb{R}^2_+ : \mathcal{X}(k_a, k_b) = x\}. \tag{5.7}$$

Note that $\mathbf{\Gamma}(x) = \mathbf{\Gamma}_a(x) \cup \mathbf{\Gamma}_b(x)$ (see Figure 2), where

$$\mathbf{\Gamma}_i(x) := \{(k_a, k_b) \in \mathbb{R}^2_+ : \mathcal{X}(k_a, k_b) = x, k_i \leq k_{-i}\}. \tag{5.8}$$

[14] We also have asymmetric equilibria.

We can show that $\mathbf{\Gamma}_a(x)$ and $\mathbf{\Gamma}_b(x)$ intersect at a unique point $\mathrm{C} = (\mathcal{K}(x), \mathcal{K}(x))$, where[15]

$$\mathcal{K}(x) := \frac{1}{(\theta_+ + \theta_-)\eta} \left(\frac{x}{\rho} - (rp + c) \right) . \tag{5.9}$$

5.2 Symmetric Closed-Loop Equilibria

An Equilibrium Condition for θ_+ and θ_-. To obtain an equilibrium strategy, we need the following condition.

Condition 5 *The pair, $\theta_+ > 0$ and $\theta_- \in (0, \theta_+]$, satisfies the following inequality bounds:*

$$\underline{g}(\theta_+/\theta_-) \leq \theta_+ + \theta_- \leq \overline{g}(\theta_+/\theta_-) \,, \tag{5.10}$$

where the upper and lower bound functions, $\overline{g}(\,\cdot\,)$ and $\underline{g}(\,\cdot\,)$, are given by

$$\overline{g}(w) = 3 - 1/w, \tag{5.11}$$

$$\underline{g}(w) = 1 + \left(1 - \frac{2}{\beta}\right) w + \frac{2}{\beta w} + \left(2 - \frac{2}{\beta}\right) \frac{w - 1}{(1 + w)^{\beta - 1} - 1} \,. \tag{5.12}$$

Intuitively, the sum $(\theta_+ + \theta_-)$ captures the effect of the industry's total capital stocks on the investment threshold $\mathcal{X}^i(k_i, k_{-i})$ and the ratio θ_+/θ_- measures the relative impact of a firm's own production compared to its competitor's production on $\mathcal{X}^i(k_i, k_{-i})$. A higher the ratio θ_+/θ_- indicates that the competitor's production has less impact on the firm's decision, causing the firm to wait longer before investing, which increases its option value. A larger sum $(\theta_+ + \theta_-)$ implies a greater impact of the industry's capital stock on the output price, thereby raising the threshold for capital investment and preserving more option values.

Figure 1 provides intuition for Condition 5. The shaded area, defined by the upper bound function $\overline{g}(w)$ (solid red line) and the lower bound function $\underline{g}(w)$ (dashed blue line), characterizes the set of closed-loop equilibria. These two lines intersect at two points, corresponding to the two roots of the equation $\overline{g}(w) - \underline{g}(w) = 0$. The lower left intersection (point A) corresponds to the $w = 1$ root with coordinates $(1, 2)$, and the upper right intersection (point B) corresponds to the other $w = w^* > 1$ root with coordinates $(w^*, 3 - 1/w^*)$, where

[15] Note that $\mathcal{K}(\,\cdot\,)$ is the inverse function of $\mathcal{X}(k, k) = \rho\,((\theta_+ + \theta_-)\eta k + rp + c)$.

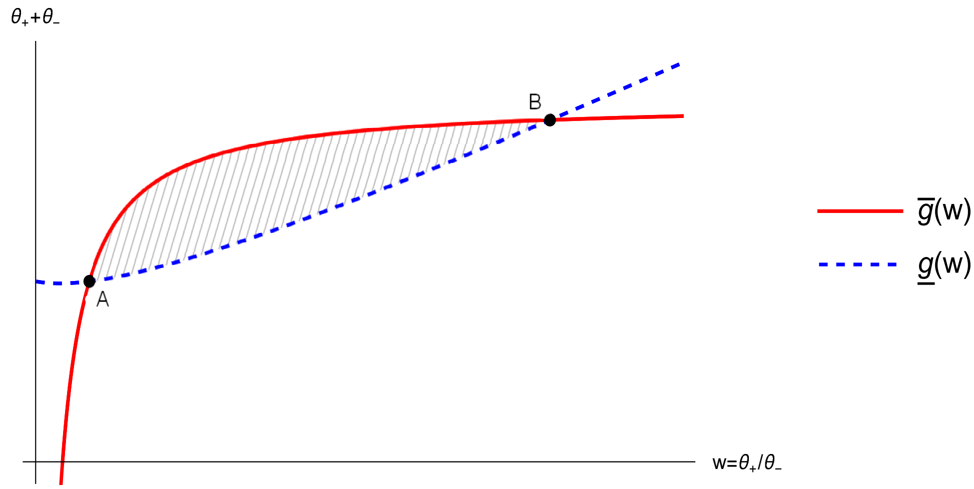

Figure 1: The shaded area, bounded by the solid red line $\overline{g}(\cdot)$ and the dashed blue $\underline{g}(\cdot)$, is the admissible region for (symmetric) closed-loop equilibria in a plane with θ_+/θ_- and $\theta_+ + \theta_-$ being the horizontal and vertical axes, respectively. The $\overline{g}(\cdot)$ and $\underline{g}(\cdot)$ functions are given in (5.11) and (5.12), respectively. Point A$= (1, 2)$ corresponds to the equilibrium with the perfectly competitive outcome ($\theta_+ = \theta_- = 1$) and point B corresponds to the equilibrium with the highest option value for both firms.

w^* is the unique root of the following equation[16]

$$h(w) := [(1+w)^{\beta-1} - 1][\beta + 2 - (\beta - 2)w] - 2(\beta - 1)w = 0\,, \quad w \geq 1\,. \tag{5.13}$$

Any pair of (θ_+, θ_-) inside the shaded area corresponds to a closed-loop equilibrium.

Later we show that Point A corresponds to the perfectly competitive equilibrium strategy (Back and Paulsen, 2009), where both firms make zero profits at all time. Point B corresponds to the closed-loop equilibrium with the highest option value.

Next, we characterize a class of Markov perfect equilibria.

Theorem 2 *Let $\mathcal{X}(k_a, k_b)$ and $\mathcal{K}(x)$ be given by (5.6) and (5.9), respectively. Let $\varphi_i :=$*

[16] The function $h(\cdot)$ is related to $\overline{g}(w) - \underline{g}(w)$ as follows: $h(w)(w-1) = (\overline{g}(w) - \underline{g}(w))\beta w[(1+w)^{\beta-1} - 1]$. It can be verified that $h(w)$ is concave for $w \geq 1$, $h(1) > 0$, and $h(w) < 0$ for sufficiently large w. Therefore, $h(w)$, $w \geq 1$ has a unique solution $w^* > 1$:

$(\boldsymbol{u}_i, \boldsymbol{v}_i)$ *be firm i's closed-loop strategy,*[17] *where*

$$\boldsymbol{u}_i(z, u_{-i}) = \left[\frac{\mathbf{1}_{k_i<k_{-i}}}{\eta\rho\theta_+} + \left(\frac{1}{\eta\rho\theta_-} - \frac{\theta_+}{\theta_-}u_{-i}\right)\mathbf{1}_{k_i\geq k_{-i}}\right]\mathbf{1}_{x\geq\mathcal{X}(k_a,k_b)}, \tag{5.14}$$

$$\boldsymbol{v}_i(z; v_{-i}) = \inf\Big\{\delta \geq 0 : \mathcal{X}\big(k_i+\delta, \max\{k_{-i}, \min\{k_{-i}+v_{-i}, \mathcal{K}(x)\}\}\big) \geq x\Big\}, \tag{5.15}$$

for any $z = (x, k_a, k_b) \in \mathbb{Z}$ *and any* $v_{-i} \geq 0$. *Under Condition 5, the closed-loop strategy pair,* $(\varphi_a, \varphi_b) \in \mathcal{S}$, *is a Markov perfect equilibrium strategy and we have:*

(i) When $\theta_+ > \theta_-$, *firm i's equilibrium investment and no-action regions are given by:*

$$\mathbf{I}_i(x) = \{(k_a, k_b) \in \mathbb{R}^2_+ : x \geq \mathcal{X}(k_a, k_b), k_i \leq \mathcal{K}(x)\} \quad \text{and} \quad \mathbf{N}_i(x) = \mathbb{R}^2_+ \setminus \mathbf{I}_i(x). \tag{5.16}$$

(ii) When $\theta_+ = \theta_-$, $\mathbf{I}_a(x) = \mathbf{I}_b(x) = \{(k_a, k_b) \in \mathbb{R}^2_+ : x \geq \mathcal{X}(k_a, k_b)\}$.

When $\theta_+ = \theta_-$, we must have $\theta_+ = \theta_- = 1$. This corresponds to the Back and Paulsen (2009)'s closed-loop equilibrium, in which firms adopt the zero-NPV rule when making investments and earn zero profits in equilibrium. We can also show that $V^i(x, k_a, k_b) = pk_i$ when firms invest, i.e., when $x \geq \mathcal{X}(k_a, k_b)$. Next, we turn to the case when firms make profits in equilibrium. This is the case for any admissible pair where $\theta_+ > \theta_-$.

5.3 Understanding Closed-loop Equilibrium Strategies

Using (5.16), we immediately obtain

$$\mathbf{I}_{ab}(x) = \{(k_a, k_b) \in \mathbb{R}^2_+ : k_a \leq \mathcal{K}(x), k_b \leq \mathcal{K}(x)\}, \tag{5.17}$$

$$\mathbf{N}_{ab}(x) = \{(k_a, k_b) \in \mathbb{R}^2_+ : x < \mathcal{X}(k_a, k_b)\}. \tag{5.18}$$

In Figure 2, we plot the equilibrium solution in the (k_a, k_b) plane. There are four mutually exclusive regions: 1.) the $\mathbf{I}_a(x) \setminus \mathbf{I}_{ab}(x)$ region defined by the triangle ABC in which only firm a invests; 2.) the $\mathbf{I}_b(x) \setminus \mathbf{I}_{ab}(x)$ region defined by the triangle CDE in which only firm b invests; 3.) the $\mathbf{I}_{ab}(x)$ region defined by the square BCDO in which both firms invest; and 4.) the $\mathbf{N}_{ab}(x)$ region to the right of ACE in which neither firm invests. The line segments AC and EC correspond to $\boldsymbol{\Gamma}_a(x)$ and $\boldsymbol{\Gamma}_b(x)$ defined in (5.8), respectively.

[17] See Section 2.1 for the definition of a closed-loop strategy. If (2.12) has no solution in $\mathbb{R}^2_+$ but admits a unique solution $(u_a, u_b) \in \mathbb{R}^2$, we can define a solution in $\mathbb{R}^2_+$ as $(\max\{u_a, 0\}, \max\{u_b, 0\})$ to allow for more admissible strategies.

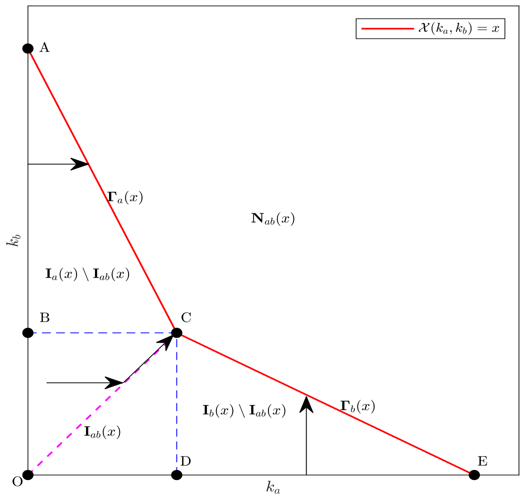

Figure 2: Equilibrium solution regions.

As shown in Figure 2, starting from $(k_a, k_b) \in \mathbf{I}_{ab}(x)$ (rectangle BCDO), the two firms will adjust their capital pair to the point C= $(\mathcal{K}(x), \mathcal{K}(x))$ instantaneously. Starting from $(k_a, k_b) \in \mathbf{I}_a(x) \setminus \mathbf{I}_{ab}(x)$ (triangle ABC), firm b will remain inactive while firm a will increase its capital instantaneously to reach the AC line segment. Similarly, starting from $(k_a, k_b) \in \mathbf{I}_b(x) \setminus \mathbf{I}_{ab}(x)$ (triangle CDE), the capital pair will move upward instantaneously to the CE line segment. In the no-action region $\mathbf{N}_{ab}(x)$, both firms remain inactive.

Next we elaborate on how to derive the firms' investment strategies in the equilibrium described above.

Investment strategy in equilibrium. We begin by examining lumpy investment. Consider any state (X_t, K_{at}, K_{bt}) at time t. First, since $\mathcal{X}(k_a, k_b)$ is increasing in k_a and k_b, we deduce from (5.15) that $\boldsymbol{v}_i(x, k_a, k_b; v_{-i}) = 0$ for any $v_{-i} \geq 0$ and $x \leq \mathcal{X}(k_a, k_b)$. It follows that

$$\Delta K_{it} = \boldsymbol{v}_i(X_t, K_{at}, K_{bt}; \Delta K_{-it}) = 0 \quad \text{for all } X_t \leq \mathcal{X}(K_{at}, K_{bt}), \tag{5.19}$$

indicating that there is no lumpy investment whenever $X_t \leq \mathcal{X}(K_{at}, K_{bt})$.

Second, noticing (5.15) and $\mathcal{X}(\mathcal{K}(x),\mathcal{K}(x)) = x$, we infer[18]

$$\Delta K_{it} = \mathcal{K}(X_t) - K_{it} > 0 \quad \text{when } (K_{at}, K_{bt}) \in [0,\mathcal{K}(X_t)) \times [0,\mathcal{K}(X_t)), \tag{5.20}$$

indicating that both firms make a lumpy investment to arrive at point $\mathbf{C}(X_t) = (\mathcal{K}(X_t),\mathcal{K}(X_t))$ in this case.

Third, we infer from (5.15) and the monotonicity of $\mathcal{X}(k_a, k_b)$ in k_a and k_b that[19]

$$\Delta K_{it} = \inf\{\delta \geq 0 : \mathcal{X}\left(K_{it} + \delta, K_{-it}\right) \geq X_t\} > 0 \text{ and } \Delta K_{-it} = 0 \tag{5.21}$$

$$\text{when } X_t > \mathcal{X}(K_{at}, K_{bt}) \text{ and } K_{-it} \geq \mathcal{K}(X_t), \tag{5.22}$$

where we have used $K_{it} < \mathcal{K}(X_t)$ for this case. This indicates that firm $-i$ does not make a lumpy investment while firm i does at time t, moving instantaneously to $\mathbf{\Gamma}_i(X_t)$.

In summary, from (5.19)-(5.22), we deduce that firm i makes a lumpy investment in the above equilibrium if and only if it is in the interior of $\mathbf{I}_i(x)$. Moreover, the lumpy investment ensures that $X_t \leq \mathcal{X}(K_{at}, K_{bt})$ for almost every $t \geq 0$.

It remains to investigate the continuous investment for $(K_{at}, K_{bt}) \in \overline{\mathbf{N}}_{ab}(X_t)$. By (5.14), we infer

$$u_{at} = u_{bt} = 0 \quad \text{when } (K_{at}, K_{bt}) \in \mathbf{N}_{ab}(X_t). \tag{5.23}$$

We only need to investigate the continuous investment strategy on $\mathbf{\Gamma}(X_t)$ (i.e., $\mathcal{X}(K_{at}, K_{bt}) = X_t$, the boundary of $\mathbf{N}_{ab}(X_t)$). Using (5.14), one can derive the following strategy on $\mathbf{\Gamma}(X_t)$:

(i) On $\mathbf{\Gamma}_i(X_t) \setminus \mathbf{C}(X_t)$,

$$dK_{it} = \frac{dM_t}{\eta\rho\theta_+} \quad \text{and} \quad dK_{-it} = 0. \tag{5.24}$$

This result follows from $\Delta K_{at} = \Delta K_{bt} = 0$ as shown in (5.19) and by solving the following system of equations for u_{it} and u_{-it} implied by (5.14):

$$u_{it} = \frac{1}{\eta\rho\theta_+}, \quad u_{-it} = \frac{1}{\eta\rho\theta_-} - \frac{\theta_+}{\theta_-}u_{it}. \tag{5.25}$$

[18] Using (5.15), one can see that for $k_a < \mathcal{K}(x)$ and $k_b < \mathcal{K}(x)$, we have $k_i + \boldsymbol{v}_i(z; v_{-i}) \geq \mathcal{K}(x)$ when $k_{-i} + v_{-i} < \mathcal{K}(x)$, and $k_i + \boldsymbol{v}_i(z; v_{-i}) = \mathcal{K}(x)$ when $k_{-i} + v_{-i} \geq \mathcal{K}(x)$. Thus, the unique solution pair(v_a, v_b) to the system of equations $v_a = \boldsymbol{v}_a(z, v_b)$ and $v_b = \boldsymbol{v}_b(z, v_a)$ is $(v_a, v_b) = (\mathcal{K}(x) - k_a, \mathcal{K}(x) - k_b)$.

[19] In the case $x > \mathcal{X}(k_a, k_b)$ and $k_{-i} \geq \mathcal{K}(x)$, we have $\boldsymbol{v}_i(z; v_{-i}) = \inf\{\delta \geq 0 : \mathcal{X}\left(k_i + \delta, k_{-i}\right) \geq x\}$ and $\boldsymbol{v}_{-i}(z; v_i) = \inf\{\delta \geq 0 : \mathcal{X}\left(k_{-i} + \delta, k_i + v_i\right) \geq x\}$ for any $v_i \geq 0$ and $v_{-i} \geq 0$.

(ii) At $\mathbf{C}(X_t)$, the two firms move symmetrically:

$$dK_{at} = dK_{bt} = \frac{dM_t}{\eta\rho(\theta_+ + \theta_-)}\,. \tag{5.26}$$

This can be derived from $\Delta K_{at} = \Delta K_{bt} = 0$ as shown in (5.19) and by solving the following system of equations for u_{at} and u_{bt} implied by (5.14):

$$u_{at} = \frac{1}{\eta\rho\theta_-} - \frac{\theta_+}{\theta_-}u_{bt}, \quad u_{bt} = \frac{1}{\eta\rho\theta_-} - \frac{\theta_+}{\theta_-}u_{at}. \tag{5.27}$$

Using (5.19)-(5.26), we conclude the following explicit expression of equilibrium capital processes for any $t > 0$:

$$K_{it} = \inf\Big\{k \geq K_{i0} : \mathcal{X}\big(k, \max\{k_{-i0}, \mathcal{K}(M_t)\}\big) \geq M_t\Big\}. \tag{5.28}$$

It can be observed that the lumpy investment only occurs at time 0 in equilibrium capital processes.

Off-equilibrium Strategy. Next, we present the dynamics of capital stocks driven by the off-equilibrium strategy implied by Theorem 2. For ease of exposition, we focus on the deviation of firm a. For any strategy pair $(\varphi_a, \varphi_b) \in \mathcal{S}$ stated in Theorem 2 and any feasible strategy pair $(\widetilde{\varphi}_a, \varphi_b) \in \mathcal{S}$, where firm a deviates from time T with $(X_t, \widetilde{K}_{at}, \widetilde{K}_{bt}) = (X_t, K_{at}, K_{bt}) \in \mathbb{Z}$ for all $t \leq T$, we denote by

$$\widetilde{K}_a := K_a^{\widetilde{\varphi}_a, \varphi_b}, \quad \widetilde{K}_b := K_b^{\widetilde{\varphi}_a, \varphi_b} \tag{5.29}$$

the two firms' capitals to highlight the dependence of $\widetilde{K}_i$ on $(\widetilde{\varphi}_a, \varphi_b)$. We first investigate the lumpy investment of firm b in response to firm a's deviation.

Using (5.15), one can deduce $\Delta\widetilde{K}_{bt}$ in the following three cases.

(i) When $\widetilde{K}_{at+} < \mathcal{K}(X_t)$, we have

$$\Delta\widetilde{K}_{bt} = \inf\{\delta \geq 0 : \ \mathcal{X}\left(\widetilde{K}_{bt} + \delta, \widetilde{K}_{at+}\right) \geq X_t\}. \tag{5.30}$$

(ii) When $\widetilde{K}_{at+} \geq \mathcal{K}(X_t) \geq \widetilde{K}_{at}$, we have

$$\Delta\widetilde{K}_{bt} = \inf\{\delta \geq 0 : \ \mathcal{X}\left(\widetilde{K}_{bt} + \delta, \mathcal{K}(X_t)\right) \geq X_t\} = \left(\mathcal{K}(X_t) - \widetilde{K}_{bt}\right)^+. \tag{5.31}$$

(iii) When $\widetilde{K}_{at} > \mathcal{K}(X_t)$, we have

$$\Delta \widetilde{K}_{bt} = \inf\{\delta \geq 0 : \ \mathcal{X}\left(\widetilde{K}_{bt} + \delta, \widetilde{K}_{at}\right) \geq X_t\}. \tag{5.32}$$

According to (5.30)-(5.32), firm b's lumpy investment ensures that $X_t \ \leq \mathcal{X}(\widetilde{K}_{at}, \widetilde{K}_{bt})$ for almost all $t \geq T$.

Next, we investigate the continuous investment strategy. Let $\widetilde{u}_{it}$ denote the continuous investment rate related to dM_t for firm i under the strategy $(\widetilde{\varphi}_a, \varphi_b)$. By (5.14)-(5.15), we infer

$$d\widetilde{K}_{bt} = 0 \quad \text{when } (\widetilde{K}_{at}, \widetilde{K}_{bt}) \in \mathbf{N}_{ab}(X_t). \tag{5.33}$$

It remains to investigate the continuous investment strategy on $\mathbf{\Gamma}(X_t)$. For illustration, we assume $M_t = X_t$, $dM_t > 0$, and $\widetilde{K}_{as}$ continuously increases with a constant rate for s close t:

$$d\widetilde{K}_{as} = \widetilde{u}_a dM_s\,, \quad \forall s \in [t, t+\epsilon], \tag{5.34}$$

where $\epsilon > 0$ is sufficiently small and $\widetilde{u}_a$ is a nonnegative constant.

Using (5.14), one can deduce $d\widetilde{K}_{bs}$ in the following three cases.

- When $(\widetilde{K}_{at}, \widetilde{K}_{bt}) \in \mathbf{\Gamma}_a(X_t) \setminus \mathbf{C}(X_t)$:

  (i) If $\widetilde{u}_a \leq \frac{1}{\eta\rho\theta_+}$, then for s close to t, $(\widetilde{K}_{as}, \widetilde{K}_{bs}) \in \mathbf{\Gamma}_a(M_s)$ and firm b moves according to

  $$d\widetilde{K}_{bs} = \left[\frac{1}{\eta\rho\theta_-} - \frac{\theta_+}{\theta_-}\widetilde{u}_a\right] dM_s. \tag{5.35}$$

  This can be derived from

  $$\widetilde{u}_{bs} = \frac{1}{\eta\rho\theta_-} - \frac{\theta_+}{\theta_-}\widetilde{u}_a \geq 0. \tag{5.36}$$

  (ii) If $\widetilde{u}_a > \frac{1}{\eta\rho\theta_+}$, then $(\widetilde{K}_{as}, \widetilde{K}_{bt}) \in \mathbf{N}_{ab}(M_s)$ for any $s \in (t, t+\epsilon)$, and thus firm b won't invest in this period.[20] This can be derived from

  $$d\mathcal{X}(\widetilde{K}_{as}, \widetilde{K}_{bs}) = \rho\eta\Big(\theta_+ \widetilde{u}_a dM_s + \theta_- d\widetilde{K}_{bs}\Big) > dM_s. \tag{5.37}$$

- When $(\widetilde{K}_{at}, \widetilde{K}_{bt}) \in \mathbf{\Gamma}_b(X_t) \setminus \mathbf{C}(X_t)$:

[20] At time t, (5.14) implies that $\widetilde{u}_b = \frac{1}{\eta\rho\theta_-} - \frac{\theta_+}{\theta_-}\widetilde{u}_a < 0$. Then by Footnote 17, we set $\widetilde{u}_{bt} = 0$.

(i) If $\widetilde{u}_a = 0$, then for any s close to t, $(\widetilde{K}_{as}, \widetilde{K}_{bs}) \in \mathbf{\Gamma}_b(M_s)$ and firm b moves according to

$$d\widetilde{K}_{bs} = \frac{1}{\eta\rho\theta_+} dM_s. \tag{5.38}$$

(ii) If $0 < \widetilde{u}_a \le \frac{1}{\eta\rho\theta_-}$, then $\widetilde{K}_{bs}$ is not well-defined.[21]

(iii) If $\widetilde{u}_a > \frac{1}{\eta\rho\theta_-}$, then $(\widetilde{K}_{as}, \widetilde{K}_{bt}) \in \mathbf{N}_{ab}(M_s)$ for any $s \in (t, t+\epsilon)$, and thus firm b won't invest in this period. This can be derived from

$$d\mathcal{X}(\widetilde{K}_{as}, \widetilde{K}_{bs}) = \rho\eta\Big(\theta_- \widetilde{u}_a dM_s + \theta_+ d\widetilde{K}_{bs}\Big) > dM_s. \tag{5.39}$$

- When $(\widetilde{K}_{at}, \widetilde{K}_{bt}) = \mathbf{C}(X_t)$, the following hold.

(i) If $\widetilde{u}_a \le \frac{1}{\eta\rho(\theta_+ + \theta_-)}$, then for any s close to t, $(\widetilde{K}_{as}, \widetilde{K}_{bs}) \in \mathbf{\Gamma}_a(M_s)$ and firm b moves according to (5.35). This can be derived from

$$\widetilde{u}_{bs} = \frac{1}{\eta\rho\theta_-} - \frac{\theta_+}{\theta_-}\widetilde{u}_a \ge \frac{1}{\eta\rho(\theta_+ + \theta_-)}. \tag{5.40}$$

(ii) If $\frac{1}{\eta\rho(\theta_+ + \theta_-)} < \widetilde{u}_a \le \frac{1}{\eta\rho\theta_-}$, then $\widetilde{K}_{bs}$ is not well-defined.[22]

(iii) If $\widetilde{u}_a > \frac{1}{\eta\rho\theta_-}$, then $(\widetilde{K}_{as}, \widetilde{K}_{bt}) \in \mathbf{N}_{ab}(M_s)$ for any $s \in (t, t+\epsilon)$ and thus firm b won't invest in this period. This can be derived from (5.39).

5.4 Equilibrium Value Functions

Next, we report closed-form expressions for value functions associated with the equilibrium strategy given in Theorem 2.

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[21] If $0 < \widetilde{u}_a \le \frac{1}{\eta\rho\theta_-}$, $(\widetilde{K}_{as}, \widetilde{K}_{bs})$ will enter $\mathbf{N}_{ab}(X_s)$ immediately, which implies that firm b won't invest at $s = t+$. However, only firm a's investment cannot ensure $(\widetilde{K}_{as}, \widetilde{K}_{bt}) \in \mathbf{N}_{ab}(X_s)$, and $\widetilde{K}_{bs}$ is not well-defined. To allow for more admissible strategies, we can replace (5.14) with

$$\boldsymbol{u}_i(z, u_{-i}) = \left[\frac{\mathbf{1}_{k_i < k_{-i}}}{\eta\rho\theta_+} + \max\left\{u_{-i}, \Big(\frac{1}{\eta\rho\theta_-} - \frac{\theta_+}{\theta_-}u_{-i}\Big)\right\}\mathbf{1}_{k_i = k_{-i}} + \Big(\frac{1}{\eta\rho\theta_-} - \frac{\theta_+}{\theta_-}u_{-i}\Big)\mathbf{1}_{k_i > k_{-i}}\right]\mathbf{1}_{x \ge \mathcal{X}(k_{a0}, k_{b0})}.$$

One can verify that the equilibrium capital process under the above response function is still given by (5.28), and for the off-equilibrium with $(\widetilde{K}_{at}, \widetilde{K}_{bt}) \in \mathbf{\Gamma}_b(X_t) \setminus \mathbf{C}(X_t)$, we have $d\widetilde{K}_{bs} = \frac{1}{\eta\rho\theta_+} dM_s$ for any s close to t.

[22] For the response function given in Footnote 21, we have $\widetilde{u}_{bs} = \widetilde{u}_a$ for any s close to t when $(\widetilde{K}_{at}, \widetilde{K}_{bt}) = \mathbf{C}(X_t)$ and $\widetilde{u}_a > \frac{1}{\eta\rho(\theta_+ + \theta_-)}$.

Theorem 3 *The value function $V^i(z)$ associated with the equilibrium strategy given in Theorem 2, where $z = (x, k_a, k_b)$, is given below.*

(i) When the demand shock is (weakly) below the trigger function $\mathcal{X}(k_a, k_b)$: $x \leq \mathcal{X}(k_a, k_b)$, i.e., in the $\overline{\mathbf{N}}_{ab}(x)$ region, $V^i(z) = U^i(z)$ where

$$U^i(z) = \Psi^i(z) + H^i(k_a, k_b)x^\beta \,, \tag{5.41}$$

where $\Psi^i(z)$ is given in (5.2), β is the constant given in (4.6), and $H^i(k_a, k_b)$, given by (5.49)-(5.51), is a function describing firm i's option value.

(ii) In the square BCDO region: $\mathbf{I}_{ab}(x)$, both firms instantly invest to the point C and

$$V^i(z) = U^i(x, \mathcal{K}(x), \mathcal{K}(x)) - p(\mathcal{K}(x) - k_i) \,. \tag{5.42}$$

(iii) In the triangle ABC region: $\mathbf{I}_a(x) \setminus \mathbf{I}_{ab}(x)$, firm b does not invest and firm a invests so that its capital stock reaches the AC line segment and is given by

$$\widehat{k}_a = \frac{1}{\eta\theta_+}\left(\frac{x}{\rho} - (rp + c)\right) - \frac{\theta_-}{\theta_+}k_b \,.$$

The equilibrium value functions for firms a and b are given by

$$V^a(z) = U^a(x, \widehat{k}_a, k_b) - p(\widehat{k}_a - k_a) \,, \tag{5.43}$$

and

$$V^b(z) = U^b(x, \widehat{k}_a, k_b) \,, \tag{5.44}$$

respectively, where $U^i(x, \widehat{k}_a, k_b)$ is given in (5.41).

By symmetry, we can obtain the solution in the $\mathbf{I}_b(x) \setminus \mathbf{I}_{ab}(x)$ region where $\mathcal{X}(k_a, k_b) < x$ and $k_a \in (\mathcal{K}(x), \widehat{\mathcal{K}}(x)]$.

Characterizing Value Function $U^i(x, k_a, k_b)$ in the Inaction Region $\mathbf{N}_{ab}(x)$.

Next we elaborate on how to obtain the closed-form solution for the value functions. We conjecture and later verify the function forms of the value functions for the firms, which satisfy Conditions 1 and 2 as given in Theorem 1. We first focus on the $\mathbf{N}_{ab}(x)$ region where $x < \mathcal{X}(k_a, k_b)$ and both firms are inactive.

As given by Condition 2, in the inaction region $\mathbf{N}_{ab}(x)$, the HJB equation (3.5) holds with $\sigma(x) = \sigma x$, $\mu(x) = \mu x$, and $F_i(x, k_a, k_b)$ given in (5.1), and its general solution takes the form of (5.41). Due to symmetry, below we only characterize firm a's option value, $H^a(k_a, k_b)$, in the inaction region.

First, as implied by (3.2), firm a sets its marginal q to its marginal cost of purchasing capital p when investing, which suggests that the following investment FOC holds on the investment boundary $\mathbf{\Gamma}_a(x)$:

$$\frac{\partial U^a(z)}{\partial k_a} = p, \quad \text{when } \mathcal{X}(k_a, k_b) = x \text{ and } k_a \leq k_b\,. \tag{5.45}$$

Second, as implied by (3.2), firm a's value should not change after a lumpy investment of its competitor, which suggests the following condition on the competitor's investment boundary $\mathbf{\Gamma}_b(x)$:

$$\frac{\partial U^a(z)}{\partial k_b} = 0, \quad \text{when } \mathcal{X}(k_a, k_b) = x \text{ and } k_a \geq k_b\,. \tag{5.46}$$

By substituting (5.41) into (5.45) and (5.46), we obtain

$$\frac{\partial H^a(k_a, k_b)}{\partial k_a} = \left(\frac{2\eta k_a + \eta k_b + rp + c}{r} - \frac{\mathcal{X}(k_a, k_b)}{r - \mu}\right)\mathcal{X}(k_a, k_b)^{-\beta}, \quad k_a \leq k_b\,, \tag{5.47}$$

$$\frac{\partial H^a(k_a, k_b)}{\partial k_b} = \frac{\eta k_a}{r}\mathcal{X}(k_a, k_b)^{-\beta}, \quad k_a \geq k_b\,. \tag{5.48}$$

As the capital stock approaches infinity, neither firm invests because all future profits are almost surely nonpositive, therefore, firm a's value equals $\Psi^a(z)$ given in (5.2) and the option value of investing is zero: $\lim_{k\to\infty} H^a(k, k) = 0$.

Third, we derive the following closed-form solution for $H^a(k, k)$:

$$H^a(k, k) = \frac{\mathcal{X}(k, k)^{1-\beta}}{\beta(\beta - 2)(r - \mu)}\left[\left(\beta - \frac{4(\beta - 1)}{\theta_+ + \theta_-}\right)k + \frac{rp + c}{\eta(\theta_+ + \theta_-)}\left(2 - \frac{4}{\theta_+ + \theta_-}\right)\right] \tag{5.49}$$

by using (5.47), (5.48), and $\lim_{k\to\infty} H^a(k, k) = 0$.

Finally, we can use (5.47) and (5.48) to obtain the closed-form solution for $H^a(k_a, k_b)$ for any $(k_a, k_b) \in \mathbb{R}_+^2$:

$$H^a(k_a, k_b) = H^a(k_b, k_b) - \int_{k_a}^{k_b} \left(\frac{2\eta k + \eta k_b + rp + c}{r} - \frac{\mathcal{X}(k, k_b)}{r - \mu} \right) \mathcal{X}(k, k_b)^{-\beta} dk, \quad k_a \le k_b, \tag{5.50}$$

$$H^a(k_a, k_b) = H^a(k_a, k_a) - \int_{k_b}^{k_a} \frac{\eta k_a}{r} \mathcal{X}(k_a, k)^{-\beta} dk, \quad k_a \ge k_b, \tag{5.51}$$

where $H^a(k, k)$ is given by (5.49).

Economics of Condition 5. To ensure that V^a satisfies the two inequalities in (3.1), we first verify that they are satisfied on the ACE boundary $\mathbf{\Gamma}(x)$. Recall that $\frac{\partial U^a(z)}{\partial k_a} = p$ for any $(k_a, k_b) \in \mathbf{\Gamma}_a(x)$ (see (5.45)) and $\frac{\partial U^a(z)}{\partial k_b} = 0$ for any $(k_a, k_b) \in \mathbf{\Gamma}_b(x)$ (see (5.46)). We thus only need to verify that $\frac{\partial U^a(z)}{\partial k_a} \le p$ for all $(k_a, k_b) \in \mathbf{\Gamma}_b(x)$ and $\frac{\partial U^a(z)}{\partial k_b} \le 0$ for all $(k_a, k_b) \in \mathbf{\Gamma}_a(x)$, i.e.,

$$\frac{\partial U^a(z)}{\partial k_a}|_{x=\mathcal{X}(k_a,k_b)} \le p, \quad k_b \le k_a, \tag{5.52}$$

$$\frac{\partial U^a(z)}{\partial k_b}|_{x=\mathcal{X}(k_a,k_b)} \le 0, \quad k_a \le k_b\,. \tag{5.53}$$

Lemma 4 in Appendix B implies that the second inequality in (5.10) is a necessary condition for (5.52) and the first inequality in (5.10) is a necessary condition for (5.53). Back and Paulsen (2009) show that the open-loop equilibrium corresponds to the pair $(\theta_+, \theta_-) = (2, 1)$, which does not satisfy Condition 5 and is thus outside of the shaded area in Figure 1. Indeed, inequality (5.52) does not hold for the open-loop equilibrium. Intuitively speaking, as firm a's marginal q exceeds its marginal cost of investing, the firm is incentivized to invest more. That is why such a deviation from the open-loop equilibrium strategy is profitable.

To ease comparison across various equilibria, we define the following set for (θ_+, θ_-):

$$\Theta := \{(\theta_+, \theta_-) \in \mathbb{R}^2 : \theta_+ \ge \theta_- > 0, \text{ subject to } (5.10)\}\,, \tag{5.54}$$

let $\varphi^{\theta_+,\theta_-}$ denote the strategy φ associated with the trigger (5.6), and let $V^i(z; \varphi^{\theta_+,\theta_-})$, $i = a, b$ denote the corresponding value function. Next, we compare value functions associated with different equilibria (associated with different choices of (θ_+, θ_-)) along the 45-degree line on the (k_a, k_b) plane in that $k_a = k_b > 0$.

Proposition 2 *Let* $\theta_+^* = \frac{3w^*-1}{1+w^*}$ *and* $\theta_-^* = \frac{3-1/w^*}{1+w^*}$. *Then* $(\theta_+^*, \theta_-^*) \in \Theta$, $(2,1) \notin \Theta$, $(1,1) \in \Theta$, *and for any* $(\theta_+, \theta_-) \in \Theta \setminus \{(\theta_+^*, \theta_-^*), (1,1)\}$, *we have*

$$V^i(x,k,k;\varphi^{2,1}) > V^i(x,k,k;\varphi^{\theta_+^*,\theta_-^*}) > V^i(x,k,k;\varphi^{\theta_+,\theta_-}) > V^i(x,k,k;\varphi^{1,1}). \tag{5.55}$$

First, the last inequality in (5.55) confirms Back and Paulsen (2009)'s result that the closed-loop equilibrium with a perfectly competitive outcome, which corresponds to the pair $(\theta_+, \theta_-) = (1,1)$, attains the lowest value among all equilibria. Second, the highest equilibrium firm value, implied by the second inequality in (5.55), is attained by a unique equilibrium strategy $\varphi^{\theta_+^*,\theta_-^*}$ that corresponds to point B in Figure 1.

Third, firm value is strictly higher in the open-loop equilibrium compared to all closed-loop equilibria, implied by the first inequality in (5.55). This is because firms in the open-loop equilibrium invest more slowly than in all closed-loop equilibria. However, the open-loop equilibrium is not Markov subgame perfect (Back and Paulsen, 2009). By contrast, in the closed-loop equilibrium associated with (θ_+^*, θ_-^*), neither firm has a profitable deviation at any t as the equilibrium is Markov subgame perfect.

6 Duopoly Solution: Nonlinear Trigger Functions

For all closed-loop equilibria (other than Back and Paulsen (2009)'s one with zero profits) presented in Section 5, firm i's marginal q is strictly lower than the marginal cost p in the region where the competitor invests in that $\frac{\partial U^i(x,k_a,k_b)}{\partial k_i} |_{x=\mathcal{X}(k_a,k_b)} < p$ for $k_i > k_{-i}$. In this section, we show that there also exist closed-loop equilibria, in which firm i's marginal q equals the marginal cost p in the region where the competitor invests in that

$$\frac{\partial U^i(x,k_a,k_b)}{\partial k_i} |_{x=\mathcal{X}(k_a,k_b)} = p \quad \text{for } k_i > k_{-i}. \tag{6.1}$$

Below we derive the equilibrium nonlinear trigger functions under regularity conditions[23] that support these new equilibria.

[23] We only consider the trigger function $\mathcal{X}(k_a,k_b)$ that is continuously differentiable on $\{(k_a,k_b) \in [\underline{k},\infty) \times [\underline{k},\infty) : k_a \neq k_b\}$ with $\mathcal{X}_{k_a}(k_a,k_b) > 0$ and $\mathcal{X}_{k_b}(k_a,k_b) > 0$, where $\underline{k} > 0$ is a constant. Without loss of generality, we assume that $\mathcal{X}(k_a,k_b)$ has a upper bound given by the monopolist's trigger $\rho(2\eta(k_a+k_b)+rp+c)$ and a lower bound given by the trigger of perfect competition equilibrium $\rho(\eta(k_a+k_b)+rp+c)$. These bounds guarantee the well-posedness of the firms' value functions under the trigger function $\mathcal{X}(k_a,k_b)$.

6.1 Equilibrium Nonlinear Trigger Functions

Using the constraint (6.1), we infer that (5.45) holds for any (k_a, k_b), as does (5.47) for any (k_a, k_b). Combining (5.47) and (5.48) and using $\frac{\partial^2 H^a(k_a,k_b)}{\partial k_a \partial k_b} = \frac{\partial^2 H^a(k_a,k_b)}{\partial k_b \partial k_a}$ for $k_a \geq k_b$,[24] we obtain

$$\rho \eta k_a \frac{\partial \mathcal{X}(k_a, k_b)}{\partial k_a} + \Big[\mathcal{X}(k_a, k_b) - \rho\big(2\eta k_a + \eta k_b + rp + c\big)\Big] \frac{\partial \mathcal{X}(k_a, k_b)}{\partial k_b} = 0, \quad k_a \geq k_b. \tag{6.2}$$

Lemma 5 in Appendix B indicates that the partial differential equation (PDE) (6.2) admits the following general solution:

$$\mathcal{X}(k_a, k_b) = \rho\Big(\eta k_a + \eta k_b + \eta \mathcal{G}(k_a, k_b) + rp + c\Big), \tag{6.3}$$

where $\mathcal{G}(k_a, k_b)$ for $k_a \geq k_b$ is a continuously differentiable function and satisfies

$$\phi\Big(k_a \mathcal{G}, k_a + k_b + \mathcal{G}\Big) = 0. \tag{6.4}$$

Here, $\phi(z_1, z_2)$ is any continuously differentiable function with $\frac{\partial \phi(z_1,z_2)}{\partial z_1} \neq 0$ or $\frac{\partial \phi(z_1,z_2)}{\partial z_2} \neq 0$.

Using (6.3) and symmetry, we obtain the following trigger function for all k_a and k_b:

$$\mathcal{X}(k_a, k_b) := \begin{cases} \rho\big(\eta k_a + \eta k_b + \eta \mathcal{G}(k_b, k_a) + rp + c\big), & \text{if } k_a \leq k_b, \\ \rho\big(\eta k_a + \eta k_b + \eta \mathcal{G}(k_a, k_b) + rp + c\big), & \text{if } k_a \geq k_b. \end{cases} \tag{6.5}$$

Next, we impose a condition on the function $\mathcal{G}(k_a, k_b)$ so that the trigger function (6.5) can yield a closed-loop equilibrium.

Condition 6 *There exists $\underline{k} > 0$ such that the continuously differentiable function $\mathcal{G}(k_a, k_b)$ given in (6.4) satisfies*

(i) for any $k_a \geq k_b \geq \underline{k}$,

$$0 < \mathcal{G}(k_a, k_b) < k_b; \tag{6.6}$$

(ii) $k_b + \mathcal{G}(k_a, k_b)$ is increasing in k_b for any $k_a \geq k_b \geq \underline{k}$.

The left inequality in (6.6) ensures that $\mathcal{X}(k_a, k_b)$ given in (6.5) is strictly higher than the trigger of the perfect competition equilibrium, i.e., the trigger given in (5.6) with $\theta_+ =$

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[24] More precisely, $H^a(k_a, k_b)$ is twice continuously differentiable in the region $k_a > k_b$, and the second-order partial derivative has a finite limit on the boundary $k_a = k_b+$. Then, we define $\frac{\partial^2 H^a(k_a,k_b)}{\partial k_a \partial k_b}\big|_{k_a=k_b} := \lim_{k_a \to k_b+0} \frac{\partial^2 H^a(k_a,k_b)}{\partial k_a \partial k_b}$ and $\frac{\partial^2 H^a(k_a,k_b)}{\partial k_b \partial k_a}\big|_{k_a=k_b} := \lim_{k_a \to k_b+0} \frac{\partial^2 H^a(k_a,k_b)}{\partial k_b \partial k_a}$.

$\theta_- = 1$. The right inequality in (6.6) ensures that $\mathcal{X}(k_a, k_b)$ given in (6.5) is strictly lower than the trigger of the open-loop equilibrium, i.e., the trigger given in (5.6) with $\theta_+ = 2$, $\theta_- = 1$. Part (ii) of Condition 6 indicates the monotonicity of $\mathcal{X}(k_a, k_b)$ in k_b when $k_a \geq k_b$. By combining this with (6.6) and considering symmetry, we can further establish the strict monotonicity of $\mathcal{X}(k_a, k_b)$ in both k_a and k_b.[25]

In the following, we set $\min(K_{a0}, K_{b0}) \geq \underline{k}$, and consider the following state space:

$$\mathbb{Z} = (0, \infty) \times [\underline{k}, \infty) \times [\underline{k}, \infty), \tag{6.7}$$

where $\underline{k}$ is defined in Condition 6. Next we present symmetric closed-loop equilibria associated with the nonlinear trigger functions $\mathcal{X}(k_a, k_b)$ as given in equation (6.5). We denote

$$\mathcal{K}(x) := \inf\{k \geq \underline{k} : \mathcal{X}(k, k) \geq x\}. \tag{6.8}$$

Moreover, the investment boundaries $\mathbf{\Gamma}(x)$ and $\mathbf{\Gamma}_i(x)$ are also defined by (5.7)-(5.8) with $\mathcal{X}(k_a, k_b)$ being given by equation (6.5). See Figure 3 for a graphical illustration for these boundaries, investment regions, and no-action regions.

6.2 Symmetric Closed-Loop Equilibria with Nonlinear Trigger Functions

Let $\varphi_i := (\boldsymbol{u}_i, \boldsymbol{v}_i)$ denote firm i's closed-loop strategy, as described below:

$$\boldsymbol{u}_i(z, u_{-i}) = \Big[\frac{\mathbf{1}_{k_i < k_{-i}}}{\vartheta_+(k_a, k_b)} + \Big(\frac{1}{\vartheta_-(k_a, k_b)} - \frac{\vartheta_+(k_a, k_b)}{\vartheta_-(k_a, k_b)} u_{-i}\Big)\mathbf{1}_{k_i \geq k_{-i}}\Big]\mathbf{1}_{x \geq \mathcal{X}(k_a, k_b)}, \tag{6.9}$$

$$\boldsymbol{v}_i(z; v_{-i}) = \inf\Big\{\delta \geq 0 : \mathcal{X}\Big(k_i + \delta, \max\{k_{-i}, \min\{k_{-i} + v_{-i}, \mathcal{K}(x)\}\}\Big) \geq x\Big\}, \tag{6.10}$$

where

$$\vartheta_+(k_a, k_b) := \begin{cases} \frac{\partial \mathcal{X}(k_a, k_b)}{\partial k_a}, & \text{if } k_a \leq k_b, \\ \frac{\partial \mathcal{X}(k_a, k_b)}{\partial k_b}, & \text{if } k_a \geq k_b, \end{cases} \tag{6.11}$$

[25] According to part (ii) of Condition 6, we have $\frac{\partial \mathcal{X}(k_a, k_b)}{\partial k_b} \geq 0$ for any $k_a \geq k_b \geq \underline{k}$. By combining (6.2) and (6.5), we find that $\frac{\partial \mathcal{X}(k_a, k_b)}{\partial k_a} / \frac{\partial \mathcal{X}(k_a, k_b)}{\partial k_b} = 1 - \frac{\mathcal{G}(k_a, k_b)}{k_a}$ for $k_a \geq k_b \geq \underline{k}$. We then infer from (6.6) that $\frac{\partial \mathcal{X}(k_a, k_b)}{\partial k_a} / \frac{\partial \mathcal{X}(k_a, k_b)}{\partial k_b} \in (0, 1)$ for any $k_a \geq k_b \geq \underline{k}$, which implies that $\frac{\partial \mathcal{X}(k_a, k_b)}{\partial k_a} > 0$ and $\frac{\partial \mathcal{X}(k_a, k_b)}{\partial k_b} > 0$ for any $k_a \geq k_b \geq \underline{k}$. By the symmetry of $\mathcal{X}(k_a, k_b)$, we conclude the strict monotonicity of $\mathcal{X}(k_a, k_b)$ in both k_a and k_b.

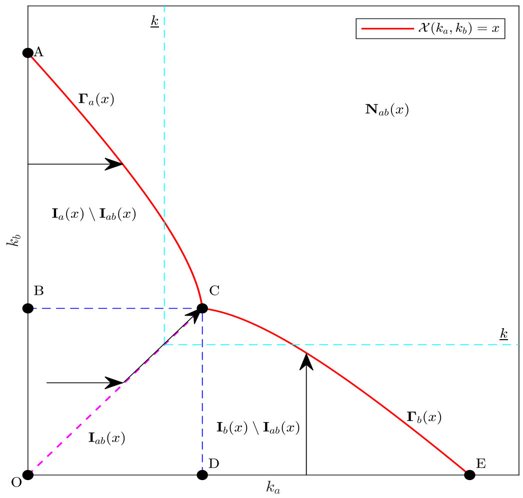

Figure 3: Equilibrium solution regions with a nonlinear trigger function.

and

$$\vartheta_-(k_a, k_b) := \begin{cases} \frac{\partial \mathcal{X}(k_a, k_b)}{\partial k_b}, & \text{if } k_a \leq k_b, \\ \frac{\partial \mathcal{X}(k_a, k_b)}{\partial k_a}, & \text{if } k_a \geq k_b. \end{cases} \tag{6.12}$$

The function ϑ_+ captures the negative effect of firm i's own production on the output price, similar to the role of $\rho\eta\theta_+$ in equation (5.6). The function ϑ_- captures the negative effect of the competitors' production on the output price, akin to the role of $\rho\eta\theta_-$ in equation (5.6). By the symmetry of $\mathcal{X}(k_a, k_b)$, we conclude the symmetry of $\vartheta_+(k_a, k_b)$ and $\vartheta_-(k_a, k_b)$.

Theorem 4 *Let $\varphi_i := (\boldsymbol{u}_i, \boldsymbol{v}_i)$ be firm i's closed-loop strategy given by (6.9)-(6.10), with $\mathcal{X}(k_a, k_b)$, $\mathcal{K}(x)$, $\vartheta_+(k_a, k_b)$ and $\vartheta_-(k_a, k_b)$ given by (6.5), (6.8), (6.11) and (6.12), respectively. Under Condition 6 with $K_{a0} \geq \underline{k}$, $K_{b0} \geq \underline{k}$, the closed-loop strategy pair $(\varphi_a, \varphi_b) \in \mathcal{S}$ is a Markov perfect equilibrium strategy.*

We can verify that the capital processes under the strategy given in Theorem 4 take the explicit form as in (5.28), but with $\mathcal{X}(k_a, k_b)$ and $\mathcal{K}(x)$ given by (6.5) and (6.8), respectively. Define firm i's investment region $\mathbf{I}_i(x)$ and no-action regions $\mathbf{N}_i(x)$ as in (5.16). Then, for the equilibrium strategy given in Theorem 4, firm i's equilibrium value $V^i(z)$ takes the same form as in Theorem 3, but with $\mathcal{X}(k_a, k_b)$ and $\mathcal{K}(x)$ given by (6.5) and (6.8), respectively, and with $H^i(k,k)$ given by:[26]

$$H^i(k,k) = -\int_k^\infty \Big(\frac{4\eta\widetilde{k} + rp + c}{r} - \frac{\mathcal{X}(\widetilde{k},\widetilde{k})}{r-\mu}\Big)\mathcal{X}(\widetilde{k},\widetilde{k})^{-\beta} d\widetilde{k}. \tag{6.13}$$

By selecting different functions ϕ and solving equation (6.4), we can derive various trigger functions and thus different sets of closed-loop equilibria. Next, we present two sets of trigger functions.

Proposition 3 *Recall the Markov perfect equilibrium strategy (φ_a, φ_b) as given in Theorem 4.*

(i) When $\phi(z_1, z_2) = z_1 - \lambda$ for a constant $\lambda > 0$, $\mathcal{G} = \lambda/k_a$ solves equation (6.4), and Condition 6 holds for any $\underline{k} > \sqrt{\lambda}$. Then $\mathcal{X}(k_a, k_b)$ given below yields a set of closed-loop equilibria:

$$\mathcal{X}(k_a, k_b) = \rho\Big(\eta k_a + \eta k_b + \lambda\frac{\eta}{\max\{k_a, k_b\}} + rp + c\Big). \tag{6.14}$$

(ii) When $\phi(z_1, z_2) = \lambda z_1 - z_2$ for a constant $\lambda > 0$, $\mathcal{G} = \frac{k_a + k_b}{\lambda k_a - 1}$ solves equation (6.4), and Condition 6 holds for any $\underline{k} > \frac{3}{\lambda}$. Then $\mathcal{X}(k_a, k_b)$ given below yields a set of closed-loop equilibria:

$$\mathcal{X}(k_a, k_b) = \rho\Big((\eta k_a + \eta k_b)\frac{\lambda \max\{k_a, k_b\}}{\lambda\max\{k_a, k_b\} - 1} + rp + c\Big). \tag{6.15}$$

7 Numerical/Quantitative Analysis

In this section, we present numerical results to compare the average q, marginal q, and the resulting equilibrium capital processes from different strategies, including the monopoly strategy, the open-loop equilibrium strategy $\varphi^{2,1}$, the perfect competition equilibrium strategy $\varphi^{1,1}$, the closed-loop equilibrium strategy with the linear trigger $\varphi^{\theta_+^*, \theta_-^*}$, the closed-loop equilibrium strategy with the nonlinear trigger φ^λ given in part (i) of Proposition 3, and the closed-loop equilibrium strategy with the nonlinear trigger φ^λ given in part (ii) of Proposition 3. The results associated with these strategies are depicted by a dotted line, a dashed line, a dash-dotted line, a solid line, a triangular line, and a squared line, respectively. The

[26] The value of $H^i(k_a, k_b)$ for $k_a \neq k_b$ also follows (5.50)-(5.51) with $H^i(k,k)$ is given by (6.13).

default parameter values are given as follows: $\mu = 0.015$, $\sigma = 0.15$, $r = 0.07$, $\eta = 0.1$, $c = 0.1$, $p = 1$, and thus $\beta = 2.333$.

7.1 Average q and Marginal q

For the monopolist, the average q is given by $\frac{V^{\mathrm{m}}(x,k)}{k}$, where $V^{\mathrm{m}}(x,k)$ is as given in (4.9) and (4.11). Given any strategy φ in the duopoly game, we define the associated average q as

$$V(x,k;\varphi) := \frac{V^a\big(x, \frac{k}{2}, \frac{k}{2}; \varphi\big) + V^b\big(x, \frac{k}{2}, \frac{k}{2}; \varphi\big)}{k},$$

where $k > 0$ represents the total capital.[27] Since symmetric equilibria are considered in the duopoly case, we only focus on firm a's marginal q defined as follows:

$$q^a(k_a, k_b; \varphi) := \frac{\partial V^a(x, k_a, k_b; \varphi)}{k_a}.$$

Comparison of Average q In Figure 4, we compare the average q under different strategies. In the left two panels, we plot the average q against x by fixing $k = 2.001$ (panel A) and $k = 20.001$ (panel C). In the right two panels, we plot the average q against k by fixing $x = 0.4$ (panel B) and $x = 3$ (panel D). For the nonlinear closed-loop equilibrium in part (i) of Proposition 3, we set $\lambda = 1$ in the top two panels and $\lambda = 100$ in the bottom two panels. For the nonlinear closed-loop equilibrium in part (ii) of Proposition 3, we set $\lambda = 3$ in the top two panels and $\lambda = 0.3$ in the bottom two panels.

We have the following observations. First, as expected, the monopoly case results in the highest average q among all strategies, while the perfect competition equilibrium yields the lowest average q. Second, consistent with Proposition 2, the average q for the open-loop equilibrium is higher than that for the closed-loop equilibrium with the linear trigger $\varphi^{\theta_+^*, \theta_-^*}$, though the difference is small. Third, while the average q for the closed-loop equilibrium

[27] We also considered the asymmetric case with average q given by $V(x,k,\alpha;\varphi) := \big(V^a(x, \alpha k, (1-\alpha)k; \varphi) + V^b(x, \alpha k, (1-\alpha)k; \varphi)\big)/k$, where $\alpha \in [0,1]$ represents the weight. However, we found the average q is insensitive to the value of $\alpha \in [0,1]$. In particular, when $k \le \mathcal{K}(x)$, $V(x,k,\alpha;\varphi)$ is independent of α. This is because $\frac{\partial V^a(x,\alpha k,(1-\alpha)k;\varphi)}{\partial k_a} = p$, $\frac{\partial V^a(x,\alpha k,(1-\alpha)k;\varphi)}{\partial k_b} = 0$, $\frac{\partial V^b(x,\alpha k,(1-\alpha)k;\varphi)}{\partial k_b} = p$, and $\frac{\partial V^b(x,\alpha k,(1-\alpha)k;\varphi)}{\partial k_a} = 0$, yielding $\frac{\partial V(x,k,\alpha;\varphi)}{\partial \alpha} = 0$.

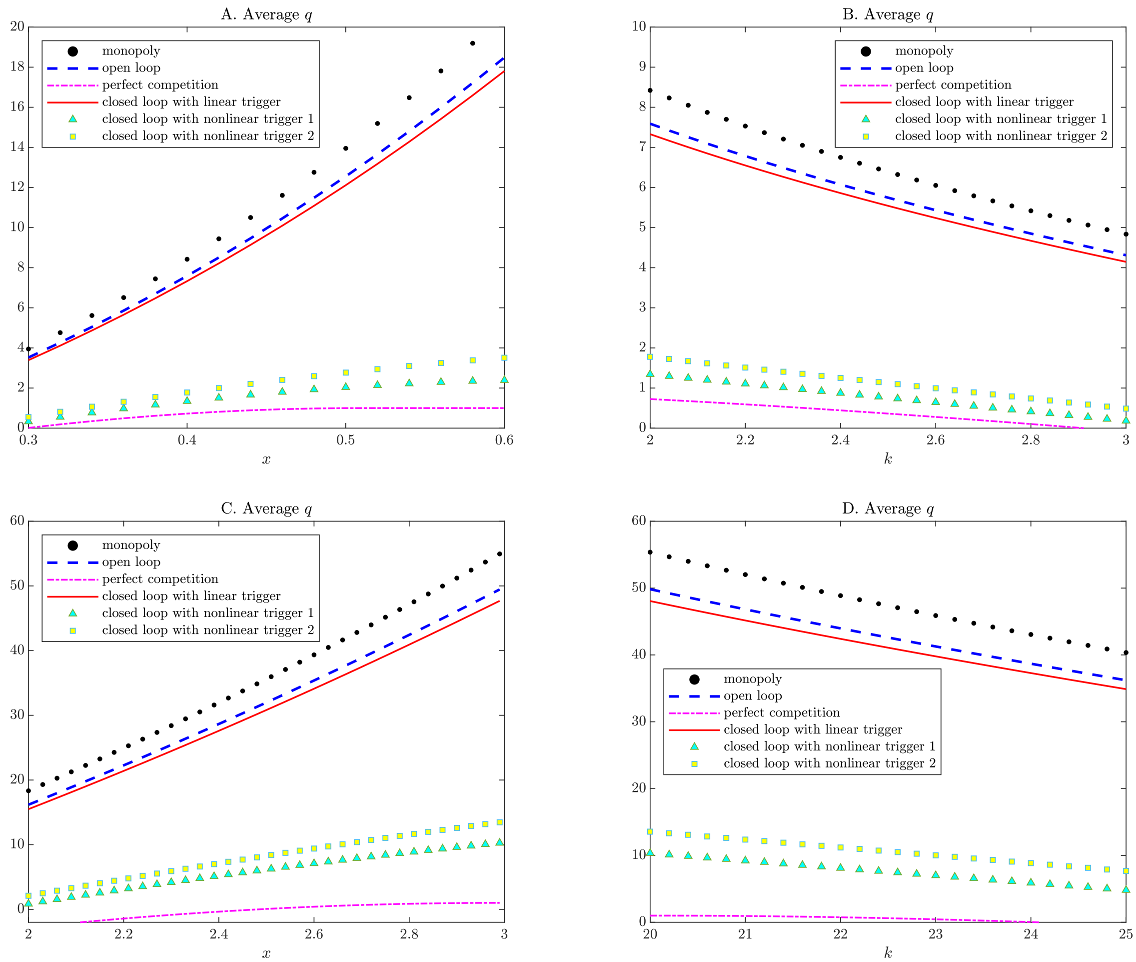

Figure 4: Average q under Different Strategies. In each panel, the marginal q under the monopoly strategy, the open-loop equilibrium strategy $\varphi^{2,1}$, the perfect competition equilibrium strategy $\varphi^{1,1}$, the closed-loop equilibrium strategy with the linear trigger $\varphi^{\theta_+^*,\theta_-^*}$, the closed-loop equilibrium strategy with the nonlinear trigger (6.14), and the closed-loop equilibrium strategy with the nonlinear trigger (6.15) are depicted by a dotted line, a dashed line, a dash-dotted line, a solid line, a triangular line, and a squared line, respectively. For the closed-loop equilibrium with the nonlinear trigger (6.14), we set $\lambda = 1$ in the top two panels and $\lambda = 100$ in the bottom two panels. For the closed-loop equilibrium with the nonlinear trigger (6.15), we set $\lambda = 3$ in the top two panels and $\lambda = 0.3$ in the bottom two panels. In the left two panels, we plot the average q against x by fixing $k = 2.001$ (panel A) and $k = 20.001$ (panel C). In the right two panels, we plot the average q against k by fixing $x = 0.4$ (panel B) and $x = 3$ (panel D). Default parameter values: $\mu = 0.015$, $\sigma = 0.15$, $r = 0.07$, $\eta = 0.1$, $c = 0.1$, $p = 1$, and thus $\beta = 2.333$.

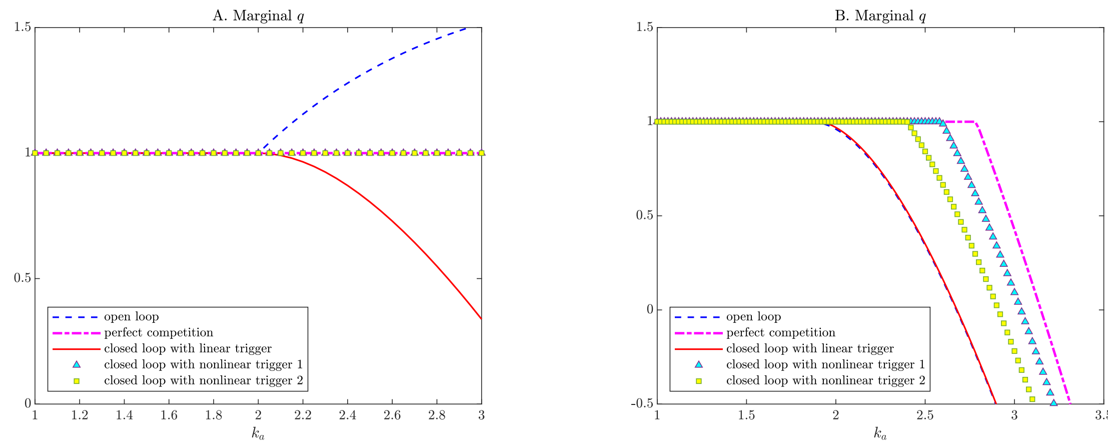

Figure 5: Firm a's Marginal q against k_a under Different Strategies. In each panel, the marginal q under the open-loop equilibrium strategy $\varphi^{2,1}$, the perfect competition equilibrium strategy $\varphi^{1,1}$, the closed-loop equilibrium strategy with the linear trigger $\varphi^{\theta_+^*,\theta_-^*}$, the closed-loop equilibrium strategy with the nonlinear trigger (6.14), and the closed-loop equilibrium strategy with the nonlinear trigger (6.15) are depicted by a dashed line, a dash-dotted line, a solid line, a triangular line, and a squared line, respectively. For the closed-loop equilibrium with the nonlinear trigger (6.14), we set $\lambda = 1$. For the closed-loop equilibrium with the nonlinear trigger (6.15), we set $\lambda = 3$. We set $k_b = 2$ and $x = \mathcal{X}(k_a, k_b)$ in panel A, and set $x = 1$, $k_a = k_b$ in panel B. Default parameter values: $\mu = 0.015$, $\sigma = 0.15$, $r = 0.07$, $\eta = 0.1$, $c = 0.1$, $p = 1$, and thus $\beta = 2.333$.

with the linear trigger $\varphi^{\theta_+^*,\theta_-^*}$ is significantly higher than that for the perfect competition equilibrium, the difference between the perfect competition equilibrium and the two closed-loop equilibria with nonlinear triggers is small. Intuitively, this is because a more restricted response requirement is imposed in the closed-loop equilibria with nonlinear triggers: when firm i deviates from the equilibrium in its investment region $\mathbf{I}_i$, the opponent's response must be at its own best interest, i.e., $\frac{\partial V^{-i}}{\partial k_{-i}} = p$. This requirement narrows the set of admissible strategies, thereby reducing firms' average q.

Comparison of Marginal q Next, we compare the marginal q under different strategies. In Figure 5, we plot the firm a's marginal q against its capital k_a. We set $k_b = 2$ and $x = \mathcal{X}(k_a, k_b)$ in panel A, and set $x = 1$, $k_a = k_b$ in panel B. For the closed-loop equilibrium with the nonlinear trigger (6.14), we set $\lambda = 1$. For the closed-loop equilibrium with the nonlinear trigger (6.15), we set $\lambda = 3$.

First, as shown in panel A of Figure 5 that the firm a's marginal q under the open-loop equilibrium can be strictly larger than the marginal cost p at the investment boundary $x = \mathcal{X}(k_a, k_b)$ when $k_a > k_b$, while the marginal q under the closed-loop equilibria is always bounded from above by the marginal cost p. Specifically, under the closed-loop equilibrium with the nonlinear triggers as in (6.14) and (6.15), the marginal q equals the marginal cost p at the whole investment boundary. In contrast, the marginal q under the closed-loop equilibrium with the linear trigger is strictly smaller than the marginal cost p at investment boundary $x = \mathcal{X}(k_a, k_b)$ with $k_a > k_b$. Second, as shown in panel B of Figure 5 that at the symmetric line $k_a = k_b$, the marginal q under the closed-loop equilibrium with the linear trigger and under the open-loop equilibrium are quite similar. In contrast, the marginal q under the closed-loop equilibria with nonlinear triggers approaches that under the perfect competition equilibrium as capital increases.

7.2 Capital Process

Now we compare the dynamics of the total capital process under different strategies. We focus on the symmetric case in which $K_{a0} = K_{b0} = K_0/2$, where K_{i0} is firm i's initial capital in the duopoly case, and K_0 represents the initial capital in the monopoly case. In the monopoly case, we have $K_t = \max\{K_0, \mathcal{K}^{\mathrm{m}}(M_t)\}$ for $t > 0$, where $\mathcal{K}^{\mathrm{m}}(x) = \frac{x/\rho-(rp+c)}{2\eta}$. In the duopoly case, we have $K_{it} = \max\{K_{i0}, \mathcal{K}(M_t)\}$, where $\mathcal{K}(\cdot)$ is given by (5.9) for the strategy $\varphi^{\theta_+,\theta_-}$ as defined in Proposition 2, $\mathcal{K}(\cdot)$ is given by (6.8) for the two set of equilibria strategy given Proposition 3.

In Figure 6, we plot the total capital process against M_t. In panel A, we set $\lambda = 1$ and $\lambda = 3$ for the closed-loop equilibria with the nonlinear triggers (6.14) and (6.15), respectively. In panel B, we set $\lambda = 100$ and $\lambda = 0.3$ for the closed-loop equilibria with the nonlinear triggers (6.14) and (6.15), respectively. Moreover, we set $K_{a0} = K_{b0} = 1.001$ in panel A and $K_{a0} = K_{b0} = 10.001$ in panel B.

First, we observe that the order of the total capital process is consistent with the order of average q illustrated in Figure 4: higher total capital corresponds to a lower average

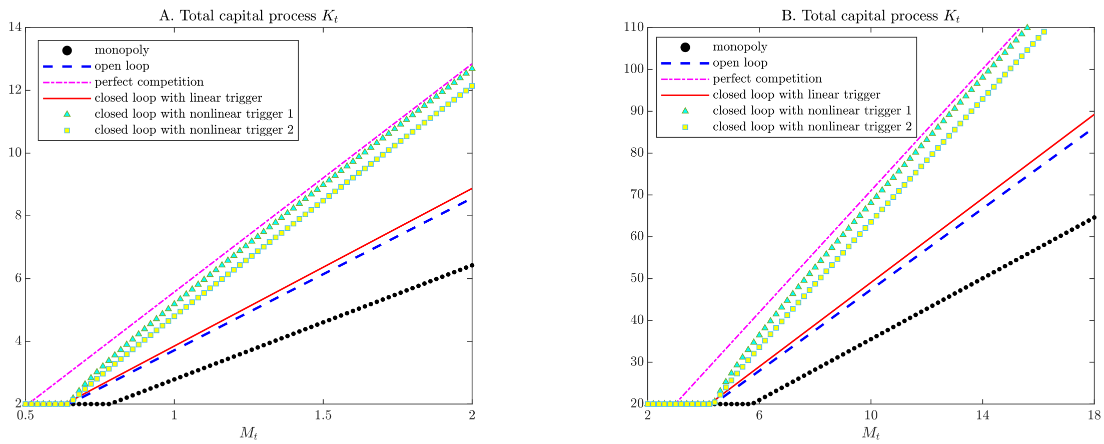

Figure 6: Total Capital Processes from Different Strategies. In each panel, the total capital for the monopoly strategy, the open-loop equilibrium strategy $\varphi^{2,1}$, the perfect competition equilibrium strategy $\varphi^{1,1}$, the closed-loop equilibrium strategy with the linear trigger $\varphi^{\theta_+^*,\theta_-^*}$, the closed-loop equilibrium strategy with the nonlinear trigger (6.14), and the closed-loop equilibrium strategy with the nonlinear trigger (6.15) are depicted by a dotted line, a dashed line, a dash-dotted line, a solid line, a triangular line, and a squared line, respectively. In panel A, we set $\lambda = 1$ and $\lambda = 3$ for the closed-loop equilibria with the nonlinear triggers (6.14) and (6.15), respectively. In panel B, we set $\lambda = 100$ and $\lambda = 0.3$ for the closed-loop equilibria with the nonlinear triggers (6.14) and (6.15), respectively. Default parameter values: $\mu = 0.015$, $\sigma = 0.15$, $r = 0.07$, $\eta = 0.1$, $c = 0.1$, $p = 1$, and thus $\beta = 2.333$. Moreover, we set $K_{a0} = K_{b0} = 1.001$ in panel A and $K_{a0} = K_{b0} = 10.001$ in panel B.

value. This result arises from the fact that higher total capital leads to reduced prices and consequently diminishes the potential profits that a firm can accrue. Second, we note that the growth rate of total capital under the nonlinear closed-loop equilibrium is even higher than that under the perfect competition. As M_t increases, total capital under the closed-loop equilibrium in part (i) of Proposition 3 gradually approaches, but consistently remains below, the total capital in the perfect competition equilibrium. This occurs because the trigger function given by (6.14) is larger than the trigger function of the perfect competition equilibrium, i.e., $\rho(\eta k_a + \eta k_b + rp + c)$, and the difference converges to zero as $\max\{k_a, k_b\}$ goes to infinity. Finally, as M_t increases, the difference between the total capital under the closed-loop equilibrium in part (ii) of Proposition 3 and the perfect competition equilibrium approaches a positive constant. This is because the trigger function given by (6.15) approaches $\rho\big(\eta k_a + \eta k_b + \eta\frac{2}{\lambda} + rp + c\big)$ when $k_a = k_b$ is sufficiently large.

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Appendices

A The Constant Elasticity Model

We extend our results to the constant elasticity model, where firm i's profit is given by

$$F_i(x, k_a, k_b) = x(k_a + k_b)^{-1/\gamma} k_i \tag{A.1}$$

with $\gamma > 1$ being a constant. When neither firm ever invests, i.e., when $dK_{at} = dK_{bt} \equiv 0$ for all t, firm i's value equals

$$\Psi^i(z) = \mathbb{E}^z \left[\int_0^\infty e^{-rt} F_i(X_t, k_a, k_b) dt \right] = x \frac{(k_a + k_b)^{-1/\gamma} k_i}{r - \mu} \tag{A.2}$$

for $z = (x, k_a, k_b)$. We denote $\psi^i(k_a, k_b) = \Psi^i(x, k_a, k_b)/x = \frac{(k_a+k_b)^{-1/\gamma} k_i}{r-\mu}$.

A.1 Monopoly Solution

First, we summarize the solution for the monopoly case in the constant elasticity model. The monopolist's profit flow function is given by $F(x, k) = xk^{1-1/\gamma}$, where k is the monopolist's capital stock.

The optimal strategy for the monopolist is to invest only when $X_t \geq \mathcal{X}^{\mathrm{m}}(K_t)$, where K_t is the monopolist's capital process. Here, $\mathcal{X}^{\mathrm{m}}(\cdot)$ is the trigger function for the monopolist and is given by

$$\mathcal{X}^{\mathrm{m}}(k) = \frac{\rho\gamma}{\gamma - 1} k^{\frac{1}{\gamma}}, \tag{A.3}$$

where ρ is a constant defined as

$$\rho = \frac{p\beta}{\beta - 1}(r - \mu), \tag{A.4}$$

with β being defined in (4.6). For convergence, we always require $\beta > \gamma$ in the constant elasticity model.

A.2 Duopoly Solution

Next, we examine the symmetric closed-loop equilibria in a duopoly. For illustration, we focus on nonlinear trigger functions and follow the analysis in Section 6.1.

Given a symmetric trigger function $\mathcal{X}(k_a, k_b)$ under certain regularity conditions,[28] we define firm i's investment region $\mathbf{I}_i(x)$ and no-action regions $\mathbf{N}_i(x)$ as in (5.16), with $\mathcal{K}(x)$ defined as in (6.8) and $\mathcal{X}(k_a, k_b)$ to be determined later. We can show that the corresponding value function for $x \leq \mathcal{X}(k_a, k_b)$ is given by (5.41), with $\Psi^i(z)$ defined by (A.2). By substituting (5.41) into (5.45) and (5.46), we obtain

$$\frac{\partial H^a(k_a, k_b)}{\partial k_a} = \Big(p - \mathcal{X}(k_a, k_b)\psi^a_{k_a}(k_a, k_b)\Big)\mathcal{X}(k_a, k_b)^{-\beta}, \quad k_a \leq k_b\,, \tag{A.5}$$

$$\frac{\partial H^a(k_a, k_b)}{\partial k_b} = -\psi^a_{k_b}(k_a, k_b)\mathcal{X}(k_a, k_b)^{1-\beta}, \quad k_a \geq k_b\,, \tag{A.6}$$

where $\psi^a_{k_a}(k_a, k_b) = (k_a + k_b)^{-\frac{1}{\gamma}-1}\frac{(1-\frac{1}{\gamma})k_a + k_b}{r-\mu}$ and $\psi^a_{k_b}(k_a, k_b) = (k_a + k_b)^{-\frac{1}{\gamma}-1}\frac{(-\frac{1}{\gamma})k_a}{r-\mu}$.

Similar to the analysis in Section 6.1, by imposing the constraint (6.1), we infer that (5.45) holds for any (k_a, k_b), as does (A.5) for any (k_a, k_b). Combining (A.5) and (A.6) and using $\frac{\partial^2 H^a(k_a,k_b)}{\partial k_a \partial k_b} = \frac{\partial^2 H^a(k_a,k_b)}{\partial k_b \partial k_a}$ for $k_a \geq k_b$,[29] we obtain

$$\frac{k_a}{k_a + k_b}\mathcal{X}(k_a, k_b)\frac{\partial \mathcal{X}(k_a, k_b)}{\partial k_a} + \left[\Big(\gamma - \frac{k_a}{k_a + k_b}\Big)\mathcal{X}(k_a, k_b) - \gamma\rho(k_a + k_b)^{\frac{1}{\gamma}}\right]\frac{\partial \mathcal{X}(k_a, k_b)}{\partial k_b} = 0 \tag{A.7}$$

for any $k_a \geq k_b$.

Lemma 8 in Appendix B indicates that the PDE (A.7) admits the following general solution locally:

$$\mathcal{X}(k_a, k_b) = \Big(\rho + \mathcal{G}(k_a, k_b)\Big)(k_a + k_b)^{\frac{1}{\gamma}}, \tag{A.8}$$

where $\mathcal{G}(k_a, k_b)$ for $k_a \geq k_b$ is a continuously differentiable function satisfying

$$\phi\Big(k_a\mathcal{G}, \frac{1}{k_a\mathcal{G}}\ln\frac{1}{(\mathcal{G} + \rho)(k_a + k_b)^{\frac{1}{\gamma}}}\Big) = 0. \tag{A.9}$$

Here, $\phi(z_1, z_2)$ is any continuously differentiable function with $\frac{\partial\phi(z_1,z_2)}{\partial z_1} \neq 0$ or $\frac{\partial\phi(z_1,z_2)}{\partial z_2} \neq 0$.

[28] We only consider the trigger function $\mathcal{X}(k_a, k_b)$ that is continuously differentiable on $\{(k_a, k_b) \in [\underline{k}, \infty) \times [\underline{k}, \infty) : k_a \neq k_b\}$ with $\mathcal{X}_{k_a}(k_a, k_b) > 0$ and $\mathcal{X}_{k_b}(k_a, k_b) > 0$, where $\underline{k} > 0$ is a constant. Without loss of generality, we assume that $\mathcal{X}(k_a, k_b)$ has a upper bound given by the monopolist's trigger $\rho\frac{\gamma}{\gamma-1}(k_a + k_b)^{\frac{1}{\gamma}}$ and a lower bound given by the trigger of perfect competition equilibrium $\rho(k_a + k_b)^{\frac{1}{\gamma}}$. These bounds guarantee the well-posedness of the firms' value functions under the trigger function $\mathcal{X}(k_a, k_b)$.

[29] See Footnote 24 for the precise definition of the second-order partial derivative of $H^a(k_a, k_b)$ at $k_a = k_b+$.

Using (A.8) and symmetry, we obtain the following trigger function:

$$\mathcal{X}(k_a,k_b) := \begin{cases} \Big(\rho+\mathcal{G}(k_b,k_a)\Big)(k_a+k_b)^{\frac{1}{\gamma}}, & \text{if } k_a \le k_b, \\ \Big(\rho+\mathcal{G}(k_a,k_b)\Big)(k_a+k_b)^{\frac{1}{\gamma}}, & \text{if } k_a \ge k_b. \end{cases} \tag{A.10}$$

Next, we impose a condition on the function $\mathcal{G}(k_a,k_b)$ so that the trigger function (A.10) yields a closed-loop equilibrium.

Condition 7 *There exists $\underline{k}>0$ such that the continuously differentiable function $\mathcal{G}(k_a,k_b)$ satisfies (A.9) for any $k_a \ge k_b \ge \underline{k}$. Moreover:*

(i) For any $k_a \ge k_b \ge \underline{k}$,

$$0 < \mathcal{G}(k_a,k_b) < \frac{\rho k_b}{\gamma(k_a+k_b)-k_b}; \tag{A.11}$$

(ii) $\big(\rho+\mathcal{G}(k_a,k_b)\big)(k_a+k_b)^{\frac{1}{\gamma}}$ is increasing in k_b for any $k_a \ge k_b \ge \underline{k}$.

The left inequality in (A.11) ensures that $\mathcal{X}(k_a,k_b)$ given in (A.10) is higher than the trigger function of the perfect competition equilibrium $\rho(k_a+k_b)^{\frac{1}{\gamma}}$. The right inequality in (A.11) ensures that $\mathcal{X}(k_a,k_b)$ given in (A.10) is lower than the trigger function of the open-loop equilibrium:

$$\mathcal{X}(k_a,k_b) < \rho\frac{1}{1-\frac{\min\{k_a,k_b\}}{\gamma(k_a+k_b)}}(k_a+k_b)^{\frac{1}{\gamma}}. \tag{A.12}$$

From parts (i) and (ii) of Condition 7, we can infer the strict monotonicity of $\mathcal{X}(k_a,k_b)$ in both k_a and k_b; see Footnote 30.

Next, we present symmetric closed-loop equilibria associated with the nonlinear trigger functions $\mathcal{X}(k_a,k_b)$ as given in equation (A.10). We define $\mathcal{K}(x)$, $\vartheta_+(k_a,k_b)$ and $\vartheta_-(k_a,k_b)$ as specified by (6.8), (6.11) and (6.12), respectively, with the trigger functions $\mathcal{X}(k_a,k_b)$ as given in equation (A.10).[30]

[30] By combining (A.7) with (A.10), we obtain $\frac{\vartheta_-(k_a,k_b)}{\vartheta_+(k_a,k_b)} = 1-\frac{\gamma(k_a+k_b)\mathcal{G}(k_a,k_b)}{k_a(\rho+\mathcal{G}(k_a,k_b))}$ for $k_a \ge k_b$. Since $\mathcal{G}(k_a,k_b) > 0$ as implied by (A.11), we have $\frac{\vartheta_-(k_a,k_b)}{\vartheta_+(k_a,k_b)} < 1$ for any $k_a \ge k_b \ge \underline{k}$. Further, since $\mathcal{G}(k_a,k_b) < \frac{\rho k_b}{\gamma(k_a+k_b)-k_b} \le \frac{\rho k_a}{\gamma(k_a+k_b)-k_a}$ as implied by (A.11), we have $\frac{\vartheta_-(k_a,k_b)}{\vartheta_+(k_a,k_b)} > 0$ for any $k_a \ge k_b \ge \underline{k}$. Hence, $\frac{\vartheta_-(k_a,k_b)}{\vartheta_+(k_a,k_b)} \in (0,1)$ for any $k_a \ge k_b \ge \underline{k}$. It follows that $\frac{\partial \mathcal{X}(k_a,k_b)}{\partial k_a} > 0$, $\frac{\partial \mathcal{X}(k_a,k_b)}{\partial k_b} > 0$ for any $k_a \ge k_b \ge \underline{k}$, or $\frac{\partial \mathcal{X}(k_a,k_b)}{\partial k_a} < 0$, $\frac{\partial \mathcal{X}(k_a,k_b)}{\partial k_b} < 0$ for any $k_a \ge k_b \ge \underline{k}$. Using Condition 7-(ii), we derive that $\frac{\partial \mathcal{X}(k_a,k_b)}{\partial k_b} \ge 0$ for any $k_a \ge k_b \ge \underline{k}$. Hence, by the symmetry of $\mathcal{X}(k_a,k_b)$, we have $\frac{\partial \mathcal{X}(k_a,k_b)}{\partial k_a} > 0$, $\frac{\partial \mathcal{X}(k_a,k_b)}{\partial k_b} > 0$ and $\frac{\vartheta_-(k_a,k_b)}{\vartheta_+(k_a,k_b)} \in (0,1)$ for any $k_a \ge \underline{k}$, $k_b \ge \underline{k}$.

Theorem 5 *Let $\varphi_i := (\boldsymbol{u}_i, \boldsymbol{v}_i)$ be firm i's closed-loop strategy given by (6.9)-(6.10), with $\mathcal{X}(k_a, k_b)$, $\mathcal{K}(x)$, $\vartheta_+(k_a, k_b)$, and $\vartheta_-(k_a, k_b)$ given by (A.10), (6.8), (6.11) and (6.12), respectively. Under Condition 7 with $K_{a0} \geq \underline{k}$ and $K_{b0} \geq \underline{k}$, the closed-loop strategy pair $(\varphi_a, \varphi_b) \in \mathcal{S}$ is a Markov perfect equilibrium strategy.*

We can verify that the capital processes under the strategy given in Theorem 5 take the explicit form as in (5.28), but with $\mathcal{X}(k_a, k_b)$ and $\mathcal{K}(x)$ given by (A.10) and (6.8), respectively. Define firm i's investment region $\mathbf{I}_i(x)$ and no-action regions $\mathbf{N}_i(x)$ as in (5.16). Then, for the equilibrium strategy given in Theorem 5, firm i's equilibrium value $V^i(z)$ takes the same form as in Theorem 3, but with $\Psi^i(z)$, $\mathcal{X}(k_a, k_b)$, $\mathcal{K}(x)$, and $H^i(k, k)$ as defined in (A.2), (A.10), (6.8) and (6.13), respectively.[31]

By choosing different function ϕ and solving equation (A.9), we can obtain various trigger functions and thus different sets of closed-loop equilibria. Next, we present two sets of trigger functions.

Proposition 4 *Recall the Markov perfect equilibrium strategy (φ_a, φ_b) as given in Theorem 5.*

(i) When $\phi(z_1, z_2) = z_1 - \lambda$ for a constant $\lambda > 0$, $\mathcal{G} = \frac{\lambda}{k_a}$ solves equation (A.9), and Condition 7 holds for any $\underline{k} > \frac{\lambda(2\gamma-1)}{\rho}$. Then $\mathcal{X}(k_a, k_b)$ given as follows yields a set of closed-loop equilibria:

$$\mathcal{X}(k_a, k_b) = \Big(\rho + \frac{\lambda}{\max\{k_a, k_b\}}\Big)(k_a + k_b)^{\frac{1}{\gamma}}. \tag{A.13}$$

(ii) When $\phi(z_1, z_2) = \frac{\ln(\lambda/z_1)}{z_1} + z_2$ for a constant $\lambda > 0$, $\mathcal{G} = \sqrt{(\frac{\rho}{2})^2 + \frac{\lambda}{k_a(k_a+k_b)^{\frac{1}{\gamma}}}} - \frac{\rho}{2}$ solves equation (A.9), and Condition 7 holds for any $\underline{k} > \frac{1}{2}\Big(\frac{(2\gamma-1)^2\lambda}{\gamma\rho^2}\Big)^{\frac{\gamma}{\gamma+1}}$. Then $\mathcal{X}(k_a, k_b)$ given as follows yields a set of closed-loop equilibria:

$$\mathcal{X}(k_a, k_b) = \rho(k_a + k_b)^{\frac{1}{\gamma}} + \frac{\lambda}{\sqrt{(\frac{\rho \max\{k_a,k_b\}}{2})^2 + \lambda \max\{k_a, k_b\}(k_a + k_b)^{\frac{-1}{\gamma}}} + \frac{\rho \max\{k_a,k_b\}}{2}}. \tag{A.14}$$

[31] The value of $H^i(k_a, k_b)$ for $k_a \neq k_b$ also follows (5.50)-(5.51) with $H^i(k, k)$ given by (6.13).

B Useful Lemmas

Lemma 1 *Assume the pair* (θ_+, θ_-) *satisfies Condition 5.*

(i) When $\theta_+ = \theta_-$*, we have* $\theta_+ = \theta_- = 1$.

(ii) When $\theta_+ > \theta_-$*, we have* $\theta_+ > 1 > \theta_-$*, and*

$$2 - \frac{\theta_+}{\theta_-} - \theta_+ < 2(\theta_- - 1). \tag{B.1}$$

(iii) The largest value of $\theta_+ + \theta_-$ *is reached only at* $\theta_+^* = \frac{3w^*-1}{1+w^*}$, $\theta_-^* = \frac{3-1/w^*}{1+w^*}$*, where* w^* *is the unique root of the function* $h(w)$*, as defined in* (5.13)*, within the region* $(1, \infty)$.

(iv) The smallest value of $\theta_+ + \theta_-$ *is reached only at* $\theta_+ = \theta_- = 1$.

(v) Moreover, the first inequality in (5.10) *is equivalent to*

$$2 + (\beta - 2)\frac{\theta_+}{\theta_-} - \beta\theta_+ \leq \frac{\beta\frac{\theta_+}{\theta_-}(2 - \frac{\theta_+}{\theta_-} - \theta_+)}{(1 + \frac{\theta_+}{\theta_-})^\beta - 1}. \tag{B.2}$$

Lemma 2 *Assume Condition 5 holds, and recall the expressions* (5.6)-(5.9), (5.16)-(5.18)*, and the function* $V^i(z)$ *defined in Theorem 3. Then, the functions* $V^a(z)$ *and* $V^b(z)$ *satisfy Conditions 1* and *2. In particular, for the case where* $\theta_+ = \theta_- = 1$*, we have*

$$V^i(x, k_a, k_b) = pk_i, \qquad x \geq \mathcal{X}(k_a, k_b), \quad (k_a, k_b) \in \mathbb{R}_+^2. \tag{B.3}$$

Lemma 3 *Assume Condition 5 holds, and recall the expressions* (5.6)-(5.9) *and* (5.16)-(5.18)*. Let the strategy* (φ_a, φ_b) *be as defined in Theorem 2, and let the function* $V^i(z)$ *be as given in Theorem 3. Then, the strategy* $(\varphi_a, \varphi_b) \in \mathcal{S}$ *and* (V^a, V^b) *satisfy Conditions 3-4 and Footnote 10.*

Lemma 4 *Let* $\mathcal{X}(k_a, k_b)$ *be as defined in* (5.6)*, where* $\theta_+ \geq \theta_- > 0$*. Let* $U^a(z)$ *and* $U^b(z)$ *be the functions given in Theorem 3. Then the second inequality in* (5.10) *is a necessary condition for* (5.52)*, and the first inequality in* (5.10) *is a necessary condition for* (5.53).

Lemma 5 *The PDE* (6.2) *admits a general solution as given by* (6.3)*. If the function* $\mathcal{G}(k_a, k_b)$ *satisfies Condition 6, then* $\mathcal{X}(k_a, k_b)$ *given by* (6.3) *satisfies* (6.2) *for any* $k_a \geq k_b \geq \underline{k}$.

Lemma 6 *Recall the expressions* (6.5), (6.8), *and* (5.16)-(5.18), *and let the function* $V^i(z)$ *be defined in Theorem 3 with* $\mathcal{X}(k_a, k_b)$, $\mathcal{K}(x)$, *and* $H^i(k, k)$ *as defined in* (6.5), (6.8) *and* (6.13), *respectively. Under Condition 6, the functions* $V^a(z)$ *and* $V^b(z)$ *satisfy Conditions 1 and 2 with* $\mathbb{Z} = \{(x, k_a, k_b) : x > 0, k_a \geq \underline{k}, k_b \geq \underline{k}\}$, *and*

$$\frac{\partial V^i(x, k_a, k_b)}{\partial k_i} = p, \qquad x \geq \mathcal{X}(k_a, k_b),\ k_a \geq \underline{k},\ k_b \geq \underline{k}. \tag{B.4}$$

Lemma 7 *Assume Condition 6 holds, and let the strategy* (φ_a, φ_b) *be as defined in Theorem 4, let the function* $V^i(z)$ *be as given in Theorem 3 but with* $\mathcal{X}(k_a, k_b)$, $\mathcal{K}(x)$, *and* $H^i(k, k)$ *as defined in* (6.5) *and* (6.8), *and* (6.13), *respectively. Then, the strategy* $(\varphi_a, \varphi_b) \in \mathcal{S}$ *and* (V^a, V^b) *satisfy Conditions 3-4 and Footnote 10 with* $\mathbb{Z} = \{(x, k_a, k_b) : x > 0, k_a \geq \underline{k}, k_b \geq \underline{k}\}$.

Lemma 8 *The PDE* (A.7) *admits a general solution as given by* (A.8)*. If the function* $\mathcal{G}(k_a, k_b)$ *satisfies Condition 7, then* $\mathcal{X}(k_a, k_b)$ *given by* (A.8) *satisfies* (A.7) *for any* $k_a \geq k_b \geq \underline{k}$.

Lemma 9 *Recall the expressions* (A.10), (6.8), *and* (5.16)-(5.18), *and let the function* $V^i(z)$ *be defined in Theorem 3 with* $\Psi^i(z)$, $\mathcal{X}(k_a, k_b)$, $\mathcal{K}(x)$, *and* $H^i(k, k)$ *as defined in* (A.2), (A.10), (6.8) *and* (6.13), *respectively. Then, the functions* $V^a(z)$ *and* $V^b(z)$ *satisfy Conditions 1 and 2 with* $\mathbb{Z} = \{(x, k_a, k_b) : x > 0, k_a \geq \underline{k}, k_b \geq \underline{k}\}$, *and* (B.4) *holds.*

Lemma 10 *Assume Condition 7 holds, let the strategy* (φ_a, φ_b) *be as defined in Theorem 5, and let the function* $V^i(z)$ *be as given in Theorem 3 but with* $\Psi^i(z)$, $\mathcal{X}(k_a, k_b)$, $\mathcal{K}(x)$, *and* $H^i(k, k)$ *as defined in* (A.2), (A.10), (6.8) *and* (6.13), *respectively. Then, the strategy* $(\varphi_a, \varphi_b) \in \mathcal{S}$ *and* (V^a, V^b) *satisfy Conditions 3-4 and Footnote 10 with* $\mathbb{Z} = \{(x, k_a, k_b) : x > 0, k_a \geq \underline{k}, k_b \geq \underline{k}\}$.

C Proofs

Proof of Theorem 1 We employ the following abbreviated notation: $V^a_{k_a}(x,k_a,k_b) := \frac{\partial V^a(x,k_a,k_b)}{\partial k_a}$ and $V^a_{k_b}(x,k_a,k_b) := \frac{\partial V^a(x,k_a,k_b)}{\partial k_b}$. Denote

$$\mathbb{W} := \{(y,k_1,k_2) \in \mathbb{Z} : (k_1,k_2) \in \mathbf{N}_{ab}(y)\},$$

where $\mathbb{Z} = \mathbb{X} \times [0,\infty) \times [0,\infty)$ denotes the state space of the game, and $\mathbf{N}_{ab}(\cdot)$ is given by (3.4). Because Condition 1 implies that $V^a_{k_a}(x,k_a,k_b)$ and $V^b_{k_b}(x,k_a,k_b)$ are continuous, the set $\mathbb{W}$ is an open subset of $\mathbb{Z}$. Let $\overline{\mathbb{W}}$ denote the closure of $\mathbb{W}$. In the following, we fix an initial state $z = (x,k_a,k_b) \in \mathbb{Z}$.

Step 1: We prove $V^a(x,k_a,k_b) \geq J^a(x,k_a,k_b;\widetilde{\varphi}_a,\varphi_b)$, where $(\widetilde{\varphi}_a,\varphi_b) \in \mathcal{S}$.

To simplify the notation, we denote $\widetilde{K}_a := K_a^{\widetilde{\varphi}_a,\varphi_b}$, $\widetilde{K}_b := K_b^{\widetilde{\varphi}_a,\varphi_b}$ as the firms' capital stocks under the off-equilibrium deviation $(\widetilde{\varphi}_a,\varphi_b)$. According to Condition 2, we have $V^a \in \mathcal{C}^{2,1,1}(\overline{\mathbb{W}})$. By Condition 4, it follows that $(X_s,\widetilde{K}_{as},\widetilde{K}_{bs}) \in \mathbb{W}$ for any $s > 0$, and $\widetilde{K}_{bs}$ is continuous in $s > 0$. Applying Itô formula to $e^{-rs}V^a(X_s,\widetilde{K}_{as},\widetilde{K}_{bs})$, $s \in (0,t]$ for $t > 0$ and taking expectation, we have

$$\begin{aligned} V^a(x,\widetilde{K}_{a0+},\widetilde{K}_{b0+}) =& \mathbb{E}^z\Big[e^{-rt}V^a(X_t,\widetilde{K}_{at},\widetilde{K}_{bt}) - \int_0^t e^{-rs}\mathcal{L}V^a(X_s,\widetilde{K}_{as},\widetilde{K}_{bs})ds \\ &- \int_0^t e^{-rs}[V^a_{k_a}(X_s,\widetilde{K}_{as},\widetilde{K}_{bs})d\widetilde{K}^C_{as} + V^a_{k_b}(X_s,\widetilde{K}_{as},\widetilde{K}_{bs})d\widetilde{K}_{bs}] \\ &- \sum_{s\in(0,t]} e^{-rs}[V^a(X_s,\widetilde{K}_{as},\widetilde{K}_{bs}) - V^a(X_s,\widetilde{K}_{as-},\widetilde{K}_{bs})]\Big], \end{aligned} \tag{C.1}$$

where $\widetilde{K}^C_a$ denotes the continuous part of $\widetilde{K}_a$, and the expectation of the local martingale term vanishes because of the regularity condition $\mathbb{E}^z\left[\int_0^t |e^{-rs}V^a_x(X_s,\widetilde{K}_{as},\widetilde{K}_{bs})\sigma(X_s)|^2 ds\right] < +\infty$, $\forall\, t \geq 0$ as in Footnote 10. According to (3.5) and (3.9), we have

$$-\mathcal{L}V^a(X_s,\widetilde{K}_{as},\widetilde{K}_{bs}) = F(X_s,\widetilde{K}_{as},\widetilde{K}_{bs}) \text{ for any } s > 0.$$

Substituting this into (C.1) and using $V^a_{k_a} \leq p$, $V^a_{k_b} \leq 0$ (Condition 1), we obtain

$$\begin{aligned} &V^a(x,\widetilde{K}_{a0+},\widetilde{K}_{b0+}) \\ \geq& \mathbb{E}^z\Big[e^{-rt}V^a(X_t,\widetilde{K}_{at},\widetilde{K}_{bt}) + \int_0^t e^{-rs}F(X_s,\widetilde{K}_{as},\widetilde{K}_{bs})ds - \int_{0+}^t e^{-rs}pd\widetilde{K}_{as}\Big]. \end{aligned} \tag{C.2}$$

By applying the condition $\lim_{n\to+\infty}\mathbb{E}^z[e^{-rt_n}V^a(X_{t_n},\widetilde{K}_{at_n},\widetilde{K}_{bt_n})]=0$ from Footnote 10, substituting t with t_n in (C.2), and letting n approach infinity, we derive the following result:

$$\begin{aligned}&V^a(x,\widetilde{K}_{a0+},\widetilde{K}_{b0+})\\ \geq&\lim_{n\to+\infty}\mathbb{E}^z\Big[\int_0^{t_n}e^{-rs}F(X_s,\widetilde{K}_{as},\widetilde{K}_{bs})ds-\int_{0+}^{t_n}e^{-rs}pd\widetilde{K}_{as}\Big]\\ =&\mathbb{E}^z\Big[\int_0^{+\infty}e^{-rs}F(X_s,\widetilde{K}_{as},\widetilde{K}_{bs})ds-\int_{0+}^{+\infty}e^{-rs}pd\widetilde{K}_{as}\Big],\end{aligned}\tag{C.3}$$

where the convergence of the first integral is due to the dominated convergence theorem and (2.14), and the second integral is well-defined by the monotone convergence theorem. Using $V^a_{k_a}\leq p$, $V^a_{k_b}\leq 0$ (Condition 1), $\widetilde{K}_{a0+}\geq k_a$, and $\widetilde{K}_{b0+}\geq k_b$, we derive

$$V^a(x,k_a,k_b)\geq V^a(x,k_a,\widetilde{K}_{b0+})\geq V^a(x,\widetilde{K}_{a0+},\widetilde{K}_{b0+})-p(\widetilde{K}_{a0+}-k_a).\tag{C.4}$$

Combining (C.4) with (C.3), we obtain

$$\begin{aligned}&V^a(x,k_a,k_b)\\ \geq&\mathbb{E}^z\Big[\int_0^{+\infty}e^{-rs}F(X_s,\widetilde{K}_{as},\widetilde{K}_{bs})ds-\int_0^{+\infty}e^{-rs}pd\widetilde{K}_{as}\Big]=J^a(x,k_a,k_b;\widetilde{\varphi}_a,\varphi_b).\end{aligned}$$

Step 2: We prove $V^a(x,k_a,k_b)=J^a(x,k_a,k_b;\varphi_a,\varphi_b)$.

Recall that (K_a,K_b) represent firms' capital stocks under the strategy (φ_a,φ_b). By replacing $\widetilde{K}_a$ and $\widetilde{K}_b$ in (C.1) with K_a and K_b, respectively, and noting that K_{as} and K_{bs} are continuous in $s>0$ (as stated in Condition 3), we obtain

$$\begin{aligned}&V^a(x,K_{a0+},K_{b0+})\\ =&\mathbb{E}^z\Big[e^{-rt}V^a(X_t,K_{at},K_{bt})-\int_0^t e^{-rs}\mathcal{L}V^a(X_s,K_{as},K_{bs})ds\\ &-\int_{0+}^t e^{-rs}[V^a_{k_a}(X_s,K_{as},K_{bs})dK_{as}+V^a_{k_b}(X_s,K_{as},K_{bs})dK_{bs}]\Big].\end{aligned}\tag{C.5}$$

According to (3.2), we have $V^a_{k_a}(x,k_1,k_2)=p$ for any $(k_1,k_2)\in\mathbf{I}_a(x)$, and $V^a_{k_b}(x,k_1,k_2)=0$

for any $(k_1, k_2) \in \mathbf{I}_b(x)$. Then, we have

$$\begin{aligned}
&\int_{0+}^{t} e^{-rs}[V^a_{k_a}(X_s, K_{as}, K_{bs})dK_{as} + V^a_{k_b}(X_s, K_{as}, K_{bs})dK_{bs}] \\
=&\int_{0+}^{t} e^{-rs}[V^a_{k_a}(X_s, K_{as}, K_{bs})\mathbf{1}_{(K_{as},K_{bs})\in\mathbf{I}_a(X_s)}dK_{as} + V^a_{k_b}(X_s, K_{as}, K_{bs})\mathbf{1}_{(K_{as},K_{bs})\in\mathbf{I}_b(X_s)}dK_{bs}] \\
=&\int_{0+}^{t} e^{-rs}pdK_{as},
\end{aligned} \tag{C.6}$$

where the first equality uses (3.8). Plugging $-\mathcal{L}V^a(X_s, K_{as}, K_{bs}) = F(X_s, K_{as}, K_{bs})$, $s > 0$ and (C.6) into (C.5), we have

$$\begin{aligned}
&V^a(x, K_{a0+}, K_{b0+}) \\
=&\mathbb{E}^z\Big[e^{-rt}V^a(X_t, K_{at}, K_{bt}) + \int_0^t e^{-rs}F(X_s, K_{as}, K_{bs})ds - \int_{0+}^t e^{-rs}pdK_{as}\Big].
\end{aligned} \tag{C.7}$$

Using the condition $\lim_{n\to+\infty}\mathbb{E}^z[e^{-rt_n}V^a(X_{t_n}, K_{at_n}, K_{bt_n})] = 0$ from Footnote 10, replacing t with t_n in (C.7), and letting n approach infinity, we derive the following result:

$$V^a(x, K_{a0+}, K_{b0+}) = \mathbb{E}^z\Big[\int_0^{+\infty} e^{-rs}F(X_s, K_{as}, K_{bs})ds - \int_{0+}^{+\infty} e^{-rs}pdK_{as}\Big], \tag{C.8}$$

where the convergence of the first integral is due to the dominated convergence theorem. Plugging (3.7) into (C.8), we have

$$\begin{aligned}
V^a(x, k_a, k_b) =&\mathbb{E}^z\Big[\int_0^{+\infty} e^{-rs}F(X_s, K_{as}, K_{bs})ds - \int_0^{+\infty} e^{-rs}pdK_{as}\Big] \\
=&J^a(x, k_a, k_b; \varphi_a, \varphi_b).
\end{aligned}$$

In sum, combining our analyses in Steps 1 and 2, we obtain $V^a(z) = J^a(z; \varphi_a, \varphi_b) \geq J^a(z; \widetilde{\varphi}_a, \varphi_b)$ for any $z = (x, k_a, k_b) \in \mathbb{Z}$ and $(\widetilde{\varphi}_a, \varphi_b) \in \mathcal{S}$. By symmetry, we also have $V^b(z) = J^b(z; \varphi_a, \varphi_b) \geq J^b(z; \varphi_a, \widetilde{\varphi}_b)$ for any $z = (x, k_a, k_b) \in \mathbb{Z}$ and $(\varphi_a, \widetilde{\varphi}_b) \in \mathcal{S}$. $\square$

Proof of Lemma 1 If $\theta_+ = \theta_-$, then by using $\overline{g}(1) = \underline{g}(1) = 1$, we observe that (5.10) simplifies to $2 \leq \theta_+ + \theta_- \leq 2$. This implies that $\theta_+ = \theta_- = 1$.

Denote

$$\widehat{g}(w) = \frac{[(1+w)^\beta - 1][(1-\frac{2}{\beta})w + \frac{2}{\beta}] + (w-2)w}{(1+w)^\beta - (1+w)}. \tag{C.9}$$

Because $\theta_+ + \theta_- = \theta_+(1 + \frac{\theta_-}{\theta_+})$ and $\underline{g}(w)/(1 + \frac{1}{w}) = \widehat{g}(w)$, the first inequality in Condition 5 is equivalent to

$$\widehat{g}\Big(\frac{\theta_+}{\theta_-}\Big) \leq \theta_+. \tag{C.10}$$

We can further verify that (C.10) is equivalent to (B.2).

Since $\beta > 2$, we have

$$\widehat{g}(w) > \frac{[(1+w)^\beta - 1][(1-\frac{2}{\beta}) + \frac{2}{\beta}] - w}{(1+w)^\beta - (1+w)} = 1, \quad w > 1. \tag{C.11}$$

Thus, for $\theta_+ > \theta_-$, we derive from (C.10) that $\theta_+ > 1$. The second inequality in (5.10) implies that

$$\theta_- \leq \frac{3-\theta_+}{1+1/\theta_+}. \tag{C.12}$$

We can show that $f(x) := \frac{3-x}{1+1/x} = \frac{x(3-x)}{x+1}$ is decreasing in $x \geq 1$ by noticing that $f'(x) = \frac{4-(x+1)^2}{(x+1)^2} < 0$ for $x > 1$. Since $\theta_+ > 1$, we have $\frac{3-\theta_+}{1+1/\theta_+} < 1$, which, together with (C.12), implies that $\theta_- < 1$.

Define $\phi(z) := ((1+z)^\beta - 1)((\beta-1)z - 2) + 2\beta z$. We observe that $\phi(w) > 0$, $\forall w > 0$, using the fact that $\phi''(w) = \beta(\beta-1)(1+w)^{\beta-2}(\beta+1)w > 0$, $\forall w > 0$, along with $\phi'(0) = 0$ and $\phi(0) = 0$. Recalling $\widehat{g}$ as defined in (C.9), we have

$$\widehat{g}(w) - \frac{4-w}{1+\frac{2}{w}} = \frac{1}{(1+w)^\beta - (1+w)} \frac{w-1}{w+2} \frac{2}{\beta} \phi(w) > 0, \quad \forall w > 1. \tag{C.13}$$

Setting $w = \frac{\theta_+}{\theta_-}$ in (C.13) and combining it with (C.10), we derive the following inequality

$$\theta_+ > \frac{4 - \frac{\theta_+}{\theta_-}}{1 + \frac{2\theta_-}{\theta_+}}, \tag{C.14}$$

when $\theta_+/\theta_- > 1$. It can then be seen that (C.14) is equivalent to (B.1).

Recall the function h as in (5.13). A straightforward calculation shows that

$$h(w)(w-1) = (\overline{g}(w) - \underline{g}(w))\beta w[(1+w)^{\beta-1} - 1].$$

It can be verified that $h''(w) < 0$ for $w \geq 1$, $h(1) > 0$, and $h(w) < 0$ for sufficiently large w. Therefore, the equation $h(w) = 0$ in $w > 1$ has a unique solution $w^* > 1$. For $w > w^*$, we have $\underline{g}(w) > \overline{g}(w)$. Thus, we have $\theta_+/\theta_- \leq w^*$. Since $\overline{g}(w) = 3 - 1/w$ is increasing for $w \geq 1$, it follows that $\overline{g}(\theta_+/\theta-) \leq 3 - 1/w^*$. From (5.10), we conclude that $\theta_+ + \theta_- \leq 3 - 1/w^*$, with equality if and only if $\theta_+/\theta- = w^*$, which implies $\theta_+ = \frac{3w^*-1}{1+w^*}$ and $\theta_- = \frac{3-1/w^*}{1+w^*}$.

We can show that $\underline{g}(w) > 2$ for $w > 1$. Hence, $\theta_+ + \theta_- > \underline{g}(\theta_+/\theta_-) > 2$ when $\theta_+ > \theta_- > 0$. Then, the smallest value of $\theta_+ + \theta_-$ is achieved only when $\theta_+ = \theta_- = 1$.

□

Proof of Lemma 2 We will prove the result for $V^a(z)$ only, as the case for $V^b(z)$ can be handled in a similar manner. To simplify notation, for any function $f \in \{V^a, U^a, H^a\}$, we denote the partial derivatives as follows: $f_{k_a}(x, k_a, k_b) = \frac{\partial f(x,k_a,k_b)}{\partial k_a}$, $f_{k_b}(x, k_a, k_b) = \frac{\partial f(x,k_a,k_b)}{\partial k_b}$, $f_{k_a k_a}(x, k_a, k_b) = \frac{\partial^2 f(x,k_a,k_b)}{\partial k_a^2}$, $f_{k_a k_b}(x, k_a, k_b) = \frac{\partial^2 f(x,k_a,k_b)}{\partial k_a \partial k_b}$, $f_{k_b k_a}(x, k_a, k_b) = \frac{\partial^2 f(x,k_a,k_b)}{\partial k_b \partial k_a}$, and $f_{k_b k_b}(x, k_a, k_b) = \frac{\partial^2 f(x,k_a,k_b)}{\partial k_b^2}$, where $H^a(k_a, k_b)$ is a function describing firm a's option value, as given by (5.49)-(5.51). Additionally, we will use the following abbreviated notation:

$$\mathcal{X}^a_{k_a} := \frac{\partial \mathcal{X}^a(k_a, k_b)}{\partial k_a} = \rho\theta_+\eta, \qquad \mathcal{X}^a_{k_b} := \frac{\partial \mathcal{X}^a(k_a, k_b)}{\partial k_b} = \rho\theta_-\eta,$$
$$\mathcal{X}^b_{k_a} := \frac{\partial \mathcal{X}^b(k_a, k_b)}{\partial k_a} = \rho\theta_-\eta, \qquad \mathcal{X}^b_{k_b} := \frac{\partial \mathcal{X}^b(k_a, k_b)}{\partial k_b} = \rho\theta_+\eta.$$

Given that $\beta > 2$, $r > 0$, and $c \geq 0$, it follows that $\mathcal{X}(k_a, k_b) > 0$ for any $(k_a, k_b) \in \mathbb{R}^2_+$. According to (5.49)-(5.51), we have $H^a \in \mathcal{C}^{1,1}(\mathbb{R}^2_+) \bigcap \mathcal{C}^{\infty,\infty}(\{(k_a, k_b) \in \mathbb{R}^2_+ : k_b \neq k_a\})$. Therefore, for $k_a \neq k_b$, we have $H^a_{k_a k_b}(k_a, k_b) = H^a_{k_b k_a}(k_a, k_b)$. By (5.41), we see that U^a belongs to $\mathcal{C}^{2,1,1}(\{(x, k_a, k_b) \in \mathbb{R}^3_+ : (k_a, k_b) \in \overline{\mathbf{N}}_{ab}(x)\})$ and satisfies (3.5), where $\mathbf{N}_{ab}(x)$ is given by (5.18). Using (5.41), (5.47), and (5.48), we obtain that

$$\frac{\partial U^a(x, k_a, k_b)}{\partial k_a}|_{x=\mathcal{X}(k_a,k_b)-} = p, \qquad k_b \geq k_a \geq 0, \tag{C.15}$$
$$\frac{\partial U^a(x, k_a, k_b)}{\partial k_b}|_{x=\mathcal{X}(k_b,k_a)-} = 0, \qquad k_a \geq k_b \geq 0. \tag{C.16}$$

By combining the above with the definition of $V^a(x, k_a, k_b)$ given in Theorem 3, we observe that $V^a_{k_a}(x, k_a, k_b)$ and $V^a_{k_b}(x, k_a, k_b)$ are continuous in $(x, k_a, k_b) \in \mathbb{R}^3_+$, particularly across the boundary $x = \mathcal{X}(k_a, k_b)$, and

$$V^a_{k_a}(x, k_a, k_b) = p, \qquad (k_a, k_b) \in \mathbf{I}_a(x), \tag{C.17}$$
$$V^a_{k_b}(x, k_a, k_b) = 0, \qquad (k_a, k_b) \in \mathbf{I}_b(x), \tag{C.18}$$

where $\mathbf{I}_i(x)$ is defined by (5.16). Consequently, $V^a \in \mathcal{C}^{0,1,1}(\mathbb{R}^3_+)$ and Condition 2 is satisfied. It remains to prove that the two inequalities in Condition 1 hold.

We will complete our proof in five steps. In the first four steps, we focus on the case

$\theta_+ > \theta_-$. Then, Lemma 1 implies that $\theta_+ > 1 > \theta_-$.

Step 1: We prove that

$$U^a_{k_b}(\mathcal{X}(k_a,k_b),k_a,k_b) < 0, \quad k_a < k_b,\ k_a \geq 0. \tag{C.19}$$

Using (5.41) and $\mathcal{X}(k_a,k_b) = \mathcal{X}^a(k_a,k_b)$ for $k_a \leq k_b$, $k_a \geq 0$, we have

$$U^a_{k_b}(\mathcal{X}^a(k_a,k_b),k_a,k_b) = -\frac{\eta k_a}{r} + H^a_{k_b}(k_a,k_b)\mathcal{X}^a(k_a,k_b)^\beta. \tag{C.20}$$

Taking the partial derivative with respect to k_b in (5.47) and using the equality $H^a_{k_a k_b}(k_a,k_b) = H^a_{k_b k_a}(k_a,k_b)$ for $k_a < k_b, k_a \geq 0$, we obtain

$$\begin{aligned} H^a_{k_a k_b}(k_a,k_b) - \frac{\mathcal{X}^a_{k_b}}{\mathcal{X}^a_{k_a}} H^a_{k_a k_a}(k_a,k_b) &= \frac{\eta}{r}\Big(1-2\frac{\mathcal{X}^a_{k_b}}{\mathcal{X}^a_{k_a}}\Big)\mathcal{X}^a(k_a,k_b)^{-\beta} \\ =&\frac{\eta}{r}\Big(1-2\frac{\mathcal{X}^a_{k_b}}{\mathcal{X}^a_{k_a}}\Big)\frac{1}{(1-\beta)\mathcal{X}^a_{k_a}}\frac{\partial(\mathcal{X}^a(k_a,k_b)^{1-\beta})}{\partial k_a}. \end{aligned}$$

Taking the integral with respect to k_a in both sides of the above equation and using the relation $\frac{\mathcal{X}^a_{k_b}}{\mathcal{X}^a_{k_a}} = \frac{\theta_-}{\theta_+}$, we obtain

$$\int H^a_{k_a k_b}(k_a,k_b)dk_a = \frac{\theta_-}{\theta_+}H^a_{k_a}(k_a,k_b) + \frac{\eta}{r}\Big(1-2\frac{\theta_-}{\theta_+}\Big)\frac{\mathcal{X}^a(k_a,k_b)^{1-\beta}}{(1-\beta)\mathcal{X}^a_{k_a}}. \tag{C.21}$$

For $k_a \leq k_b, k_a \geq 0$, we derive from (C.21) that

$$\begin{aligned} H^a_{k_b}(k_a,k_b) =& H^a_{k_b}(k_b,k_b) - \frac{\theta_-}{\theta_+}[H^a_{k_a}(k_b,k_b) - H^a_{k_a}(k_a,k_b)] \\ &- \frac{\eta}{r}\Big(1-2\frac{\theta_-}{\theta_+}\Big)\frac{\mathcal{X}^a(k_b,k_b)^{1-\beta} - \mathcal{X}^a(k_a,k_b)^{1-\beta}}{(1-\beta)\mathcal{X}^a_{k_a}}. \end{aligned} \tag{C.22}$$

Denote $\varpi(k_a,k_b) = \frac{2\eta k_a + \eta k_b + pr + c}{r} - \frac{\mathcal{X}^a(k_a,k_b)}{r-\mu}$. Substituting (5.47) into (C.22), we have

$$\begin{aligned} H^a_{k_b}(k_a,k_b)\mathcal{X}^a(k_b,k_b)^\beta =& \frac{\eta k_b}{r} - \frac{\theta_-}{\theta_+}\left[\varpi(k_b,k_b) - \varpi(k_a,k_b)\Big(\frac{\mathcal{X}^a(k_b,k_b)}{\mathcal{X}^a(k_a,k_b)}\Big)^\beta\right] \\ &- \frac{\eta}{r}\Big(1-2\frac{\theta_-}{\theta_+}\Big)\frac{\mathcal{X}^a(k_b,k_b) - \mathcal{X}^a(k_a,k_b)\Big(\frac{\mathcal{X}^a(k_b,k_b)}{\mathcal{X}^a(k_a,k_b)}\Big)^\beta}{(1-\beta)\mathcal{X}^a_{k_a}} \\ =& \Big(\frac{\mathcal{X}^a(k_b,k_b)}{\mathcal{X}^a(k_a,k_b)}\Big)^\beta\left[\frac{\theta_-}{\theta_+}\varpi(k_a,k_b) + \frac{\eta}{r}(1-2\frac{\theta_-}{\theta_+})\frac{\mathcal{X}^a(k_a,k_b)}{(1-\beta)\mathcal{X}^a_{k_a}}\right] \\ &- \left[\frac{\theta_-}{\theta_+}\varpi(k_b,k_b) + \frac{\eta}{r}(1-2\frac{\theta_-}{\theta_+})\frac{\mathcal{X}^a(k_b,k_b)}{(1-\beta)\mathcal{X}^a_{k_a}} - \frac{\eta k_b}{r}\right]. \end{aligned} \tag{C.23}$$

Plugging (C.23) into (C.20), for $k_a \le k_b$, $k_a \ge 0$, we have

$$U^a_{k_b}(\mathcal{X}^a(k_a,k_b),k_a,k_b) = \left[\frac{\theta_-}{\theta_+}\varpi(k_a,k_b) + \frac{\eta}{r}\Big(1-2\frac{\theta_-}{\theta_+}\Big)\frac{\mathcal{X}^a(k_a,k_b)}{(1-\beta)\mathcal{X}^a_{k_a}} - \frac{\eta k_a}{r}\right] - \Big(\frac{\mathcal{X}^a(k_a,k_b)}{\mathcal{X}^a(k_b,k_b)}\Big)^{\beta}\left[\frac{\theta_-}{\theta_+}\varpi(k_b,k_b) + \frac{\eta}{r}\Big(1-2\frac{\theta_-}{\theta_+}\Big)\frac{\mathcal{X}^a(k_b,k_b)}{(1-\beta)\mathcal{X}^a_{k_a}} - \frac{\eta k_b}{r}\right]. \tag{C.24}$$

Denote

$$\mathcal{G}(k_a,k_b) = (2-\frac{\theta_+}{\theta_-})\eta k_a + \eta k_b + pr + c - (\theta_+\eta k_a + \theta_-\eta k_b + pr + c)\frac{1}{\beta-1}(\beta + \frac{1}{\theta_-} - \frac{2}{\theta_+}). \tag{C.25}$$

A straightforward calculation shows that $\frac{\theta_-}{\theta_+}\varpi(k_a,k_b) + \frac{\eta}{r}(1-2\frac{\theta_-}{\theta_+})\frac{\mathcal{X}^a(k_a,k_b)}{(1-\beta)\mathcal{X}^a_{k_a}} - \frac{\eta k_a}{r} = \mathcal{G}(k_a,k_b)\frac{\theta_-}{r\theta_+}$. Then we derive from (C.24) that for $k_a \le k_b$, $k_a \ge 0$,

$$U^a_{k_b}(\mathcal{X}^a(k_a,k_b),k_a,k_b) = \frac{\theta_-}{r\theta_+}\left[\mathcal{G}(k_a,k_b) - \Big(\frac{\mathcal{X}^a(k_a,k_b)}{\mathcal{X}^a(k_b,k_b)}\Big)^{\beta}\mathcal{G}(k_b,k_b)\right]. \tag{C.26}$$

To prove (C.19), we first prove that

$$U^a_{k_b}(\mathcal{X}^a(0,k_b),0,k_b) < 0, \quad k_b \ge 0. \tag{C.27}$$

To achieve this, we define

$$\mathcal{H}(k) := \Big(\frac{\mathcal{X}^a(k,k)}{\mathcal{X}^a(0,k)}\Big)^{\beta}\mathcal{G}(0,k) - \mathcal{G}(k,k), \tag{C.28}$$

and conclude that (C.27) is equivalent to $\mathcal{H}(k_b) < 0$ for any $k_b \ge 0$, using (C.26) and $\mathcal{X}^a(k_a,k_b) > 0$. Now, we prove that $\mathcal{H}(k) < 0$, $\forall\ k > 0$. Let $\mathcal{Y}(k) := \frac{\mathcal{X}^a(k,k)}{\mathcal{X}^a(0,k)}$. Then $\mathcal{H}(k) = \mathcal{Y}(k)^\beta\mathcal{G}(0,k) - \mathcal{G}(k,k)$. Given that $\theta_+/\theta_- > 1$, $\theta_+ \ge 1$, and (B.2), we have

$$\mathcal{G}_{k_a} := \frac{\partial\mathcal{G}(k_a,k_b)}{\partial k_a} = \Big(2-\frac{\theta_+}{\theta_-}-\theta_+\Big)\eta\frac{\beta}{\beta-1} < 0, \tag{C.29}$$
$$\mathcal{G}_{k_b} := \frac{\partial\mathcal{G}(k_a,k_b)}{\partial k_b} = (2+(\beta-2)\frac{\theta_+}{\theta_-} - \beta\theta_+)\eta\frac{1}{\beta-1}\frac{\theta_-}{\theta_+} < 0,$$
$$\mathcal{G}(0,0) = \frac{pr+c}{(\beta-1)\theta_+}\Big(2-\frac{\theta_+}{\theta_-}-\theta_+\Big) < 0.$$

It follows that

$$\mathcal{G}(k_a,k_b) < 0, \quad (k_a,k_b) \in \mathbb{R}^2_+. \tag{C.30}$$

Because $\mathcal{Y}(k) = 1 + \frac{\theta_+}{\theta_-} - \frac{\theta_+}{\theta_-}\frac{\mathcal{X}^a(0,0)/\mathcal{X}^a_{k_b}}{k+\mathcal{X}^a(0,0)/\mathcal{X}^a_{k_b}}$, $\mathcal{Y}(0) = 1$, and $\mathcal{Y}'(0) = \frac{\theta_+}{\theta_-}\frac{1}{\mathcal{X}^a(0,0)/\mathcal{X}^a_{k_b}}$, we infer

$$\mathcal{H}'(0) = \beta\mathcal{Y}'(0)\mathcal{G}(0,0) - \mathcal{G}_{k_a} = \beta\theta_+\frac{\eta}{pr+c}\mathcal{G}(0,0) - \mathcal{G}_{k_a} = 0 \tag{C.31}$$

and

$$\begin{aligned}\mathcal{H}''(k) =&\beta\mathcal{Y}(k)^{\beta-2}\mathcal{Y}'(k)^2\left[\Big(\beta-1+\frac{\mathcal{Y}(k)\mathcal{Y}''(k)}{\mathcal{Y}'(k)^2}\Big)\mathcal{G}(0,k)+2\frac{\mathcal{Y}(k)}{\mathcal{Y}'(k)}\mathcal{G}_{k_b}\right]\\ =&\beta\mathcal{Y}(k)^{\beta-2}\mathcal{Y}'(k)^2\left[(\beta-1)\mathcal{G}(0,k)+\frac{\mathcal{Y}(k)\mathcal{Y}''(k)}{\mathcal{Y}'(k)^2}\mathcal{G}\Big(0,-\frac{\mathcal{X}^a(0,0)}{\mathcal{X}^a_{k_b}}\Big)\right],\end{aligned} \tag{C.32}$$

where the second equality is because $\mathcal{G}(0,k)=\mathcal{G}\big(0,-\frac{\mathcal{X}^a(0,0)}{\mathcal{X}^a_{k_b}}\big)+(k+\frac{\mathcal{X}^a(0,0)}{\mathcal{X}^a_{k_b}})\mathcal{G}_{k_b}$ and $\frac{\mathcal{Y}(k)\mathcal{Y}''(k)}{\mathcal{Y}'(k)^2}(k+\frac{\mathcal{X}^a(0,0)}{\mathcal{X}^a_{k_b}})+2\frac{\mathcal{Y}(k)}{\mathcal{Y}'(k)}=0$. A direct calculation yields that $\frac{\mathcal{Y}(0)\mathcal{Y}''(0)}{\mathcal{Y}'(0)^2}=-2\frac{\theta_-}{\theta_+}$, and

$$\begin{aligned}&(\beta-1)\mathcal{G}(0,0)+\frac{\mathcal{Y}(0)\mathcal{Y}''(0)}{\mathcal{Y}'(0)^2}\mathcal{G}\Big(0,-\frac{\mathcal{X}^a(0,0)}{\mathcal{X}^a_{k_b}}\Big)\\ =&\Big(\beta-1-2\frac{\theta_-}{\theta_+}\Big)\mathcal{G}(0,0)+2\frac{\theta_-}{\theta_+}\frac{\mathcal{X}^a(0,0)}{\mathcal{X}^a_{k_b}}\mathcal{G}_{k_b}\\ =&\frac{pr+c}{\theta_+}\Big[2-\frac{\theta_+}{\theta_-}-\theta_++2(1-\theta_-)\Big]<0,\end{aligned} \tag{C.33}$$

where the inequality utilizes (B.1). By combining (C.32), (C.33), $\mathcal{Y}(0)=1$, and $\mathcal{Y}'(0)>0$, we can conclude that $\mathcal{H}''(0)<0$.

A direct calculation shows that $\frac{\mathcal{Y}(k)\mathcal{Y}''(k)}{\mathcal{Y}'(k)^2}$ and $\mathcal{G}(0,k)$ are affine functions of k. Consequently, the expression in the bracket in (C.32) is linear in k. Then we conclude from $\mathcal{H}''(0)<0$, (C.32), $\mathcal{Y}'(k)^2>0$, and $\mathcal{Y}(k)>0$, $\forall k\geq 0$ that either $\mathcal{H}''(k)<0$, $\forall\ k\geq 0$, or there exists $\tilde{k}>0$ such that $\mathcal{H}''(k)<0$, $\forall k\in[0,\tilde{k})$ and $\mathcal{H}''(k)>0$, $\forall\ k>\tilde{k}$. Recall that we have shown $\mathcal{H}'(0)=0$ in (C.31). Then $\mathcal{H}'(k)<0$, $\forall\ k>0$, or there exists $k_0\geq 0$ such that $\mathcal{H}'(k)<0$, $\forall k\in(0,k_0)$ and $\mathcal{H}'(k)>0$, $\forall\ k>k_0$. Thus, $\mathcal{H}(k)<\max(\mathcal{H}(0),\lim_{\widehat{k}\to+\infty}\mathcal{H}(\widehat{k}))$ for any $k>0$. Note that $\lim_{k\to+\infty}\mathcal{Y}(k)=1+\frac{\theta_+}{\theta_-}$ and

$$\begin{aligned}\lim_{k\to+\infty}\frac{\mathcal{H}(k)}{k}&=\Big(1+\frac{\theta_+}{\theta_-}\Big)^\beta\mathcal{G}_{k_b}-(\mathcal{G}_{k_a}+\mathcal{G}_{k_b})\\ =&\frac{\theta_-}{\theta_+}\frac{\eta}{\beta-1}\left[\Big[\big(1+\frac{\theta_+}{\theta_-}\big)^\beta-1\Big][2+(\beta-2)\frac{\theta_+}{\theta_-}-\beta\theta_+]-\beta\frac{\theta_+}{\theta_-}\Big(2-\frac{\theta_+}{\theta_-}-\theta_+\Big)\right]\leq 0,\end{aligned} \tag{C.34}$$

where the inequality utilizes (B.2) and the condition $\theta_+\geq\theta_->0$. Since $\mathcal{H}(0)=0$, it follows that $\mathcal{H}(k)<0$, $\forall\ k>0$. Therefore, we have completed the proof of (C.27).

Given that (C.29) implies that $\frac{\partial^2}{\partial k_a^2}\mathcal{G}(k_a,k_b)=0$, we derive from (C.26) that

$$\frac{\partial^2}{\partial k_a^2}U^a_{k_b}(\mathcal{X}^a(k_a,k_b),k_a,k_b)=-\frac{\theta_-}{r\theta_+}\beta(\beta-1)\Big(\frac{\mathcal{X}^a(k_a,k_b)}{\mathcal{X}^a(k_b,k_b)}\Big)^\beta\Big(\frac{\mathcal{X}^a_{k_a}}{\mathcal{X}^a(k_a,k_b)}\Big)^2\mathcal{G}(k_b,k_b)>0,$$

for any $k_a\in[0,k_b]$, where the inequality uses (C.30). The convexity of $U^a_{k_b}(\mathcal{X}^a(k_a,k_b),k_a,k_b)$

with respect to k_a implies that

$$\begin{aligned}&U^a_{k_b}(\mathcal{X}^a(k_a,k_b),k_a,k_b) < U^a_{k_b}(\mathcal{X}^a(0,k_b),0,k_b) \vee U^a_{k_b}(\mathcal{X}^a(k_b,k_b),k_b,k_b)\\ =&U^a_{k_b}(\mathcal{X}^a(0,k_b),0,k_b) \vee 0 = 0, \qquad \forall k_a \in (0,k_b),\end{aligned} \tag{C.35}$$

where the first equality uses $U^a_{k_b}(\mathcal{X}^a(k_b,k_b),k_b,k_b) = 0$ and the second equality uses (C.27). By combining (C.27) and (C.35), we can see that (C.19) holds.

Step 2: We prove that

$$V^a_{k_b}(x,k_a,k_b) < 0, \quad (k_a,k_b) \in \mathbf{N}_b(x) \setminus \{(0,0)\},\ x > 0. \tag{C.36}$$

Recall that for any $x \in (0, \mathcal{X}(k_a,k_b)]$ and $(k_a,k_b) \in \mathbb{R}^2_+$, $V^a(x,k_a,k_b)$ is given by (5.41). Consequently, $V^a_{k_b}(x,k_a,k_b) = -\frac{\eta k_a}{r} + H^a_{k_b}(k_a,k_b)x^\beta$, which is linear in $x^\beta > 0$.

For $0 \le k_a < k_b$, since $V^a_{k_b}(\mathcal{X}(k_a,k_b),k_a,k_b) < 0$ by (C.19), and $\lim_{x\to 0+} V^a_{k_b}(x,k_a,k_b) = -\frac{\eta k_a}{r} \le 0$, it follows that $V^a_{k_b}(x,k_a,k_b) < 0$ for any $x \in (0,\mathcal{X}(k_a,k_b))$. For $k_a \ge k_b \ge 0$ with $k_a > 0$, since $V^a_{k_b}(\mathcal{X}(k_a,k_b),k_a,k_b) = 0$ by (5.46), and $\lim_{x\to 0+} V^a_{k_b}(x,k_a,k_b) = -\frac{\eta k_a}{r} < 0$, it follows that $V^a_{k_b}(x,k_a,k_b) < 0$ for any $x \in (0,\mathcal{X}(k_a,k_b))$. Thus, we conclude that $V^a_{k_b}(x,k_a,k_b) < 0$ for any $(k_a,k_b) \in \mathbf{N}_{ab}(x) \setminus \{(0,0)\}$, $x > 0$.

In addition, for any $(k_a,k_b) \in \mathbf{I}_a(x) \setminus \mathbf{I}_{ab}(x)$, it follows from equation (5.43) that

$$V^a_{k_b}(x,k_a,k_b) = U^a_{k_b}(x,\widehat{k}_a,k_b) < 0, \tag{C.37}$$

where $\widehat{k}_a = \frac{1}{\eta\theta_+}\left(\frac{x}{\rho} - (rp+c)\right) - \frac{\theta_-}{\theta_+}k_b$, the equality is derived from (C.15), and the inequality follows from (C.19). Since $\mathbf{N}_{ab}(x) \cup (\mathbf{I}_a(x) \setminus \mathbf{I}_{ab}(x)) \supseteq \mathbf{N}_b(x)$, we infer (C.36).

Step 3: We prove that

$$U^a_{k_a}(\mathcal{X}(k_a,k_b),k_a,k_b) < p, \quad k_a > k_b \ge 0. \tag{C.38}$$

Using (5.41) and $\mathcal{X}(k_a,k_b) = \mathcal{X}^b(k_b,k_a)$ for $k_a \ge k_b \ge 0$, we have

$$\begin{aligned}&U^a_{k_a}(\mathcal{X}(k_b,k_a),k_a,k_b) - p = U^a_{k_a}(\mathcal{X}^b(k_b,k_a),k_a,k_b) - p\\ =&\frac{\mathcal{X}^b(k_b,k_a)}{r-\mu} + H^a_{k_a}(k_a,k_b)\mathcal{X}^b(k_b,k_a)^\beta - \frac{2\eta k_a + \eta k_b + pr + c}{r}.\end{aligned} \tag{C.39}$$

Taking the partial derivative with respect to k_a in (5.48) and using $\mathcal{X}(k_a,k_b) = \mathcal{X}^b(k_b,k_a)$

for $k_a \geq k_b \geq 0$, we obtain the following result:

$$\begin{aligned} H^a_{k_b k_a}(k_a, k_b) =& \frac{\eta}{r}[\mathcal{X}^b(k_b, k_a)^{-\beta} - \beta k_a \mathcal{X}^b_{k_a} \mathcal{X}^b(k_b, k_a)^{-1-\beta}] \\ =& \frac{\eta}{r}\Big[\frac{1}{(1-\beta)\mathcal{X}^b_{k_b}} \frac{\partial(\mathcal{X}^b(k_b, k_a)^{1-\beta})}{\partial k_b} + k_a \frac{\mathcal{X}^b_{k_a}}{\mathcal{X}^b_{k_b}} \frac{\partial(\mathcal{X}^b(k_b, k_a)^{-\beta})}{\partial k_b}\Big] \end{aligned}$$

for $k_a \geq k_b \geq 0$. Integrating the above equation with respect to k_b, we have

$$\int H^a_{k_a k_b}(k_a, k_b) dk_b = \frac{\eta}{r}\left[\frac{\mathcal{X}^b(k_b, k_a)^{1-\beta}}{(1-\beta)\mathcal{X}^b_{k_b}} + k_a \frac{\theta_-}{\theta_+} \mathcal{X}^b(k_b, k_a)^{-\beta}\right]. \tag{C.40}$$

For $k_a \geq k_b \geq 0$, we derive from (C.40) that

$$\begin{aligned} H^a_{k_a}(k_a, k_b) &= H^a_{k_a}(k_a, k_a) - \int_{k_b}^{k_a} H^a_{k_a k_b}(k_a, k) dk \\ &= \Big(\frac{2\eta k_a + \eta k_a + pr + c}{r} - \frac{\mathcal{X}^b(k_a, k_a)}{r-\mu}\Big) \mathcal{X}^b(k_a, k_a)^{-\beta} \\ &\quad - \frac{\eta}{r}\left[\frac{\mathcal{X}^b(k_a, k_a)^{1-\beta} - \mathcal{X}^b(k_b, k_a)^{1-\beta}}{(1-\beta)\mathcal{X}^b_{k_b}} + k_a \frac{\theta_-}{\theta_+}\big(\mathcal{X}^b(k_a, k_a)^{-\beta} - \mathcal{X}^b(k_b, k_a)^{-\beta}\big)\right], \end{aligned} \tag{C.41}$$

where the second equality utilizes (5.47) with $k_a = k_b$ and $\mathcal{X}^a(k_a, k_a) = \mathcal{X}^b(k_a, k_a)$. Plugging (C.41) into (C.39), we have

$$\begin{aligned} & U^a_{k_a}(\mathcal{X}^b(k_b, k_a), k_a, k_b) - p \\ =& \Big(\frac{\mathcal{X}^b(k_b, k_a)}{\mathcal{X}^b(k_a, k_a)}\Big)^\beta \left(\frac{(2 - \frac{\theta_-}{\theta_+})\eta k_a + \eta k_a + pr + c}{r} - \mathcal{X}^b(k_a, k_a)\Big(\frac{1}{r-\mu} + \frac{\eta}{r(1-\beta)\mathcal{X}^b_{k_b}}\Big)\right) \\ & - \left(\frac{(2 - \frac{\theta_-}{\theta_+})\eta k_a + \eta k_b + pr + c}{r} - \mathcal{X}^b(k_b, k_a)\Big(\frac{1}{r-\mu} + \frac{\eta}{r(1-\beta)\mathcal{X}^b_{k_b}}\Big)\right). \end{aligned} \tag{C.42}$$

Denote

$$\mathcal{B}(k_a, k_b) = (2 - \frac{\theta_-}{\theta_+})\eta k_a + \eta k_b + pr + c - (\theta_+ \eta k_b + \theta_- \eta k_a + pr + c)\frac{1}{\beta - 1}(\beta - \frac{1}{\theta_+}). \tag{C.43}$$

Then for $k_a \geq k_b \geq 0$, (C.42) can be simplified as

$$U^a_{k_a}(\mathcal{X}^b(k_b, k_a), k_a, k_b) - p = \frac{1}{r}\left[\Big(\frac{\mathcal{X}^b(k_b, k_a)}{\mathcal{X}^b(k_a, k_a)}\Big)^\beta \mathcal{B}(k_a, k_a) - \mathcal{B}(k_a, k_b)\right]. \tag{C.44}$$

Denote

$$G(k_a, k_b) := \Big(\frac{\mathcal{X}^b(k_b, k_a)}{\mathcal{X}^b(k_a, k_a)}\Big)^\beta \mathcal{B}(k_a, k_a) - \mathcal{B}(k_a, k_b). \tag{C.45}$$

According to (C.44), we observe that (C.38) is equivalent to

$$G(k_a, k_b) < 0, \quad k_a > k_b \geq 0. \tag{C.46}$$

Denote $\psi(k_a) := [(2 - \frac{\theta_-}{\theta_+})\eta k_a + pr + c - (\theta_- \eta k_a + pr + c)\frac{1}{\beta-1}(\beta - \frac{1}{\theta_+})]\frac{\beta-1}{\beta}\frac{1}{\eta(\theta_+-1)}$. We have $\mathcal{B}(k_a, \psi(k_a)) = 0$. Given that $\theta_+ > 1$, we find $\mathcal{B}_{k_b} = \eta\frac{\beta}{\beta-1}(1-\theta_+) < 0$, and $\mathcal{B}(k_a, -\frac{\eta\theta_- k_a + pr + c}{\eta\theta_+}) = 2(1-\frac{\theta_-}{\theta_+})\eta k_a + (pr+c)(1-\frac{1}{\theta_+}) > 0$. From $\mathcal{B}_{k_b} < 0$, $\mathcal{B}(k_a, \psi(k_a)) = 0$, and $\mathcal{B}(k_a, -\frac{\eta\theta_- k_a + pr + c}{\eta\theta_+}) > 0$, we conclude that $-\frac{\eta\theta_- k_a + pr + c}{\eta\theta_+} < \psi(k_a)$. Since $\mathcal{X}^b(-\frac{\eta\theta_- k_a + pr + c}{\eta\theta_+}, k_a) = 0$, it follows that $\mathcal{X}^b(k_b, k_a) > 0$ for $k_b \geq \psi(k_a)$ and $k_a \geq 0$. Additionally, from $rp + c > 0$, we deduce that $\mathcal{X}^b(k_b, k_a) > 0$ for $k_b \geq 0$ and $k_a \geq 0$. Therefore, $G(k_a, k_b)$, as given by (C.45), is well-defined for any $k_b \in [0 \wedge \psi(k_a), k_a)$ with $k_a > 0$.

For any $k_b \in [0 \wedge \psi(k_a), k_a)$, $k_a > 0$, we have

$$\begin{aligned} G_{k_b}(k_a, k_a) =& \beta\frac{\mathcal{X}^b_{k_b}}{\mathcal{X}^b(k_a, k_a)}\mathcal{B}(k_a, k_a) - \eta\frac{\beta}{\beta-1}(1-\theta_+) \\ =& \frac{\mathcal{X}^b_{k_b}}{\mathcal{X}^b(k_a, k_a)}\beta\eta k_a\Big[3 - \frac{\theta_-}{\theta_+} - \theta_- - \theta_+\Big] \geq 0, \end{aligned} \tag{C.47}$$

where the inequality follows from the second inequality in (5.10). Additionally, we have

$$G_{k_b k_b}(k_a, k_b) = \beta(\beta-1)\Big(\frac{\mathcal{X}^b_{k_b}}{\mathcal{X}^b(k_a, k_a)}\Big)^2\Big(\frac{\mathcal{X}^b(k_b, k_a)}{\mathcal{X}^b(k_a, k_a)}\Big)^{\beta-2}\mathcal{B}(k_a, k_a). \tag{C.48}$$

Fix $k_a \geq 0$. If $\mathcal{B}(k_a, k_a) = 0$, then from $\mathcal{B}_{k_b} < 0$, we derive that $\mathcal{B}(k_a, k_b) > 0$ for $k_a > k_b$, and thus $G(k_a, k_b) = -\mathcal{B}(k_a, k_b) < 0$. If $\mathcal{B}(k_a, k_a) < 0$, then we derive from (C.48) that $G_{k_b k_b}(k_a, k_b) < 0$, which, together with (C.47), implies that $G_{k_b}(k_a, k_b) > G_{k_b}(k_a, k_a) \geq 0$ for $k_b \in [0 \wedge \psi(k_a), k_a)$. Hence, we have $G(k_a, k_b) < G(k_a, k_a) = 0$ for any $k_b \in [0 \wedge \psi(k_a), k_a)$.

To prove (C.46), it suffices to consider any given $k_a \geq 0$ such that $\mathcal{B}(k_a, k_a) > 0$. From (C.48), we deduce that $G(k_a, k_b)$ is convex in k_b, which implies that

$$G(k_a, k_b) < G(k_a, k_a) \vee G(k_a, 0 \wedge \psi(k_a)) = 0 \vee G(k_a, 0 \wedge \psi(k_a)), \tag{C.49}$$

for any $k_b \in (0 \wedge \psi(k_a), k_a)$. Setting $k_b = 0 \wedge \psi(k_a)$ in (C.45), we obtain

$$\begin{aligned} G(k_a, 0 \wedge \psi(k_a)) =& \Big(\frac{\mathcal{X}^b(0 \wedge \psi(k_a), k_a)}{\mathcal{X}^b(k_a, k_a)}\Big)^\beta \mathcal{B}(k_a, k_a) - \mathcal{B}(k_a, 0 \wedge \psi(k_a)) \\ \leq& \Big(\frac{\mathcal{X}^b(0 \wedge \psi(k_a), k_a)}{\mathcal{X}^b(k_a, k_a)}\Big)^\beta [\mathcal{B}(k_a, k_a) - \mathcal{B}(k_a, 0 \wedge \psi(k_a))] < 0, \quad k_a > 0, \end{aligned} \tag{C.50}$$

where the first inequality uses $\frac{\mathcal{X}^b(0 \wedge \psi(k_a), k_a)}{\mathcal{X}^b(k_a, k_a)} \leq 1$ and $\mathcal{B}(k_a, 0 \wedge \psi(k_a)) \geq \mathcal{B}(k_a, \psi(k_a)) = 0$, and the second inequality follows from $\mathcal{B}_{k_b} < 0$ and $k_a > 0$. Combining (C.49) with (C.50), we conclude that $G(k_a, k_b) < 0$ for any $k_b \in [0 \wedge \psi(k_a), k_a)$ and $k_a > 0$.

Therefore, (C.46) holds. Referring to (C.44) and the definition of $G(k_a, k_b)$ in (C.45), we see that (C.38) holds.

Step 4: We prove that

$$V^a_{k_a}(x, k_a, k_b) < p, \quad (k_a, k_b) \in \mathbf{N}_a(x),\ x > 0. \tag{C.51}$$

For $k_a \geq k_b \geq 0$, we have

$$\begin{aligned} &\mathcal{X}^b(k_b, k_a) - \rho[2\eta k_a + \eta k_b + pr + c] = \rho[(\theta_+ - 1)\eta k_b + (\theta_- - 2)\eta k_a] \\ \leq &\rho[(\theta_+ - 1)\eta k_b + (\theta_- - 2)\eta k_b] = \rho\eta k_b(\theta_- + \theta_+ - 3) \leq 0, \end{aligned} \tag{C.52}$$

where the first inequality holds due to $\theta_- < 1$, and the second inequality follows from $\theta_- + \theta_+ - 3 \leq -\frac{\theta_-}{\theta_+} < 0$ as implied by the second inequality in (5.10). For $k_b \geq k_a \geq 0$, we have

$$\begin{aligned} &\mathcal{X}^a(k_a, k_b) - \rho[2\eta k_a + \eta k_b + pr + c] = \rho[(\theta_+ - 2)\eta k_a + (\theta_- - 1)\eta k_b] \\ \leq &\rho[(\theta_+ - 2)\eta k_a + (\theta_- - 1)\eta k_a] = \rho\eta k_a(\theta_+ + \theta_- - 3) \leq 0, \end{aligned} \tag{C.53}$$

where the first inequality uses $\theta_- < 1$, and the second inequality uses $\theta_+ + \theta_- - 3 \leq -\frac{\theta_-}{\theta_+} < 0$ as given in the second inequality in (5.10). Combining (5.6), (C.52), and (C.53), we have

$$\mathcal{X}(k_a, k_b) \leq \rho[2\eta k_a + \eta k_b + pr + c], \quad (k_a, k_b) \in \mathbb{R}^2_+. \tag{C.54}$$

Next, for each fixed $(k_a, k_b) \in \mathbb{R}^2_+$, we prove that $V^a_{k_a}(x, k_a, k_b) < p$ for all $x \in [0, \mathcal{X}(k_a, k_b))$. Using (5.41), we obtain

$$V^a_{k_a}(x, k_a, k_b) = \frac{x}{r - \mu} - \frac{2\eta k_a + \eta k_b + c}{r} + H^a_{k_a}(k_a, k_b)x^\beta, \quad x \in [0, \mathcal{X}(k_a, k_b)). \tag{C.55}$$

If $H^a_{k_a}(k_a, k_b) \geq 0$, then using $r > \mu$, we see that $V^a_{k_a}(x, k_a, k_b)$ is increasing in $x \in (0, \mathcal{X}(k_a, k_b))$ and

$$V^a_{k_a}(x, k_a, k_b) < V^a_{k_a}(\mathcal{X}(k_a, k_b), k_a, k_b) \leq p, \tag{C.56}$$

for any $x \in (0, \mathcal{X}(k_a, k_b))$, where the second inequality uses (5.45) and (C.38).

If $H^a_{k_a}(k_a, k_b) < 0$, then $\frac{x}{r-\mu} + H^a_{k_a}(k_a, k_b)x^\beta$ is first increasing and then decreasing in the region $x > 0$. Denote $\hat{x} := \arg\max_{x \in [0, \mathcal{X}(k_a, k_b)]} \frac{x}{r-\mu} + H^a_{k_a}(k_a, k_b)x^\beta$. We have $\hat{x} > 0$.

If $\hat{x} = \mathcal{X}(k_a, k_b)$, then $\frac{x}{r-\mu} + H^a_{k_a}(k_a, k_b)x^\beta$ is increasing in $x \in [0, \mathcal{X}(k_a, k_b)]$. Hence, we conclude from (C.55) that $V^a_{k_a}(x, k_a, k_b) < V^a_{k_a}(\mathcal{X}(k_a, k_b), k_a, k_b) \le p$, $\forall\ x \in (0, \mathcal{X}(k_a, k_b))$. If $\hat{x} < \mathcal{X}(k_a, k_b)$, then we have $\hat{x} \in (0, \mathcal{X}(k_a, k_b))$ and $\frac{\partial V^a_{k_a}(x,k_a,k_b)}{\partial x} \mid_{x=\hat{x}} = 0$, which implies that $\frac{1}{r-\mu} + \beta H^a_{k_a}(k_a, k_b)\hat{x}^{\beta-1} = 0$. It follows that

$$\frac{\hat{x}}{r-\mu} + H^a_{k_a}(k_a, k_b)\hat{x}^\beta = \frac{\beta-1}{\beta}\frac{\hat{x}}{r-\mu} < \frac{\beta-1}{\beta}\frac{\mathcal{X}(k_a, k_b)}{r-\mu} \le \frac{2\eta k_a + \eta k_b + pr + c}{r}, \quad \text{(C.57)}$$

where the first inequality holds because of $r - \mu > 0$ and $\beta > 2$, and the second inequality uses (C.54). Combining (C.57) with (C.55), we have $V^a_{k_a}(x, k_a, k_b) \le V^a_{k_a}(\hat{x}, k_a, k_b) < p$ for any $x \in (0, \mathcal{X}(k_a, k_b))$.

Therefore, we have shown $V^a_{k_a}(x, k_a, k_b) < p$, for all $x \in (0, \mathcal{X}(k_a, k_b))$ and $(k_a, k_b) \in \mathbb{R}^2_+$, i.e., for any $(k_a, k_b) \in \mathbf{N}_{ab}(x)$ with $x \ge 0$. Using (C.38), we have $V^a_{k_a}(x, k_a, k_b) = V^a_{k_a}(x, k_a, \widehat{k}_b) = U^a_{k_a}(x, k_a, \widehat{k}_b) < p$ for any $(k_a, k_b) \in \mathbf{I}_b(x) \setminus \mathbf{I}_{ab}(x)$, where $\widehat{k}_b$ is uniquely determined by the following linear equation:

$$\mathcal{X}(k_a, \widehat{k}_b) = x. \quad \text{(C.58)}$$

Note that $\mathbf{N}_{ab}(x) \cup (\mathbf{I}_b(x) \setminus \mathbf{I}_{ab}(x)) = \mathbf{N}_a(x)$. Then (C.51) holds.

In summary, by combining our analyses in Steps 1-4 with (C.17)-(C.18), we have verified Condition 1.

Step 5: For the case $\theta_+ = \theta_-$, we prove that (B.3) holds.

For the case $\theta_+ = \theta_-$, we derive from Lemma 1 that $\theta_+ = \theta_- = 1$.

Observe that (C.39)-(C.44) also apply when $\theta_+ = \theta_- = 1$. In particular, for $\mathcal{B}(k_a, k_b)$ as defined in (C.43), we have $\mathcal{B}(k_a, k_b) \equiv 0$. Substituting this into (C.44) and recalling $\mathcal{X}(k_a, k_b) = \mathcal{X}^b(k_b, k_a)$ for $k_a \ge k_b \ge 0$, we find that $U^a_{k_a}(\mathcal{X}(k_a, k_b), k_a, k_b) = p$ for any $k_a > k_b \ge 0$. Thus $V^a_{k_a}(x, k_a, k_b) = V^a_{k_a}(x, k_a, \widehat{k}_b) = U^a_{k_a}(x, k_a, \widehat{k}_b) = p$ for any $(k_a, k_b) \in \mathbf{I}_b(x) \setminus \mathbf{I}_{ab}(x)$, where $\widehat{k}_b$ is uniquely determined by (C.58). Combining this with (C.17), we have

$$V^a_{k_a}(x, k_a, k_b) = p, \quad x \ge \mathcal{X}(k_a, k_b) \quad \text{(C.59)}$$

for any $(k_a, k_b) \in \mathbb{R}^2_+$.

Similarly, note that (C.20)-(C.26) also hold for the case $\theta_+ = \theta_- = 1$. In particular, for $\mathcal{G}(k_a, k_b)$ as defined in (C.25), we have $\mathcal{G}(k_a, k_b) = 0$ for all k_a and k_b. Substituting it into (C.26) and recalling $\mathcal{X}(k_a, k_b) = \mathcal{X}^a(k_b, k_a)$ for $k_b \geq k_a \geq 0$, we have $U^a_{k_b}(\mathcal{X}(k_a, k_b), k_a, k_b) = 0$. Thus, for any $(k_a, k_b) \in \mathbf{I}_a(x) \setminus \mathbf{I}_{ab}(x)$, we derive from (5.43) that $V^a_{k_b}(x, k_a, k_b) = U^a_{k_b}(x, \widehat{k}_a, k_b) = 0$, where $\widehat{k}_a = \frac{1}{\eta\theta_+}\left(\frac{x}{\rho} - (rp + c)\right) - \frac{\theta_-}{\theta_+}k_b$. Combining this with (C.18), we have

$$V^a_{k_b}(x, k_a, k_b) = 0, \quad x \geq \mathcal{X}(k_a, k_b) \tag{C.60}$$

for any $(k_a, k_b) \in \mathbb{R}^2_+$.

Substituting $\theta_+ = \theta_- = 1$ into (5.49), we have

$$H^a(k, k) = -\frac{\mathcal{X}(k, k)^{1-\beta}}{\beta(r - \mu)}k. \tag{C.61}$$

Substituting (C.61) and $\mathcal{X}(k_a, k_b) = \rho(\eta k_a + \eta k_b + rp + c)$ into (5.41) with $k_a = k_b = k \geq 0$ $x = \mathcal{X}(k, k)$, we have $V^a(\mathcal{X}(k, k), k, k) = pk$. Using (C.59) and (C.18), we derive $V^a_{k_a}(\mathcal{X}(k, k), k, k) = pk$. Combining this with (C.59) and (C.60), we obtain $V^a(x, k_a, k_b) = pk_a$ for any $x \geq \mathcal{X}(k_a, k_b)$ and $(k_a, k_b) \in \mathbb{R}^2_+$. Due to symmetry, we see that (B.3) holds.

Finally, by following similar procedures in Step 2 and Step 4, we can demonstrate that (3.1) holds for the case $\theta_+ = \theta_- = 1$.

□

Proof of Lemma 3 We complete our proof in four steps.

Step 1: We prove that $(\varphi_a, \varphi_b) \in \mathcal{S}$.

According to (5.19)-(5.28), Definition 1-(i) holds. By (5.28), we have $K_{it} \leq k_{i0} \vee \mathcal{K}(M_t)$, $t \geq 0$. Since $\mathcal{K}(x)$, as given by (5.9), is a linear function, there exists a constant $C_1 > 0$ such that $K_{it} \in [k_{i0}, C_1(1 + M_t)]$, $\forall\, t \geq 0$, $i = a, b$. Then, for any finite $T > 0$ and $n > 1$, a sufficient condition for $\mathbb{E}^z[|K_{iT}|^n] < \infty$ is

$$\mathbb{E}^z[|M_T|^n] < \infty. \tag{C.62}$$

Using $F_i(x, k_a, k_b) = (x - \eta(k_a + k_b))k_i - ck_i$, $K_{it} \in [k_{i0}, C_1(1 + M_t)]$, and $X_t \in [0, M_t]$, $\forall\, t \geq 0$, we can see there exists a constant $C_2 > 0$ such that $|F_i(X_t, K_{at}, K_{bt})| \leq C_2(1 + |M_t|^2)$, $\forall\, t \geq 0$.

Thus, a sufficient condition for (2.14) is

$$\int_0^{+\infty} \mathbb{E}^z[e^{-rt}|M_t|^2]dt < +\infty. \tag{C.63}$$

According to (2.16) in Chapter 4 of Friedman (2012), we can show (C.62) holds for any $T > 0$ and $n > 1$. For any initial value $z = (x_0, k_{a0}, k_{b0})$ and $X_t = x_0 + \int_0^t \mu X_s ds + \int_0^t \sigma X_s d\mathcal{W}_s$, there exists a constant $C_3 > 0$ such that

$$\begin{aligned}
&\mathbb{E}^z[M_t^2] \leq 3x_0^2 + 3\mathbb{E}\Big[\Big(\int_0^t |\mu| X_s ds\Big)^2\Big] + 3\mathbb{E}\Big[\sup_{z\in[0,t]} \Big(\int_0^z \sigma X_s d\mathcal{W}_s\Big)^2\Big] \\
\leq& 3x_0^2 + 3t\mathbb{E}\Big[\int_0^t \mu^2 X_s^2 ds\Big] + C_3\mathbb{E}\Big[\int_0^t \sigma^2 X_s^2 ds\Big] \\
=& 3x_0^2 + (3\mu^2 t + C_3\sigma^2)x_0^2 \int_0^t e^{(\sigma^2+2\mu)s} ds,
\end{aligned} \tag{C.64}$$

where the first inequality holds because $|x_1 + x_2 + x_3|^2 \leq 3(|x_1|^2 + |x_2|^2 + |x_3|^2)$ for any $x_1, x_2, x_3 \in \mathbb{R}$, and the second inequality follows from the Cauchy inequality and the Burkholder-Davis-Gundy Inequality. Since $\beta > 2$, we have $r > \sigma^2 + 2\mu$. Combining this with $r > 0$, we conclude from (C.64) that (C.63) holds. Therefore, the conditions in Definition 1-(ii) are satisfied, and $\varphi = (\varphi_a, \varphi_b) \in \mathcal{S}$.

Step 2: We prove that Condition 3 holds.

Using (5.19)-(5.22), (C.17)-(C.18), and symmetry, we observe that (3.7) holds. According to the explicit expression of the capital process as given by (5.28), we have $(K_{at}, K_{bt}) \in \overline{\mathbf{N}}_{ab}(X_t)$ for any $t > 0$, and K_{it} is continuous in $t > 0$ and satisfies (3.8).

Step 3: We prove that Condition 4 holds.

For any strategy pair $(\varphi_a, \varphi_b) \in \mathcal{S}$ as stated in Theorem 2, and any other feasible strategy pair $(\widetilde{\varphi}_a, \varphi_b) \in \mathcal{S}$, where firm a deviates from time 0, we denote

$$\widetilde{K}_a := K_a^{\widetilde{\varphi}_a, \varphi_b}, \quad \widetilde{K}_b := K_b^{\widetilde{\varphi}_a, \varphi_b} \tag{C.65}$$

as the two firms' capitals to emphasize the dependence of $\widetilde{K}_i$ on $(\widetilde{\varphi}_a, \varphi_b)$.

According to (5.30)-(5.32), we observe that firm b's lumpy investment ensures that $X_t \leq \mathcal{X}(\widetilde{K}_{at}, \widetilde{K}_{bt})$ for almost every $t > 0$. Next, we show that $X_t \leq \mathcal{X}(\widetilde{K}_{at}, \widetilde{K}_{bt})$ for every $t > 0$. For the sake of contradiction, assume there exists a $t > 0$ such that $X_t > \mathcal{X}(\widetilde{K}_{at}, \widetilde{K}_{bt})$. Define

$$\Delta t := \max\{\delta \in [0, t] : X_s > \mathcal{X}(\widetilde{K}_{at}, \widetilde{K}_{bt}), \quad \forall s \in [t - \delta, t]\}. \tag{C.66}$$

Since X_s is continuous, we have $\Delta t > 0$ almost surely. It follows that for any $s \in (t - \Delta t, t]$,

$$X_s > \mathcal{X}(\widetilde{K}_{at}, \widetilde{K}_{bt}) \geq \mathcal{X}(\widetilde{K}_{as}, \widetilde{K}_{bs}). \tag{C.67}$$

where the second inequality follows because $\mathcal{X}(k_a, k_b)$ is increasing in $k_a \geq 0$ and $k_b \geq 0$, and $\widetilde{K}_a$, $\widetilde{K}_b$ are nondecreasing process. This leads to a contradiction, as $X_s \leq \mathcal{X}(\widetilde{K}_{as}, \widetilde{K}_{bs})$ for almost every $s > 0$.

Therefore, $X_t \leq \mathcal{X}(\widetilde{K}_{at}, \widetilde{K}_{bt})$ for every $t > 0$, i.e., (3.9) holds. Using (5.15), we can deduce that $\Delta \widetilde{K}_{bt} = 0$ when $X_t \leq \mathcal{X}(\widetilde{K}_{at}, \widetilde{K}_{bt})$. It follows that $\widetilde{K}_{bt}$ is continuous for every $t > 0$.

Step 4: We prove that V^a and V^b satisfy the regularity conditions in Footnote 10.

Due to symmetry, we only prove the regularity conditions for V^a. Consider any $(\widetilde{\varphi}_a, \varphi_b) \in \mathcal{S}$ and recall the notation (C.65). Also, recall $H^a(k_a, k_b)$ as given by (5.49)-(5.51). Denote

$$\begin{aligned}
\mathcal{H}(k) &:= |H^a(k,k)| + k\Big(\frac{2\eta k + \eta k + pr + c}{r} + \frac{\mathcal{X}(k,k)}{r-\mu}\Big)\mathcal{X}(0,k)^{-\beta}, \\
\widetilde{\mathcal{H}}(k) &:= |H^a(k,k)| + \frac{\eta k^2}{r}\mathcal{X}(k,0)^{-\beta}.
\end{aligned}$$

We first show that $\lim_{k\to+\infty} \mathcal{H}(k) = 0$, $\lim_{k\to+\infty} \widetilde{\mathcal{H}}(k) = 0$, and

$$|H^a(k_a, k_b)| \leq \mathcal{H}(k_b)\mathbf{1}_{k_a \leq k_b} + \widetilde{\mathcal{H}}(k_a)\mathbf{1}_{k_a > k_b}, \quad \forall (k_a, k_b) \in \mathbb{R}_+^2. \tag{C.68}$$

Since $\lim_{k\to+\infty} H^a(k,k) = 0$, $\beta > 2$, and $\mathcal{X}(k,k)$, $\mathcal{X}(0,k)$ are positive and linearly increasing in $k \geq 0$, we have $\lim_{k\to+\infty} \mathcal{H}(k) = 0$. By (5.47), for any $k_b \geq k_a$, $k_a \geq 0$, we have

$$|H^a(k_a, k_b)| = |H^a(k_b, k_b) - \int_{k_a}^{k_b} \Big(\frac{2\eta y + \eta k_b + rp + c}{r} - \frac{\mathcal{X}(y, k_b)}{r - \mu}\Big)\mathcal{X}(y, k_b)^{-\beta} dy| \leq \mathcal{H}(k_b),$$

where the inequality holds because $\mathcal{X}(k_a, k_b) > 0$ is increasing in $k_a \in [0, k_b]$. Because $\lim_{k\to+\infty} H^a(k,k) = 0$, $\beta > 2$, and $\mathcal{X}(k, 0)$ is positive and linearly increasing in $k \geq 0$, we have $\lim_{k\to+\infty} \widetilde{\mathcal{H}}(k) = 0$. By (5.48), for any $k_a > k_b \geq 0$, we have

$$|H^a(k_a, k_b)| = |H^a(k_a, k_a) - \int_{k_b}^{k_a} \frac{\eta k_a}{r}\mathcal{X}(k_a, y)^{-\beta} dy| \leq \widetilde{\mathcal{H}}(k_a),$$

where the inequality holds because $\mathcal{X}(k_a, k_b)$ is increasing in $k_b \in [0, k_a]$. Thus, we can see (C.68) holds.

Fix any $(k_a, k_b) \in \overline{\mathbf{N}}_{ab}(x)$, i.e., $x \leq \mathcal{X}(k_a, k_b)$. If $k_a \leq k_b$, then $\mathcal{X}(k_b, k_b) \geq \mathcal{X}(k_a, k_b) \geq x$, which implies that $k_b \geq \mathcal{K}(x)$. If $k_a \geq k_b$, then $\mathcal{X}(k_a, k_a) \geq \mathcal{X}(k_a, k_b) \geq x$, which implies that $k_a \geq \mathcal{K}(x)$. Hence, we have

$$k_b \mathbf{1}_{k_a \leq k_b} + k_a \mathbf{1}_{k_a > k_b} \geq \mathcal{K}(x). \tag{C.69}$$

Because $\lim_{k\to\infty} \mathcal{H}(k) = 0$, $\lim_{k\to\infty} \widetilde{\mathcal{H}}(k) = 0$, and $\lim_{x\to+\infty} \mathcal{K}(x) = \infty$, for any $\delta > 0$, there exists $x_\delta > 0$ such that $|\mathcal{H}(k_b)| \leq \delta$ and $|\widetilde{\mathcal{H}}(k_a)| \leq \delta$ for any $k_a \geq \mathcal{K}(x_\delta)$ and $k_b \geq \mathcal{K}(x_\delta)$. Combining (C.68) and (C.69), we see that $k_b \mathbf{1}_{k_a \leq k_b} + k_a \mathbf{1}_{k_a > k_b} \geq \mathcal{K}(x_\delta)$ and $|H^a(k_a, k_b)| \leq \delta$ when $\mathcal{X}(k_a, k_b) \geq x_\delta$. Denote $\tau_\delta := \inf\{t \geq 0 : X_t \geq x_\delta\}$. Then for any $t > 0$,

$$|\mathbb{E}^z[e^{-rt} H^a(\widetilde{K}_{at}, \widetilde{K}_{bt}) X_t^\beta \mathbf{1}_{t \geq \tau_\delta}]| \leq \delta |\mathbb{E}^z[e^{-rt} X_t^\beta]| = \delta x^\beta, \tag{C.70}$$

where the inequality arises because $\mathcal{X}(\widetilde{K}_{at}, \widetilde{K}_{bt}) \geq \mathcal{X}(\widetilde{K}_{a\tau_\delta}, \widetilde{K}_{b\tau_\delta}) \geq X_{\tau_\delta} = x_\delta$, $\forall\ t \geq \tau_\delta$ (as stated by (3.9)) together with $|H^a(k_a, k_b)| \leq \delta$ for $\mathcal{X}(k_a, k_b) \geq x_\delta$, and the equality holds because $e^{-rt} X_t^\beta$ is a martingale, as shown in (4.6). Since $H^a(k_a, k_b)$ is continuous in $(k_a, k_b) \in \mathbb{R}_+^2$, we infer that $H^a(k_a, k_b)$ is bounded in the compact set $\{(k_a.k_b) \in \mathbb{R}_+^2 : \mathcal{X}(k_a, k_b) \leq x_\delta\}$. Since we have shown $|H^a(k_a, k_b)| \leq \delta$ when $\mathcal{X}(k_a, k_b) \geq x_\delta$, we see that $H^a(k_a, k_b)$ is bounded in $(k_a, k_b) \in \mathbb{R}_+^2$. Recalling that $r > 0$ and $X_t \geq 0$, $\forall t \geq 0$, we infer that $\lim_{t\to+\infty} \mathbb{E}^z[e^{-rt} H^a(\widetilde{K}_{at}, \widetilde{K}_{bt}) X_t^\beta \mathbf{1}_{t < \tau_\delta}] = 0$. Combining this with (C.70), we have $\lim_{t\to+\infty} \mathbb{E}^z[e^{-rt} H^a(\widetilde{K}_{at}, \widetilde{K}_{bt}) X_t^\beta] \leq \delta x^\beta$. Since $\delta > 0$ is arbitrary, we obtain

$$\lim_{t\to\infty} \mathbb{E}^z[e^{-rt} H^a(\widetilde{K}_{at}, \widetilde{K}_{bt}) X_t^\beta] = 0. \tag{C.71}$$

According to (C.64), $r > 0$ and $r > \sigma^2 + 2\mu$, we have

$$\lim_{t\to\infty} \mathbb{E}^z[e^{-rt} M_t^2] = 0. \tag{C.72}$$

Since $(\widetilde{\varphi}_a, \varphi_b) \in \mathcal{S}$, we derive from (2.14) and Tonelli's Theorem that

$$\int_0^\infty \mathbb{E}^z[e^{-rt} |F_i(X_t, \widetilde{K}_{at}, \widetilde{K}_{bt})|] dt < \infty.$$

It follows that there exists a sequence $\{t_n\}$ of real numbers such that $\lim_{n\to\infty} t_n = \infty$ and

$$\lim_{n\to\infty} \mathbb{E}^z[e^{-rt_n} F_i(X_{t_n}, \widetilde{K}_{at_n}, \widetilde{K}_{bt_n})] = 0. \tag{C.73}$$

Since $F_i(x, k_a, k_b) = k_a(x - \frac{\eta}{4} k_a) - k_a(\frac{3\eta}{4} k_a + \eta k_b + c) \leq k_a(x - \frac{\eta}{4} k_a)$ for any $(x, k_a, k_b) \in \mathbb{R}_+^3$, we derive from (C.73) that

$$\liminf_{n\to\infty} \mathbb{E}[e^{-rt_n} \widetilde{K}_{at_n}(X_{t_n} - \frac{\eta}{4} \widetilde{K}_{at_n})] \geq 0. \tag{C.74}$$

In addition, we have

$$
\begin{aligned}
0 \geq & \limsup_{n\to\infty} \mathbb{E}^z \left[e^{-rt_n} \big(-\frac{1}{\eta} X_{t_n}^2 + \widetilde{K}_{at_n} (X_{t_n} - \frac{\eta}{4} \widetilde{K}_{at_n}) \big) \right] \\
= & \limsup_{n\to\infty} \mathbb{E}^z \left[e^{-rt_n} \widetilde{K}_{at_n} \big(X_{t_n} - \frac{\eta}{4} \widetilde{K}_{at_n} \big) \right], \qquad (C.75)
\end{aligned}
$$

where the first inequality holds because $-\frac{1}{\eta} X_{t_n}^2 + \widetilde{K}_{at_n}(X_{t_n} - \frac{\eta}{4}\widetilde{K}_{at_n}) = \frac{-1}{\eta}[X_{t_n} - \frac{\eta}{2}\widetilde{K}_{at_n}]^2$, and the equality is due to (C.72). Hence, we derive from (C.74) and (C.75) that

$$
\lim_{n\to\infty} \mathbb{E}^z \left[e^{-rt_n} \widetilde{K}_{at_n} \big(X_{t_n} - \frac{\eta}{4} \widetilde{K}_{at_n} \big) \right] = 0. \qquad (C.76)
$$

Combining (C.76), (C.73), and $F_i(x, k_a, k_b) = k_a(x - \frac{\eta}{4}k_a) - k_a(\frac{3\eta}{4}k_a + \eta k_b + c)$, we conclude that $\lim_{n\to\infty} \mathbb{E}^z[e^{-rt_n}\widetilde{K}_{at_n}(\frac{3\eta}{4}\widetilde{K}_{at_n} + \eta \widetilde{K}_{bt_n} + c)] = 0$. Since $\widetilde{K}_a \geq 0$, $\widetilde{K}_b \geq 0$, $\eta > 0$, $c \geq 0$, we have

$$
\begin{aligned}
& \lim_{n\to\infty} \mathbb{E}^z[e^{-rt_n} \widetilde{K}_{at_n}^2] = 0, \qquad (C.77) \\
& \lim_{n\to\infty} \mathbb{E}^z[e^{-rt_n} \widetilde{K}_{at_n} \widetilde{K}_{bt_n}] = 0. \qquad (C.78)
\end{aligned}
$$

Combining (C.77) with (C.72) and using the Cauchy inequality, we obtain

$$
\lim_{n\to\infty} \mathbb{E}^z[e^{-rt_n} \widetilde{K}_{at_n} X_{t_n}] = 0. \qquad (C.79)
$$

Since we have shown that $(\widetilde{K}_{at}, \widetilde{K}_{bt}) \in \overline{\mathbf{N}}_{ab}(X_t)$ for any $t > 0$, and $V^a(X_t, \widetilde{K}_{at}, \widetilde{K}_{bt})$ is given by (5.41), we derive from (C.71) and (C.77)-(C.79) that $\lim_{n\to\infty} \mathbb{E}^z[e^{-rt_n} V^a(X_{t_n}, \widetilde{K}_{at_n}, \widetilde{K}_{bt_n})] = 0$.

Using (5.41), we have $V_x^a(x, k_a, k_b) = \frac{k_a}{r-\mu} + \beta H^a(k_a, k_b) x^{\beta-1}$ for any $(k_a, k_b) \in \overline{\mathbf{N}}_{ab}(x)$. Thus, there exists a constant $C_1 > 0$ such that for any $(k_a, k_b) \in \overline{\mathbf{N}}_{ab}(x)$,

$$
\begin{aligned}
|V_x^a(x, k_a, k_b) x|^2 \leq & 2 \left|\frac{k_a x}{r-\mu}\right|^2 + 2\beta^2 |H^a(k_a, k_b)|^2 x^{2\beta} \\
\leq & C_1(k_a^4 + x^4 + x^{2\beta}) \leq C_1(k_a^4 + 1 + 2x^{2\beta}), \qquad (C.80)
\end{aligned}
$$

where the second inequality holds because we have shown that $H^a(k_a, k_b)$ is bounded for $(k_a, k_b) \in \mathbb{R}_+^2$, and the last inequality uses $\beta > 2$ and $x^4 < 1 + x^{2\beta}$. Since $\widetilde{K}_{at} \geq 0$ is nondecreasing in t, and $\mathbb{E}^z[|\widetilde{K}_{at}|^n] < +\infty$ for any $t > 0$, $n > 1$ by Definition 1-(i), we have $\mathbb{E}^z[\int_0^t |\widetilde{K}_{as}|^4 ds] \leq t\mathbb{E}^z[|\widetilde{K}_{at}|^4] < +\infty$, $\forall t > 0$. In addition, it is straightforward to calculate that $\mathbb{E}^z[X_t^{2\beta}] = e^{\beta t(2\mu + (2\beta-1)\sigma^2)}$ and then $\mathbb{E}^z[\int_0^t X_s^{2\beta} ds] < \infty$, $\forall t > 0$. We then conclude from $(\widetilde{K}_{at}, \widetilde{K}_{bt}) \in \overline{\mathbf{N}}_{ab}(X_t)$, $t > 0$, and (C.80) that $\mathbb{E}^z \left[\int_0^t |e^{-rs} V_x^a(X_s, \widetilde{K}_{as}, \widetilde{K}_{bs}) \sigma(X_s)|^2 ds \right] <$

$+\infty$, $\forall\, t \geq 0$. Therefore, we have proved the regularity conditions in Footnote 10.

□

Proof of Lemma 4 Note that (C.39)-(C.44) hold for any constants $\theta_+ \geq \theta_- > 0$. Recall the function $G(k_a, k_b)$ defined in (C.45) and the equalities in (C.47). If the second inequality in (5.10) is reversed, then we derive from (C.47) that $G_{k_b}(k_a, k_a) < 0$. Since (C.45) implies $G(k_a, k_a) = 0$, there exists $\epsilon \in (0, k_a)$ such that $G(k_a, k_b) > 0$ for any $k_b \in (k_a - \epsilon, k_a)$. Combining (C.44) and the definition of $G(k_a, k_b)$ in (C.45), we have

$$U^a_{k_a}(\mathcal{X}(k_a, k_b), k_a, k_b) > p \tag{C.81}$$

for any $k_b \in (k_a - \epsilon, k_a)$. Therefore, the second inequality in (5.10) is a necessary condition for (5.52).

Note that (C.20)-(C.26) hold for any constants $\theta_+ \geq \theta_- > 0$. Recall the function $\mathcal{H}(k)$ defined in (C.28). If the first inequality in (5.10) is reversed, then Lemma 1 implies that the inequality (B.2) is reversed. Then, we derive from the second equality in (C.34) that $\lim_{k\to+\infty} \frac{\mathcal{H}(k)}{k} > 0$. It follows that $\mathcal{H}(k_b) > 0$ for some $k_b > 0$. Combining (C.26) and (C.28), we obtain $U^a_{k_b}(\mathcal{X}^a(0, k_b), 0, k_b) > 0$. Therefore, the first inequality in (5.10) is a necessary condition for (5.53).

□

Proof of Theorem 2 Using Theorem 1 and Lemmas 2-3, we can see that $(\varphi_a, \varphi_b) \in \mathcal{S}$ is a Markov perfect equilibrium strategy.

In the proof of Lemma 2, we have shown that for the case $\theta_+ > \theta_-$, we have (C.17)-(C.18), (C.36), and (C.51), where $\mathbf{I}_i(x)$, $\mathbf{N}_i(x)$, $\mathbf{I}_{ab}(x)$, and $\mathbf{N}_{ab}(x)$ are given by (5.16)-(5.18). Hence, $\mathbf{I}_i(x)$ and $\mathbf{N}_i(x)$ are consistent with (3.2) and (3.3).

□

Proof of Theorem 3 The proof is straightforward thanks to Theorems 1-2 and Lemmas 2-3.

□

Proof of Proposition 2 Due to symmetry, we only prove (5.55) for $i = a$.

For any $\theta_+ \geq \theta_- > 0$ and strategy $\varphi^{\theta_+,\theta_-}$, the corresponding firm value is characterized by Theorem 3. In particular, the option value $H^i(k_a, k_b)$ is given by (5.49)-(5.51).

We can see that $H^a(k,k)$ given by (5.49) depends on (θ_+, θ_-) only via $\theta_+ + \theta_-$. We denote it as $H^a(k,k;\theta_+ + \theta_-)$. First, we show that $H^a(k,k;\theta_+ + \theta_-)$ is increasing in $\theta_+ + \theta_- \in (0,4]$. To this end, we define $f(\theta) := -\frac{(\theta k + \frac{rp+c}{\eta})^{1-\beta}}{\theta}\Big[(4(\beta-1)-\beta\theta)k + \frac{rp+c}{\eta}(\frac{4}{\theta}-2)\Big]$. We can see that $H^a(k,k;\theta) = (\rho\eta)^{1-\beta}\frac{f(\theta)}{\beta(\beta-2)(r-\mu)}$. A direct calculation shows that

$$f'(\theta) = \frac{(\theta k + \frac{rp+c}{\eta})^{-\beta}}{\theta^3}(4-\theta)\Bigg[(\theta k)^2(\beta-1)\beta + 2\theta k\beta\frac{rp+c}{\eta} + 2\Big(\frac{rp+c}{\eta}\Big)^2\Bigg].$$

Thanks to $\beta > 2$ and $r > 0$, we see that $f'(\theta) > 0$ for any $\theta \in [0,4)$, and $f'(\theta) < 0$ for any $\theta > 4$. Hence, $H^a(k,k;\theta)$ is increasing in $\theta \in (0,4]$.

Second, we can see $\mathcal{K}(x)$ given by (5.9) is decreasing in $\theta_+ + \theta_-$ for any $x \geq \rho(rp+c)$. To highlight the dependence of $\mathcal{K}(x)$ on $\theta_+ + \theta_-$, we denote it as $\mathcal{K}(x;\theta_+ + \theta_-)$.

Next, we fix $\theta_+ \geq \theta_- > 0$ and $\widetilde{\theta}_+ \geq \widetilde{\theta}_- > 0$ with $4 \geq \theta := \theta_+ + \theta_- > \widetilde{\theta} := \widetilde{\theta}_+ + \widetilde{\theta}_-$. For any $k \geq \mathcal{K}(x;\widetilde{\theta})$, we conclude from (5.41) that

$$V^a(x,k,k;\varphi^{\theta_+,\theta_-}) - V^a(x,k,k;\varphi^{\widetilde{\theta}_+,\widetilde{\theta}_-}) = \Big[H^a(k,k;\theta) - H^a(k,k;\widetilde{\theta})\Big]x^\beta > 0. \tag{C.82}$$

For $k \in [\mathcal{K}(x;\theta), \mathcal{K}(x;\widetilde{\theta})]$, we conclude from Theorem 5.4-(ii) that

$$\begin{aligned} &V^a(x,\mathcal{K}(x;\widetilde{\theta}),\mathcal{K}(x;\widetilde{\theta});\varphi^{\widetilde{\theta}_+,\widetilde{\theta}_-}) - V^a(x,k,k;\varphi^{\widetilde{\theta}_+,\widetilde{\theta}_-}) = p[\mathcal{K}(x;\widetilde{\theta}) - k] \\ \geq &V^a(x,\mathcal{K}(x;\widetilde{\theta}),\mathcal{K}(x;\widetilde{\theta});\varphi^{\theta_+,\theta_-}) - V^a(x,k,k;\varphi^{\theta_+,\theta_-}), \end{aligned} \tag{C.83}$$

where the equality follows from $V^a_{k_a}(x,k,k;\varphi^{\widetilde{\theta}_+,\widetilde{\theta}_-}) = p$ and $V^a_{k_b}(x,k,k;\varphi^{\widetilde{\theta}_+,\widetilde{\theta}_-}) = 0$ for $k \in [\mathcal{K}(x;\theta), \mathcal{K}(x;\widetilde{\theta})]$, and the inequality follows from $V^a_{k_a}(x,k,k;\varphi^{\theta_+,\theta_-}) \leq p$, $V^a_{k_b}(x,k,k;\varphi^{\theta_+,\theta_-}) \leq 0$. Then we derive from (C.83) that for $k \in [\mathcal{K}(x;\theta), \mathcal{K}(x;\widetilde{\theta})]$,

$$\begin{aligned} &V^a(x,k,k;\varphi^{\theta_+,\theta_-}) - V^a(x,k,k;\varphi^{\widetilde{\theta}_+,\widetilde{\theta}_-}) \\ \geq &V^a(x,\mathcal{K}(x;\widetilde{\theta}),\mathcal{K}(x;\widetilde{\theta});\varphi^{\theta_+,\theta_-}) - V^a(x,\mathcal{K}(x;\widetilde{\theta}),\mathcal{K}(x;\widetilde{\theta});\varphi^{\widetilde{\theta}_+,\widetilde{\theta}_-}) > 0, \end{aligned} \tag{C.84}$$

where the second inequality uses (C.82). Since $V^a_{k_a}(x,k,k;\varphi^{\theta_+,\theta_-}) = V^a_{k_a}(x,k,k;\varphi^{\widetilde{\theta}_+,\widetilde{\theta}_-}) = p$, and $V^a_{k_b}(x,k,k;\varphi^{\theta_+,\theta_-}) = V^a_{k_b}(x,k,k;\varphi^{\widetilde{\theta}_+,\widetilde{\theta}_-}) = 0$ for any $k \in [0,\mathcal{K}(x;\theta)]$, we derive from (C.84) that $V^a(x,k,k;\varphi^{\theta_+,\theta_-}) > V^a(x,k,k;\varphi^{\widetilde{\theta}_+,\widetilde{\theta}_-})$ for any $k \in [0,\mathcal{K}(x;\theta)]$.

For any $(\theta_+, \theta_-) \in \Theta \setminus \{(\theta_+^*, \theta_-^*), (1,1)\}$, we have

$$3 > \theta_+^* + \theta_-^* > \theta_+ + \theta_- > 2,$$

where the first inequality uses $\theta_+^* + \theta_-^* = 3 - \frac{1}{w^*} < 3$, the second inequality uses Lemma 1-(iii), and the last inequality uses Lemma 1-(iv). Then, we conclude that (5.55) holds.

□

Proof of Lemma 5 Denote

$$\mathcal{G}(k_a, k_b) = \frac{\mathcal{X}(k_a, k_b) - \rho\big(\eta k_a + \eta k_b + rp + c\big)}{\rho\eta} \tag{C.85}$$

for $k_a \geq k_b > 0$. Then, (6.2) is equivalent to

$$k_a \mathcal{G}_{k_a}(k_a, k_b) + \Big(\mathcal{G}(k_a, k_b) - k_a\Big)\mathcal{G}_{k_b}(k_a, k_b) + \mathcal{G}(k_a, k_b) = 0. \tag{C.86}$$

Note that (C.86) is a quasilinear first-order PDE, and the coefficient of $\mathcal{G}_{k_a}(k_a, k_b)$ is $k_a > 0$. The general (local) solution of (C.86) can be obtained by solving the following characteristic equations (see e.g., Chapter 1 in McOwen (2004)):

$$\frac{dk_a}{k_a} = \frac{dk_b}{\mathcal{G} - k_a} = -\frac{d\mathcal{G}}{\mathcal{G}}. \tag{C.87}$$

Solving (C.87) yields the following two independent solutions:

$$k_a + k_b + \mathcal{G} = \lambda_1, \quad \mathcal{G} = \frac{\lambda_2}{k_a}, \tag{C.88}$$

where λ_1 and λ_2 are constants. Then, the general solution to the PDE (C.86) is characterized by (6.4). Hence, the PDE (6.2) admits the general solution (6.3) locally.

In addition, if $\mathcal{G}(k_a, k_b)$ satisfies (6.4) for any $k_a \geq k_b \geq \underline{k} > 0$, by taking partial derivatives with respect to k_a and k_b in (6.4), we obtain

$$\phi_{z_1}\Big(k_a\mathcal{G}, k_a + k_b + \mathcal{G}\Big)(\mathcal{G} + k_a\mathcal{G}_{k_a}) + \phi_{z_2}\Big(k_a\mathcal{G}, k_a + k_b + \mathcal{G}\Big)\big(1 + \mathcal{G}_{k_a}\big) = 0, \tag{C.89}$$

$$\phi_{z_1}\Big(k_a\mathcal{G}, k_a + k_b + \mathcal{G}\Big)k_a\mathcal{G}_{k_b} + \phi_{z_2}\Big(k_a\mathcal{G}, k_a + k_b + \mathcal{G}\Big)\big(1 + \mathcal{G}_{k_b}\big) = 0. \tag{C.90}$$

Since $\phi_{z_1} \neq 0$ or $\phi_{z_2} \neq 0$, it follows from (C.89) and (C.90) that

$$(\mathcal{G} + k_a\mathcal{G}_{k_a})\big(1 + \mathcal{G}_{k_b}\big) = \big(1 + \mathcal{G}_{k_a}\big)k_a\mathcal{G}_{k_b}. \tag{C.91}$$

Simplifying the above equation, we can see that (C.86) holds for any $k_a \geq k_b \geq \underline{k}$.

□

Proof of Lemma 6 Similar to the proof of Lemma 2, we only prove the case for $V^a(z)$ as the case for $V^b(z)$ can be treated similarly. Similarly, it is straightforward to see that $V^i \in \mathcal{C}^{0,1,1}(\mathbb{R}^3_+)$ and that Condition 2, (C.17), and (C.18) are satisfied. Hence, we only need to prove the two inequalities in Condition 1 and (B.4).

In the following, we only focus on $k_a \geq \underline{k}$ and $k_b \geq \underline{k}$. Next, we complete our proof in three steps.

Step 1: We prove that (B.4) *holds.*

Lemma 5 implies that $\mathcal{X}(k_a, k_b)$ as given by (6.3) satisfies (6.2) for any $k_a \geq k_b \geq \underline{k} > 0$. Denote $\mathcal{Y}(k_a, k_b) = \left(\frac{2\eta k_a + \eta k_b + rp + c}{r} - \frac{\mathcal{X}(k_a,k_b)}{r-\mu}\right)\mathcal{X}(k_a, k_b)^{-\beta}$. A direct calculation shows that (6.2) is equivalent to the following equation:

$$\frac{\partial \mathcal{Y}(k_a, k_b)}{\partial k_b} = \frac{\partial\left[\frac{\eta k_a}{r}\mathcal{X}(k_a, k_b)^{-\beta}\right]}{\partial k_a}, \quad k_a > k_b \geq \underline{k}. \tag{C.92}$$

Using (5.51), we obtain (5.48). Substituting (5.48) into (C.92) gives $\frac{\partial \mathcal{Y}(k_a,k_b)}{\partial k_b} = \frac{\partial}{\partial k_a}[\frac{\partial H^a(k_a,k_b)}{\partial k_b}] = \frac{\partial}{\partial k_b}[\frac{\partial H^a(k_a,k_b)}{\partial k_a}]$. It follows that

$$\frac{\partial H^a(k_a, k_b)}{\partial k_a} = \mathcal{Y}(k_a, k_b) + C(k_a), \quad k_a > k_b \geq \underline{k}, \tag{C.93}$$

for some function $C(k_a)$.

Using (5.50), we obtain (5.47), i.e., $\frac{\partial H^a(k_a,k_b)}{\partial k_a} = \mathcal{Y}(k_a, k_b)$ for $k_a \leq k_b$, which, together with (C.93) and the continuity of $\frac{\partial H^a(k_a,k_b)}{\partial k_a}$ and $\mathcal{Y}(k_a, k_b)$, implies that $C(k_a) = 0$. Hence, (5.47) also holds for $k_a > k_b \geq \underline{k}$. Then, we derive from (5.41) that

$$\frac{\partial U^a(x, k_a, k_b)}{\partial k_a}\,|_{x=\mathcal{X}(k_a,k_b)} = p \;\text{ for } k_a > k_b \geq \underline{k}. \tag{C.94}$$

Therefore, $\frac{\partial V^a(x,k_a,k_b)}{\partial k_a} = \frac{\partial V^a(x,k_a,\widehat{k}_b)}{\partial k_a} = \frac{\partial U^a(x,k_a,\widehat{k}_b)}{\partial k_a} = p$ for any $(k_a, k_b) \in \mathbf{I}_b(x) \setminus \mathbf{I}_{ab}(x)$, where $\widehat{k}_b$ is uniquely determined by (C.58) with $\mathcal{X}(k_a, k_b)$ as defined in (6.5). Similar to the proof in Lemma 2, we have (C.17). Therefore, we have proved (B.4).

Step 2: We prove that

$$V^a_{k_a}(x, k_a, k_b) < p, \quad (k_a, k_b) \in \mathbf{N}_{ab}(x), \; x > 0, \; k_a \geq \underline{k}, \; k_b \geq \underline{k}. \tag{C.95}$$

For $k_a \geq k_b \geq \underline{k}$, we have

$$\mathcal{X}(k_a, k_b) - \rho(2\eta k_b + \eta k_a + pr + c) = \rho\eta\Big(\mathcal{G}(k_a, k_b) - k_b\Big) < 0, \tag{C.96}$$

where the inequality follows from the second inequality in (6.6). Using (C.96) and the symmetry of $\mathcal{X}(k_a, k_b)$, for $k_a \leq k_b$, we have

$$\mathcal{X}(k_a, k_b) - \rho(2\eta k_a + \eta k_b + pr + c) < 0. \tag{C.97}$$

For $k_a \geq k_b \geq \underline{k}$, we have

$$\mathcal{X}(k_a, k_b) < \rho(2\eta k_b + \eta k_a + pr + c) \leq \rho(2\eta k_a + \eta k_b + pr + c), \tag{C.98}$$

where the first inequality uses (C.96), and the second inequality uses $k_a \geq k_b$. Combining (C.97) and (C.98), we infer

$$\mathcal{X}(k_a, k_b) \leq \rho(2\eta k_a + \eta k_b + pr + c) \tag{C.99}$$

for any $k_a \geq \underline{k}$, $k_b \geq \underline{k}$.

By employing similar arguments as in Step 4 of the proof of Lemma 2, we can show that (C.95) holds by using (C.99) and $V^a_{k_a}(x, k_a, k_b) \mid_{x=\mathcal{X}(k_a,k_b)}= p$.

Step 3: We prove that

$$U^a_{k_b}(\mathcal{X}(k_a, k_b), k_a, k_b) < 0, \quad k_b > k_a \geq \underline{k}. \tag{C.100}$$

Using (5.41), we have $U^a_{k_b}(\mathcal{X}(k_a, k_b), k_a, k_b) = -\frac{\eta k_a}{r} + H^a_{k_b}(k_a, k_b)\mathcal{X}(k_a, k_b)^{\beta}$ and thus (C.100) is equivalent to $H^a_{k_b}(k_a, k_b) < \mathcal{H}(k_a, k_b)$, where $\mathcal{H}(k_a, k_b) := \frac{\eta k_a}{r}\mathcal{X}(k_a, k_b)^{-\beta}$. Additionally, by using (5.48), we have $U^a_{k_b}(\mathcal{X}(k_a, k_b), k_a, k_b) = 0$ for $k_a \geq k_b \geq \underline{k}$ and thus $H^a_{k_b}(k, k) = \mathcal{H}(k, k)$. As a result, (C.100) is equivalent to

$$\int_{k_a}^{k_b} \frac{\partial H^a_{k_b}(k, k_b)}{\partial k_a} dk > \int_{k_a}^{k_b} \frac{\partial \mathcal{H}(k, k_b)}{\partial k_a} dk, \quad k_b > k_a \geq \underline{k}. \tag{C.101}$$

Then, we only need to show that $H^a_{k_a k_b}(k_a, k_b) > \frac{\partial \mathcal{H}(k_a, k_b)}{\partial k_a}$ for any $k_b > k_a \geq \underline{k}$. Using (5.47), we have

$$\begin{aligned} &H^a_{k_a k_b}(k_a, k_b) - \frac{\partial \mathcal{H}(k_a, k_b)}{\partial k_a} \\ =&\frac{(\beta - 1)\Big[\mathcal{X}_{k_b}(k_a, k_b)\big(\mathcal{X}(k_a, k_b) - \rho(2\eta k_a + \eta k_b + pr + c)\big) + \rho\eta\mathcal{X}_{k_a}(k_a, k_b)k_a\Big]}{(r - \mu)\mathcal{X}(k_a, k_b)^{\beta+1}}. \end{aligned} \tag{C.102}$$

For $k_b > k_a$, we have

$$\begin{aligned}
&\mathcal{X}(k_a,k_b) - \rho(2\eta k_a + \eta k_b + pr + c) + \rho\eta\frac{\mathcal{X}_{k_a}(k_a,k_b)}{\mathcal{X}_{k_b}(k_a,k_b)}k_a \\
=&\rho\left([2 - \frac{\vartheta_-(k_a,k_b)}{\vartheta_+(k_a,k_b)}]\eta k_b + \eta k_a\right) - \rho\left([2 - \frac{\vartheta_+(k_a,k_b)}{\vartheta_-(k_a,k_b)}]\eta k_a + \eta k_b\right) \\
=&\rho\eta\left([1 - \frac{\vartheta_-(k_a,k_b)}{\vartheta_+(k_a,k_b)}]k_b - [1 - \frac{\vartheta_+(k_a,k_b)}{\vartheta_-(k_a,k_b)}]k_a\right) > 0,
\end{aligned} \tag{C.103}$$

where the first equality uses $\mathcal{X}(k_a,k_b) = \rho\left([2 - \frac{\vartheta_-(k_a,k_b)}{\vartheta_+(k_a,k_b)}]\eta k_b + \eta k_a + rp + c\right)$ for $k_b > k_a$, and the inequality uses $\frac{\vartheta_-(k_a,k_b)}{\vartheta_+(k_a,k_b)} \in (0,1)$; see Footnote 25.

Substituting (C.103) into (C.102) gives $H^a_{k_a k_b}(k_a,k_b) > \frac{\partial \mathcal{H}(k_a,k_b)}{\partial k_a}$ for any $k_b > k_a \geq \underline{k}$. Hence, (C.100) holds.

Finally, by following the same arguments as in Step 2 of the proof of Lemma 2, we can prove that $V^a_{k_b}(x,k_a,k_b) < 0$ for any $(k_a,k_b) \in \mathbf{N}_b(x)$, $k_a \geq \underline{k}$, $k_b \geq \underline{k}$, and $x > 0$, by using (C.100) and (C.18).

□

Proof of Lemma 7 The proof is similar to that of Lemma 3. □

Proof of Theorem 4 We can see that Theorem 1 also holds for the region $\mathbb{Z}$ defined in (6.7). Hence, using Theorem 1, Lemmas 6 and 7, we can see that Theorem 4 holds.

□

Proof of Proposition 3 We only need to verify that Condition 6 holds.

We first consider the case $\phi(z_1,z_2) = z_1 - \lambda$ for a constant $\lambda > 0$. Then, $\mathcal{G} = \frac{\lambda}{k_a}$ solves the equation (6.4). For $\underline{k} > \sqrt{\lambda}$, we have $\lambda < k_a k_b$ for any $k_a \geq \underline{k}$, $k_b \geq \underline{k}$. Hence, Condition 6-(i) holds. Condition 6-(ii) is obvious.

Next, we consider the case $\phi(z_1,z_2) = \lambda z_1 - z_2$ for a constant $\lambda > 0$. Then, $\mathcal{G}(k_a,k_b) = \frac{k_a+k_b}{\lambda k_a - 1}$ solves equation (6.4). For $\underline{k} > \frac{3}{\lambda}$, we have $\lambda k_a - 1 > 0$ for any $k_a \geq \underline{k}$. Then, for any $k_a \geq \underline{k}$, $k_b \geq \underline{k}$, we have $\mathcal{G}(k_a,k_b) > 0$ and

$$\frac{\mathcal{G}(k_a,k_b)}{k_b} = \frac{k_a+k_b}{(\lambda k_a - 1)k_b} \leq \frac{k_a + \underline{k}}{(\lambda k_a - 1)\underline{k}} \leq \frac{2\underline{k}}{(\lambda \underline{k} - 1)\underline{k}} = \frac{2}{\lambda \underline{k} - 1} < 1, \tag{C.104}$$

where the first inequality follows from the fact that $\frac{k_a+k_b}{(\lambda k_a-1)k_b}$ is decreasing in $k_b \in [\underline{k}, k_a]$, the second inequality follows from the fact that $\frac{k_a+\underline{k}}{(\lambda k_a-1)\underline{k}}$ is decreasing in $k_a \geq \underline{k}$, and the last inequality uses $\underline{k} > \frac{3}{\lambda}$. Hence, Condition 6-(i) holds. Since $\frac{\partial \mathcal{G}(k_a,k_b)}{\partial k_b} = \frac{1}{\lambda k_a - 1} > 0$, Condition 6-(ii) is obvious. Then, Condition 6 holds.

□

Proof of Lemma 8 Denote

$$\mathcal{G}(k_a, k_b) = \mathcal{X}(k_a, k_b)(k_a + k_b)^{-1/\gamma} - \rho. \tag{C.105}$$

One can show that (A.7) is equivalent to

$$\begin{aligned} &\mathcal{G}_{k_a}(k_a, k_b)k_a\big[\mathcal{G}(k_a, k_b) + \rho\big] + \mathcal{G}_{k_b}(k_a, k_b)\Big[\mathcal{G}(k_a, k_b)\big(\gamma(k_a + k_b) - k_a\big) - \rho k_a\Big] \\ &+\mathcal{G}(k_a, k_b)(\mathcal{G}(k_a, k_b) + \rho) = 0. \end{aligned} \tag{C.106}$$

Let $g(k_a, k) = \mathcal{G}(k_a, k_b)k_a$, where $k = k_a + k_b$. Then (C.106) is equivalent to

$$g_{k_a}(k_a, k)(g(k_a, k)k_a + \rho k_a^2) + g_k(k_a, k)g(k_a, k)\gamma k = 0. \tag{C.107}$$

Note that (C.107) is a quasilinear first-order PDE, and the coefficients of $g_{k_a}(k_a, k)$ and $g_k(k_a, k)$ cannot be zero simultaneously given $k \geq k_a > 0$ and $\gamma > 1$. The general (local) solution of (C.107) can be obtained by solving the following characteristic equations (see e.g., Chapter 1 in McOwen (2004)):

$$\begin{cases} \frac{dk_a}{gk_a+\rho k_a^2} = \frac{dk}{g\gamma k}, \\ dg = 0. \end{cases} \tag{C.108}$$

Solving (C.108) yields the following two independent solutions:

$$\frac{1}{g} \ln \frac{k_a}{(g + \rho k_a)k^{\frac{1}{\gamma}}} = \lambda_1, \quad g = \lambda_2, \tag{C.109}$$

where λ_1, λ_2 are constants. Then, the general solution to the PDE (C.106) is characterized by (A.9). Hence, the PDE (A.7) admits a general solution as given by (A.8).

In addition, if $\mathcal{G}(k_a, k_b)$ satisfies (A.9) for any $k_a \geq k_b \geq \underline{k} > 0$, then for $k = k_a + k_b$, $g = g(k_a, k) = G(k_a, k_b)k_a$ satisfies

$$\phi\Big(g, \frac{1}{g} \ln \frac{k_a}{(g + \rho k_a)k^{\frac{1}{\gamma}}}\Big) = 0. \tag{C.110}$$

Denote $u(k_a,k,z) = \frac{1}{z}\ln\frac{k_a}{(z+\rho k_a)k^{\frac{1}{\gamma}}}$. Taking partial derivatives with respect to k_a and k in (C.110), we have

$$\phi_{z_1}\Big(g,u(k_a,k,g)\Big)g_{k_a} + \phi_{z_2}\Big(g,u(k_a,k,g)\Big)\big(u_{k_a}+u_z g_{k_a}\big) = 0, \tag{C.111}$$

$$\phi_{z_1}\Big(g,u(k_a,k,g)\Big)g_{k} + \phi_{z_2}\Big(g,u(k_a,k,g)\Big)\big(u_{k}+u_z g_{k}\big) = 0, \tag{C.112}$$

where $g = g(k_a,k)$. Using $\phi_{z_1}\neq 0$ or $\phi_{z_2}\neq 0$, we derive from (C.111) and (C.112) that $\big(u_k+u_z g_k\big)g_{k_a} = \big(u_{k_a}+u_z g_{k_a}\big)g_k$, which is equivalent to

$$g_{k_a}u_k = g_k u_{k_a}. \tag{C.113}$$

Substituting $u_{k_a}(k_a,k,g) = \frac{1}{g}(\frac{1}{k_a}-\frac{\rho}{g+\rho k_a})$ and $u_k(k_a,k,g) = \frac{-1}{g\gamma k}$ into (C.113), we obtain (C.107) and thus (C.106) holds for any $k_a\geq k_b\geq \underline{k}$.

□

Proof of Lemma 9 Similar to the proof of Lemma 2, we only prove the case for $V^a(z)$ as the case for $V^b(z)$ can be treated similarly. Similarly, it is straightforward to see that $V^i\in\mathcal{C}^{0,1,1}(\mathbb{R}^3_+)$ and that Condition 2, (C.17), and (C.18) are satisfied. The proof of (B.4) is similar to that in Lemma 6. Hence, we only need to prove the two inequalities in Condition 1.

Next, we complete our proof in three steps.

Step 1: We prove that (C.95) *holds.*

Using (5.41) and (A.2), we obtain

$$V^a_{k_a}(x,k_a,k_b) = \psi^a_{k_a}(k_a,k_b)x + H^a_{k_a}(k_a,k_b)x^\beta,\quad x\in(0,\mathcal{X}(k_a,k_b)], \tag{C.114}$$

where $\psi^a_{k_a}(k_a,k_b) = (k_a+k_b)^{-\frac{1}{\gamma}-1}\frac{(1-\frac{1}{\gamma})k_a+k_b}{r-\mu}$. Using $r>\mu$ and $\gamma>1$, we have $\psi^a_{k_a}(k_a,k_b)>0$. If $H^a_{k_a}(k_a,k_b)\geq 0$, then $V^a_{k_a}(x,k_a,k_b)$ is increasing in $x\in[0,\mathcal{X}(k_a,k_b)]$. Hence, for any $x\in(0,\mathcal{X}(k_a,k_b))$,

$$V^a_{k_a}(x,k_a,k_b) < V^a_{k_a}(\mathcal{X}(k_a,k_b),k_a,k_b) = p, \tag{C.115}$$

where the equality uses (B.4).

If $H^a_{k_a}(k_a,k_b)<0$, then $\psi^a_{k_a}(k_a,k_b)x + H^a_{k_a}(k_a,k_b)x^\beta$ is first increasing and then decreasing in the region $x\geq 0$. Denote $\hat{x} := \arg\max_{x\in[0,\mathcal{X}(k_a,k_b)]}[\psi^a_{k_a}(k_a,k_b)x + H^a_{k_a}(k_a,k_b)x^\beta]$. We have

$\hat{x} > 0$. If $\hat{x} = \mathcal{X}(k_a, k_b)$, then $\psi^a_{k_a}(k_a, k_b)x + H^a_{k_a}(k_a, k_b)x^\beta$ is increasing in $x \in [0, \mathcal{X}(k_a, k_b)]$. Hence, we conclude from (C.114) that $V^a_{k_a}(x, k_a, k_b) < V^a_{k_a}(\mathcal{X}(k_a, k_b), k_a, k_b) = p$ for all $x \in (0, \mathcal{X}(k_a, k_b))$. If $\hat{x} < \mathcal{X}(k_a, k_b)$, then we have $\hat{x} \in (0, \mathcal{X}(k_a, k_b))$ and $\frac{\partial V^a_{k_a}(x,k_a,k_b)}{\partial x} \mid_{x=\hat{x}} = 0$, which implies that $\psi^a_{k_a}(k_a, k_b) + \beta H^a_{k_a}(k_a, k_b)\hat{x}^{\beta-1} = 0$. It follows that

$$\begin{aligned}\psi^a_{k_a}(k_a, k_b)\hat{x} + H^a_{k_a}(k_a, k_b)\hat{x}^\beta &= \frac{\beta-1}{\beta}\psi^a_{k_a}(k_a, k_b)\hat{x} \\ &< \frac{\beta-1}{\beta}(k_a+k_b)^{-\frac{1}{\gamma}-1}\frac{(1-\frac{1}{\gamma})k_a + k_b}{r-\mu}\mathcal{X}(k_a, k_b) \leq p,\end{aligned} \tag{C.116}$$

where the first inequality holds because of $\psi^a_{k_a}(k_a, k_b) = (k_a + k_b)^{-\frac{1}{\gamma}-1}\frac{(1-\frac{1}{\gamma})k_a+k_b}{r-\mu} > 0$ and $\beta > \gamma > 1$, and the second inequality uses (A.12) and (A.4). Combining (C.116) with (C.114), we have $V^a_{k_a}(x, k_a, k_b) \leq V^a_{k_a}(\hat{x}, k_a, k_b) < p$ for any $x \in (0, \mathcal{X}(k_a, k_b))$. Therefore, (C.95) holds.

Step 2: We prove that (C.100) holds.

Using (5.41) and (A.2), we have

$$U^a_{k_b}(x, k_a, k_b) = \psi^a_{k_b}(k_a, k_b)x + H^a_{k_b}(k_a, k_b)x^\beta, \quad x \in (0, \mathcal{X}(k_a, k_b)], \tag{C.117}$$

where $\psi^a_{k_b}(k_a, k_b) = (k_a + k_b)^{-\frac{1}{\gamma}-1}\frac{(-\frac{1}{\gamma})k_a}{r-\mu}$. Thus, (C.100) is equivalent to $H^a_{k_b}(k_a, k_b) < \mathcal{H}(k_a, k_b)$ for $k_b > k_a \geq \underline{k}$, where $\mathcal{H}(k_a, k_b) := -\psi^a_{k_b}(k_a, k_b)\mathcal{X}(k_a, k_b)^{1-\beta}$. Additionally, we can derive from (A.6) that $H^a_{k_b}(k_a, k_b) = \mathcal{H}(k_a, k_b)$ for $k_a = k_b$. As a result, (C.100) is equivalent to (C.101).

Then, we only need to show that $H^a_{k_a k_b}(k_a, k_b) > \frac{\partial \mathcal{H}(k_a,k_b)}{\partial k_a}$ for any $k_b > k_a \geq \underline{k}$. Using (A.5), we have

$$\begin{aligned}&H^a_{k_a k_b}(k_a, k_b) - \frac{\partial \mathcal{H}(k_a, k_b)}{\partial k_a} \\ =&\frac{(\beta-1)\Big[\mathcal{X}(k_a, k_b)\big[\mathcal{X}_{k_b}(k_a, k_b)\psi^a_{k_a}(k_a, k_b) - \mathcal{X}_{k_a}(k_a, k_b)\psi^a_{k_b}(k_a, k_b)\big] - p\frac{\beta}{\beta-1}\mathcal{X}_{k_b}(k_a, k_b)\Big]}{\mathcal{X}(k_a, k_b)^{\beta+1}} \\ =&\frac{(\beta-1)\Big[\frac{\mathcal{X}(k_a,k_b)}{(k_a+k_b)^{\frac{1}{\gamma}}}\Big[\mathcal{X}_{k_b}(k_a, k_b) + (\mathcal{X}_{k_a}(k_a, k_b) - \mathcal{X}_{k_b}(k_a, k_b))\frac{k_a}{\gamma(k_a+k_b)}\Big] - \rho\mathcal{X}_{k_b}(k_a, k_b)\Big]}{(r-\mu)\mathcal{X}(k_a, k_b)^{\beta+1}} \\ =&\frac{(\beta-1)\mathcal{X}_{k_b}(k_a, k_b)\Big[\frac{\mathcal{X}(k_a,k_b)}{(k_a+k_b)^{\frac{1}{\gamma}}}\Big[1 + \big(\frac{\vartheta_+(k_a,k_b)}{\vartheta_-(k_a,k_b)} - 1\big)\frac{k_a}{\gamma(k_a+k_b)}\Big] - \rho\Big]}{(r-\mu)\mathcal{X}(k_a, k_b)^{\beta+1}},\end{aligned} \tag{C.118}$$

where the second equality uses $\psi^a_{k_a}(k_a,k_b)=(k_a+k_b)^{-\frac{1}{\gamma}-1}\frac{(1-\frac{1}{\gamma})k_a+k_b}{r-\mu}$ and $\psi^a_{k_b}(k_a,k_b)=(k_a+k_b)^{-\frac{1}{\gamma}-1}\frac{(-\frac{1}{\gamma})k_a}{r-\mu}$, and the last equality uses $\vartheta_+(k_a,k_b)=\mathcal{X}_{k_a}(k_a,k_b)$, $\vartheta_-(k_a,k_b)=\mathcal{X}_{k_b}(k_a,k_b)$ for $k_b>k_a$.

Since (A.11) implies $\mathcal{G}(k_a,k_b)>0$, we have $\mathcal{X}(k_a,k_b)>\rho(k_a+k_b)^{\frac{1}{\gamma}}$ for any $k_a\geq\underline{k}$, $k_b\geq\underline{k}$. Plugging it and $\frac{\vartheta_-(k_a,k_b)}{\vartheta_+(k_a,k_b)}\in(0,1)$ (see Footnote 30) into (C.118), we have $H^a_{k_ak_b}(k_a,k_b)>\frac{\partial\mathcal{H}(k_a,k_b)}{\partial k_a}$ for any $k_b>k_a\geq\underline{k}$. Hence, (C.100) holds.

Step 3: We prove that

$$V^a_{k_b}(x,k_a,k_b)<0,\quad (k_a,k_b)\in\mathbf{N}_b(x),\ k_a\geq\underline{k},\ k_b\geq\underline{k},\ x>0. \tag{C.119}$$

We can see from (C.117) that for $x\in(0,\mathcal{X}(k_a,k_b)]$, $U^a_{k_b}(x,k_a,k_b)/x=\psi^a_{k_b}(k_a,k_b)+H^a_{k_b}(k_a,k_b)x^{\beta-1}$ is linear in $x^{\beta-1}>0$ and satisfies $\lim_{x\to0+}U^a_{k_b}(x,k_a,k_b)/x=\psi^a_{k_b}(k_a,k_b)=(k_a+k_b)^{-\frac{1}{\gamma}-1}\frac{(-\frac{1}{\gamma})k_a}{r-\mu}<0$. Using (5.46) and (C.100), we have $\lim_{x\to\mathcal{X}(k_a,k_b)-}U^a_{k_b}(x,k_a,k_b)/x\leq 0$. Hence, we have $V^a_{k_b}(x,k_a,k_b)=U^a_{k_b}(x,k_a,k_b)<0$ for any $x\in(0,\mathcal{X}(k_a,k_b))$.

In addition, for any $(k_a,k_b)\in\mathbf{I}_a(x)\setminus\mathbf{I}_{ab}(x)$ with $k_a\geq\underline{k}$, $k_b\geq\underline{k}$, it follows from equation (5.43) that

$$V^a_{k_b}(x,k_a,k_b)=U^a_{k_b}(x,\widehat{k}_a,k_b)<0, \tag{C.120}$$

where $\widehat{k}_a$ is determined by $\mathcal{X}(\widehat{k}_a,k_b)=x$, the equality is derived from (C.15), and the inequality follows from (C.100). Since $\mathbf{N}_{ab}(x)\cup(\mathbf{I}_a(x)\setminus\mathbf{I}_{ab}(x))\supseteq\mathbf{N}_b(x)$, we infer (C.119).

□

Proof of Lemma 10 We only prove that $(\varphi_a,\varphi_b)\in\mathcal{S}$. The proof of Conditions 3-4 and Footnote 10 is similar to that of Lemma 3.

According to Footnote 30, we have $\vartheta_+(k_a,k_b)>\vartheta_-(k_a,k_b)>0$. Then, we derive from (6.9) that there exists a unique pair $(u_a,u_b)\in\mathbb{R}^2_+$ satisfying (2.12). In addition, we can see that (5.19)-(5.28) hold with $\eta\rho\theta_+$ and $\eta\rho\theta_-$ replaced by $\vartheta_+(k_a,k_b)$ and $\vartheta_-(k_a,k_b)$, respectively. Hence, Definition 1-(i) holds.

By (5.28), we have $K_{it}\leq K_{i0}\vee\mathcal{K}(M_t)$, $t\geq0$. Recall that Condition 7 implies $\mathcal{X}(k_a,k_b)\geq\rho(k_a+k_b)^{\frac{1}{\gamma}}$, and $\mathcal{K}(x)$ is defined in (6.8). It follows that there exists a constant $C_1>0$ such that $K_{it}\in[K_{i0},C_1(1+|M_t|^\gamma)]$, $\forall\ t\geq0$, $i=a,b$. Using $F_i(x,k_a,k_b)=x(k_a+k_b)^{-1/\gamma}k_i$,

$K_{it} \in [K_{i0}, C_1(1+|M_t|^\gamma)]$ and $X_t \in [0, M_t]$, $\forall\, t \geq 0$, we can see there exists a constant $C_2 > 0$ such that $|F_i(X_t, K_{at}, K_{bt})| \leq C_2(1+|M_t|^{\gamma+1})$, $\forall\, t \geq 0$. Recalling (C.62), (C.63), and (C.64), we can see the conditions in Definition 1-(ii) are satisfied, and $\varphi = (\varphi_a, \varphi_b) \in \mathcal{S}$.

□

Proof of Theorem 5 We can see that Theorem 1 also holds for the region $\mathbb{Z}$ defined in (6.7). Hence, using Theorem 1, Lemmas 9 and 10, we can see that Theorem 5 holds.

□

Proof of Proposition 4 We only need to verify that Condition 7 holds.

We first consider the case $\phi(z_1, z_2) = z_1 - \lambda$ for a constant $\lambda > 0$. Then, $\mathcal{G} = \frac{\lambda}{k_a}$ solves the equation (A.9). Since $\frac{\rho k_a k_b}{\gamma(k_a+k_b)-k_b} = \frac{\rho}{\gamma/k_b+(\gamma-1)/k_a}$ is increasing in $k_a \geq \underline{k}$ and $k_b \geq \underline{k}$, we have

$$\frac{\rho k_a k_b}{\gamma(k_a+k_b)-k_b} \geq \frac{\rho \underline{k}^2}{2\gamma\underline{k}-\underline{k}} = \frac{\rho\underline{k}}{2\gamma-1} > \lambda, \tag{C.121}$$

where the second inequality uses $\underline{k} > \frac{\lambda(2\gamma-1)}{\rho}$. Hence, Condition 7-(i) holds. Using $\gamma > 1$ and $\rho > 0$, we can see that $\big(\rho + \mathcal{G}(k_a, k_b)\big)(k_a+k_b)^{\frac{1}{\gamma}} = \big(\rho + \frac{\lambda}{k_a}\big)(k_a+k_b)^{\frac{1}{\gamma}}$ is increasing in k_b and then Condition 7-(ii) holds.

Next, we consider the case $\phi(z_1, z_2) = \frac{\ln(\lambda/z_1)}{z_1} + z_2$ for a constant $\lambda > 0$. Then, $\mathcal{G} = \sqrt{(\frac{\rho}{2})^2 + \frac{\lambda}{k_a(k_a+k_b)^{\frac{1}{\gamma}}}} - \frac{\rho}{2}$ solves the equation (A.9). Since $\lambda > 0$, we can see $\mathcal{G} > 0$ for any $k_a \geq \underline{k}$, $k_b \geq \underline{k}$. Next, we prove the second inequality in Condition 7-(i), which is equivalent to

$$\sqrt{\Big(\frac{\rho}{2}\Big)^2 + \frac{\lambda}{k_a(k_a+k_b)^{\frac{1}{\gamma}}}} < \frac{\rho k_b}{\gamma(k_a+k_b)-k_b} + \frac{\rho}{2}. \tag{C.122}$$

Using $\frac{\rho k_b}{\gamma(k_a+k_b)-k_b} + \frac{\rho}{2} = \frac{\rho}{2}\frac{\gamma(k_a+k_b)+k_b}{\gamma(k_a+k_b)-k_b}$, one can see (C.122) is equivalent to

$$\Big(\frac{\rho}{2}\Big)^2 + \frac{\lambda}{k_a(k_a+k_b)^{\frac{1}{\gamma}}} < \Big(\frac{\rho}{2}\Big)^2\Big(\frac{\gamma(k_a+k_b)+k_b}{\gamma(k_a+k_b)-k_b}\Big)^2. \tag{C.123}$$

Using $\Big(\frac{\gamma(k_a+k_b)+k_b}{\gamma(k_a+k_b)-k_b}\Big)^2 - 1 = \frac{4\gamma(k_a+k_b)k_b}{(\gamma(k_a+k_b)-k_b)^2}$, one can see (C.123) is equivalent to

$$\lambda < \gamma\rho^2\frac{k_a k_b (k_a+k_b)^{1+\frac{1}{\gamma}}}{(\gamma(k_a+k_b)-k_b)^2}. \tag{C.124}$$

We prove that $\frac{k_a k_b (k_a+k_b)^{1+\frac{1}{\gamma}}}{(\gamma(k_a+k_b)-k_b)^2}$ is increasing in k_a and k_b in the region $k_a \geq k_b \geq \underline{k}$. Denote $f(k_a,k_b) := \ln\left(\frac{k_a k_b (k_a+k_b)^{1+\frac{1}{\gamma}}}{(\gamma(k_a+k_b)-k_b)^2}\right) = \ln k_a + \ln k_b + (1+\frac{1}{\gamma})\ln(k_a+k_b) - 2\ln(\gamma(k_a+k_b)-k_b)$. We have

$$\frac{\partial f(k_a,k_b)}{\partial k_a} = \frac{1}{k_a} + \left(1+\frac{1}{\gamma}\right)\frac{1}{k_a+k_b} - \frac{2\gamma}{\gamma(k_a+k_b)-k_b}, \tag{C.125}$$

$$\frac{\partial f(k_a,k_b)}{\partial k_b} = \frac{1}{k_b} + \left(1+\frac{1}{\gamma}\right)\frac{1}{k_a+k_b} - \frac{2(\gamma-1)}{\gamma(k_a+k_b)-k_b}. \tag{C.126}$$

We can rewrite $\frac{\partial f(k_a,k_b)}{\partial k_a}$ as follows:

$$\frac{\partial f(k_a,k_b)}{\partial k_a} = \frac{k_a^2 + (\gamma-1)k_b^2 + (\gamma-1-\frac{1}{\gamma})k_a k_b}{k_a(k_a+k_b)(\gamma(k_a+k_b)-k_b)}. \tag{C.127}$$

For any $k_a \geq k_b \geq \underline{k}$, we have $\frac{\partial\left(k_a^2+(\gamma-1)k_b^2+(\gamma-1-\frac{1}{\gamma})k_a k_b\right)}{\partial k_a} = 2k_a + (\gamma-1-\frac{1}{\gamma})k_b \geq 2k_b + (\gamma-1-\frac{1}{\gamma})k_b > 0$, and $(k_a^2+(\gamma-1)k_b^2+(\gamma-1-\frac{1}{\gamma})k_a k_b)\ |_{k_a=k_b} = (2\gamma-1-\frac{1}{\gamma})k_b^2 > 0$. Then, we derive from (C.127) that $\frac{\partial f(k_a,k_b)}{\partial k_a} > 0$. Comparing (C.125) with (C.126), we have $\frac{\partial f(k_a,k_b)}{\partial k_b} \geq \frac{\partial f(k_a,k_b)}{\partial k_a} > 0$ for any $k_a \geq k_b \geq \underline{k}$.

Hence, $f(k_a,k_b)$, and consequently $\frac{k_a k_b (k_a+k_b)^{1+\frac{1}{\gamma}}}{(\gamma(k_a+k_b)-k_b)^2}$, are increasing in k_a and k_b in the region $k_a \geq k_b \geq \underline{k}$. It follows that (C.124) holds for any $k_a \geq k_b \geq \underline{k}$ if and only if (C.124) holds for $k_a = k_b = \underline{k}$, which is equivalent to $\underline{k} > \frac{1}{2}\left(\frac{(2\gamma-1)^2\lambda}{\gamma\rho^2}\right)^{\frac{\gamma}{\gamma+1}}$. Hence, Condition 7-(i) holds for any $\underline{k} > \frac{1}{2}\left(\frac{(2\gamma-1)^2\lambda}{\gamma\rho^2}\right)^{\frac{\gamma}{\gamma+1}}$.

Since $\mathcal{G}(k_a,k_b) = \frac{\lambda(k_a+k_b)^{\frac{-1}{\gamma}}}{\sqrt{(\frac{\rho k_a}{2})^2+\lambda k_a(k_a+k_b)^{\frac{-1}{\gamma}}}+\frac{\rho k_a}{2}}$, we have

$$\mathcal{G}(k_a,k_b)(k_a+k_b)^{\frac{1}{\gamma}} = \frac{\lambda}{\sqrt{(\frac{\rho k_a}{2})^2+\lambda k_a(k_a+k_b)^{\frac{-1}{\gamma}}}+\frac{\rho k_a}{2}}, \tag{C.128}$$

which is increasing in k_b. Hence, Condition 7-(ii) holds.

□