Valuing Private Equity

Morten Sorensen
Columbia Business School and NBER

Neng Wang
Columbia Business School and NBER

Jinqiang Yang
The School of Finance, Shanghai University of Finance and Economics

We investigate whether the performance of private equity (PE) investments is sufficient to compensate investors (LPs) for risk, long-term illiquidity, management, and incentive fees charged by the general partner (GP). We analyze the LPs' portfolio-choice problem and find that management fees, carried interest, and illiquidity are costly, and GPs must generate substantial alpha to compensate LPs for bearing these costs. Debt is cheap and reduces these costs, potentially explaining the high leverage of buyout transactions. Conventional interpretations of PE performance measures appear optimistic. On average, LPs may just break even, net of management fees, carry, risk, and costs of illiquidity. (*JEL* G11, G23, G24)

While it is generally accepted that the average return of private equity (PE) funds exceeds the return on the market (e.g., Harris, Jenkinson, and Kaplan 2011), it remains controversial whether this outperformance is sufficient to compensate investors (LPs) for the costs of risks and long-term illiquidity. Moreover, there is little formal analysis of these costs to help guide LPs' investment decisions, despite their large and increasing PE allocations (e.g., Yale University's endowment currently targets a 34% PE allocation).

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To evaluate the costs of PE investments, we analyze the portfolio-choice problem of a risk-averse LP that invests in a risk-free asset, and public and private equity. Our model captures four key institutional features of PE investments.[1] First, PE investments are illiquid and long term. PE funds have ten-year maturities and the secondary market for PE positions is opaque, making it difficult for LPs to rebalance their PE investments. Second, PE investments are risky. Part of this risk is spanned by publicly-traded liquid assets and hence commands the standard risk premium for systematic risk. The remaining part of this risk, however, is not spanned by the market, due to illiquidity, and the LP requires an additional premium for holding this risk. Third, the management of the PE fund is delegated to a general partner (GP), who receives both an annual management fee, typically 1.5%–2% of the committed capital, and a performance-based incentive fee (carried interest), typically 20% of profits. Intuitively, management fees resemble a fixed-income stream and the carried interest resembles a call option on the fund's underlying portfolio companies. Fourth, to compensate the LP for bearing the unspanned risk as well as management and performance fees, the GP must generate sufficient risk-adjusted excess return (alpha) by effectively managing the fund's assets.

We first consider the full-spanning case. In this case, the risk of the PE investment is fully spanned by publicly-traded securities, and there is no cost of illiquidity. This full-spanning case provides a powerful and tractable framework for valuing the GP's compensation, including management and incentive fees (carried interest). Even with full spanning, our pricing formula differs from the standard Black-Scholes option pricing formula, because our model must allow for the GP's value-adding skill (alpha), while the Black-Scholes (1973) formula has no room for risk-adjusted excess returns for any security. Quantitatively, in present value terms, we find that both management fees and carried interest contribute to the GP's total compensation. For the LP, the cost of management fees is between one-half and two-thirds of the total cost of fees, which is in line with previous findings by Metrick and Yasuda (2010).

More generally, when the PE investment has non-spanned risk, our model allows us to quantify the cost of illiquidity. We derive a non-linear differential equation for the LP's certainty-equivalent valuation and obtain analytical solutions for the optimal hedging portfolio and consumption rules. Intuitively, when markets are incomplete and the risk of the PE investment is not fully spanned by the market, the standard law-of-one-price does not hold. Unlike the standard Black-Scholes (1973) formula, our framework incorporates alpha, management fees, carried interest, illiquidity, and the non-linear pricing of unspanned risk.

After calibrating the model, we calculate the discount in the LP's valuation of the PE investment that is due to illiquidity and the break-even alpha, which

[1] See Gompers and Lerner (2002) and Metrick and Yasuda (2010, 2011) for detailed discussions of the institutional features of these investments.

we define as the alpha that the GP must generate to compensate the LP for the costs of illiquidity and the GP's compensation. Using both measures, we find that the cost of illiquidity is large. As a benchmark, the total magnitude of the cost of illiquidity is comparable to the total cost of the GP's compensation, including both management fees and carried interest.

The break-even alpha is the LP's additional cost of capital of the PE investment, due to illiquidity and the GP's compensation, beyond the standard CAPM-implied cost of capital. Interestingly, this cost does not increase monotonically with the investment horizon. Instead, it follows a U-shaped pattern. For shorter horizons, the cost increases due to an optionality effect of the GP's carried interest. At intermediate horizons, the cost is lower. At longer horizons, the cost increases again due to the long-term cost of illiquidity.

We assume that leverage is provided by diversified investors. The LP cannot trade the debt used to leverage the PE asset. If it were possible, the LP would want a short position because the debt is expensive for the LP as it is priced without illiquidity premium and shorting the debt partially hedges the non-spanned risk for the LP.

Quantitatively, we find that without leverage, the break-even alpha is 3.08% annually with the typical 2/20 compensation contract. Interestingly, leverage reduces this alpha. To illustrate, Axelson et al. (2011) report a historical average debt to equity (D/E) ratio of 3.0 for buyout (BO) transactions. In our baseline calibration, increasing the D/E ratio to 3.0 reduces the break-even alpha from 3.08% to 2.06% per year. The benefits of leverage are twofold: First, for a given size of the LP's investment, leverage increases the total size of PE assets for which the GP generates alpha, effectively reducing the fees per dollar of unlevered assets. Second, leverage transfers some of the risk of the PE investment to diversified creditors. This may provide an answer to the "PE leverage puzzle" from Axelson et al. (2011). They find that the credit market is the primary predictor of leverage used in PE transactions, and that PE funds appear to use as much leverage as tolerated by the market.[2] This behavior is inconsistent with standard theories of capital structure (see also Axelson, Stromberg, and Weisbach 2009). In our model it is optimal.

To summarize, of the LP's total costs of the PE investment, approximately 50% are due to illiquidity, 25% are due to management fees, and the remaining 25% are due to carried interest. In present value terms, when the LP has a total committed capital of \$125, resulting in an initial investment of \$100, the LP's cost of the GP's compensation is 50.97, and the LP's cost of illiquidity is 40.48. Hence, GPs need to create substantial value to cover these costs.

[2] In their conclusion, Axelson et al. (2011) state that "the factors that predict capital structure in public companies have no explanatory power for buyouts. Instead, the main factors that do affect the capital structure of buyouts are the price and availability of debt; when credit is abundant and cheap, buyouts become more leveraged [...] Private equity practitioners often state that they use as much leverage as they can to maximize the expected returns on each deal. The main constraint they face, of course, is the capital market, which limits at any particular time how much private equity sponsors can borrow for any particular deal."

Our model produces tractable expressions for the performance measures used in practice. Given the difficulties of estimating traditional risk and return measures such as CAPM alphas and betas, some alternative performance measures have been adopted, such as the internal rate of return (IRR) and the public market equivalent (PME). While these alternative measures are easier to compute, they are more difficult to interpret. Harris, Jenkinson, and Kaplan (2011) report a value-weighted average PME of 1.27 and conclude that "buyout funds have outperformed public markets in the 1980s, 1990s, and 2000s."[3] Whether or not this outperformance is sufficient to compensate LPs for the illiquidity and other frictions can be evaluated within our model. Given the break-even alpha, we calculate the corresponding break-even values of the IRR and PME measures. We find that these break-even values are close to their empirical counterparts. Our baseline calibration gives a break-even PME of 1.30, suggesting that the empirical average of 1.27 is just sufficient for LPs to break even on average.[4] While the exact break-even values depend on the specific calibration, a more general message is that the traditional interpretation of these performance measures may be misleading.

Our analysis relates to the literature about valuation and portfolio choice with illiquid assets, such as restricted stocks, executive compensation, non-traded labor income, illiquid entrepreneurial businesses, and hedge fund lock-ups. For example, Svensson and Werner (1993), Duffie et al. (1997), Koo (1998), and Viceira (2001) study consumption and portfolio choices with non-tradable labor income. Kahl, Liu, and Longstaff (2003) analyze a continuous-time portfolio choice model with restricted stocks. Chen, Miao, and Wang (2010) and Wang, Wang, and Yang (2012) study entrepreneurial firms under incomplete markets. For hedge funds, Goetzmann, Ingersoll, and Ross (2003), Panageas and Westerfield (2009), and Lan, Wang, and Yang (2013) analyze the impact of management fees and high-water mark based incentive fees on leverage and valuation. Ang, Papanikolaou, and Westerfield (2012) analyze a model with an illiquid asset that can be traded and rebalanced at Poisson arrival times. We are unaware, though, of any existing model that captures the illiquidity, managerial skill (alpha), and compensation of PE investments. Capturing these institutional features in a model that is sufficiently tractable to evaluate actual PE performance is a main contribution of this study.

[3] As explained in the text, the PME is calculated by dividing the present value (PV) of the cash flows distributed to the LP by the PV of the cash flows paid by the LP, where the PV is calculated using the realized market return as the discount rate. A PME exceeding one is typically interpreted as outperformance relative to the market.

[4] Kaplan and Schoar (2005) find substantial persistence in the performance of subsequent PE funds managed by the same PE firm, indicating that PE firms differ in their quality and ability to generate returns. Lerner, Schoar, and Wongsunwai (2007) find systematic variation in PE performance across LP types, suggesting that LPs differ in their ability to identify and access high-quality PE firms. Hence, some specific LPs may consistently outperform (or underperform) the average.

1. Model

An institutional investor with an infinite horizon invests in three assets: a risk-free asset, public equity, and private equity. The risk-free asset and public equity represent the standard investment opportunities as in the classic Merton (1971) model. The risk-free asset pays a constant interest rate r. Public equity can be interpreted as the public market portfolio, and its value, S_t, follows the geometric Brownian motion (GBM):

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dB_t^S, \tag{1}$$

where B_t^S is a standard Brownian motion, and μ_S and σ_S are the constant drift and volatility parameters. The Sharpe ratio for the public equity is:

$$\eta = \frac{\mu_S - r}{\sigma_S}. \tag{2}$$

1.1 Private equity

The PE investment is structured as follows. At time 0, the investor, acting as a limited partner (LP), commits X_0 (the committed capital) to a PE fund. This PE fund is managed by a PE firm, which acts as the general partner (GP) of the fund. Of this committed capital, only I_0 is invested immediately. The remaining $X_0 - I_0$ is retained by the LP to cover subsequent management fees over the life of the fund. We provide expressions for $X_0 - I_0$ and I_0 below, when we discuss management fees.

At time 0, the PE fund receives the initial investment of I_0 from the LP. The fund leverages this initial investment with D_0 of debt to acquire at total of $A_0 = I_0 + D_0$ worth of underlying portfolio companies. In reality, a typical PE fund gradually acquires 10-20 portfolio companies, but for tractability we do not model these individual acquisitions. Instead, we refer to the entire portfolio of the fund's companies collectively as the PE asset, and we assume that the entire portfolio is acquired at time 0. In practice, the leverage is obtained at the portfolio company level, not by the fund. Hence, D_0 represents the total debt of the portfolio companies after they have been acquired by the PE fund, and the initial value of the PE asset, A_0, is the total unlevered value of these portfolio companies. Let $l = D_0/I_0$ denote the initial D/E ratio. For example, with invested capital of $I_0 = \$100$ and leverage of $l = 3$, the fund acquires $A_0 = \$400$ worth of companies and finances part of these acquisitions by imposing $D_0 = \$300$ of debt on these companies.

The PE fund has maturity T. In practice, PE funds have ten-year horizons, which can sometimes be extended by a few additional years. When the fund matures, the fund and the PE asset are liquidated and the proceeds are distributed to the creditors, the GP, and the LP, who will not invest in another one. After maturity, the LP will only invest in the risk-free asset and public equity, reducing the LP's problem to the standard Merton (1971) portfolio problem.

It is important to distinguish the LP's partnership interest from the underlying PE asset. The LP's partnership interest is the LP's claim on the PE fund, including the obligation to pay the remaining management fees and the right to the eventual proceeds from the sale of the PE asset, net of the GP's carried interest.[5] In contrast, the PE asset represents the total unlevered value of the corporate assets of the underlying portfolio companies, owned by the fund. To consistently evaluate the effects of illiquidity, risk, leverage, and GP's value-adding activities (alpha), we specify the value process of the underlying PE asset. Given this value process, we can then value the LP's partnership interest as an illiquid claim on the underlying PE asset.

1.1.1 Risk of PE asset. The PE asset is illiquid, and the PE fund must hold it to maturity, T. As a starting point, we assume that the maturity equals the life of the PE fund. Below, we also consider shorter maturities, corresponding to the holding periods of the individual portfolio companies. The value of the PE asset is the total unlevered value of the portfolio companies. Between times 0 and T, the value of the PE asset, A_t, follows the GBM:

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma_A dB_t^A, \tag{3}$$

where B_t^A is a standard Brownian motion, μ_A is the drift, and σ_A is the volatility. At time T, the PE asset is liquidated for total proceeds of A_T, and these proceeds are divided among the creditors, LP, and GP according to the "waterfall" structure specified below.

To capture its systematic risk, the return on the PE asset is correlated with the return on the public market, and the correlation between the B_t^S and B_t^A processes is denoted ρ. When $|\rho| < 1$, the two processes are not perfectly correlated; the risk of the PE asset is not fully spanned by the market, and the LP cannot fully hedge the risk of the PE investment by dynamically trading the public equity and risk-free asset.

The unlevered beta (or asset beta) of the PE asset is given as:

$$\beta = \frac{\rho \sigma_A}{\sigma_S}. \tag{4}$$

This is the unlevered beta of the return on the entire PE asset relative to the return on the public market. In contrast, the levered beta (or equity beta) of the levered return on the LP's partnership interest may be several times greater than the unlevered beta, β. It is useful, though, to define the systematic risk in terms of the asset beta of the underlying PE asset, because this asset beta

[5] In practice, PE funds have several LPs, which typically share the value of the fund *pro rata*. We interpret X_0 as the LP's share of the total fund. If the total committed capital to the PE fund is $\overline{X}_0$, and a given LP owns the share s of the fund, then $X_0 = s\overline{X}_0$. Alternatively, we can interpret the LP in the model as representing the aggregate collection of LPs.

can be assumed constant, and it enables us to consistently evaluate the effects of changes in compensation structure and leverage, while accounting for the implied changes in the risks of the LP's partnership interest.

The total volatility of the PE asset is σ_A. The fraction of this volatility that is spanned by the public market is $\rho\sigma_A$. The remaining unspanned volatility is denoted ϵ, given as:

$$\epsilon=\sqrt{\sigma_A^2-\rho^2\sigma_A^2}=\sqrt{\sigma_A^2-\beta^2\sigma_S^2}. \tag{5}$$

The unspanned volatility introduces an additional risk into the LP's overall portfolio. The spanned and unspanned volatilities play distinct roles in the valuation of the LP's partnership interest, and the LP requires different risk premia for bearing these two risks.[6]

1.1.2 Return of PE asset. An important feature of our model is that it allows the value of the underlying PE asset to appreciate faster than the overall market and earn an excess risk-adjusted return, called alpha.[7] Formally, alpha is defined as:

$$\alpha=\mu_A-r-\beta(\mu_S-r). \tag{6}$$

Intuitively, the alpha is the CAPM risk-adjusted excess return of the portfolio companies that are managed by the PE fund. This value may arise from improved governance, management efficiency (e.g., Jensen, 1989), or from the non-spanned illiquidity risk, as we show later.

1.1.3 Discussion. We define the alpha and beta relative to the public market portfolio and not relative to the LP's entire portfolio, which also contains the partnership interest in the PE fund. With appropriate data, the alpha and beta can be estimated by regressing the return earned on PE transactions on the return of the public market portfolio. Empirical studies of PE performance also measure PE performance relative to the public market portfolio, and our definitions allow us to use existing estimates to calibrate our model. Defining alphas and betas relative to the LP's entire portfolio, containing both public and private equity, is impractical because the value of the LP's partnership interest is difficult to estimate and may require a different valuation framework, as indicated by our analysis.

The alpha and beta are defined in terms of the performance of the underlying unlevered PE asset, not the levered performance of the LP's partnership interest,

[6] We assume that $X_0>0$, and that the GP cannot short the investment in the PE fund. Further, we assume that the LP cannot trade the debt used to leverage the PE asset. The LP would want a short position in this debt to partially hedge the non-spanned risk.

[7] Our model can also accommodate value created by levering the PE asset with "cheap" debt. In this paper, however, we only consider debt priced in equilibrium. Ivashina and Kovner (2010) provide empirical evidence of the use of cheap debt financing in PE transactions.

which is obviously the performance the LP cares about. Defining the alpha and beta for the PE asset is useful for a number of reasons: (1) it provides a way to assess the value that GPs must create for the portfolio companies that they manage, independently of the leverage of these companies; (2) it allows us to fix the underlying real value-generating process of the PE asset, and evaluate the effects of changes in management compensation, leverage, risk aversion, and illiquidity; and (3) it makes the model easier to calibrate, because we can use estimates for the performance of individual portfolio companies.

1.2 Capital structure and waterfall

The GP's compensation consists of management fees and incentive fees, which are also known as "carried interest" (we use the terms "incentive fees," "performance fees," and "carried interest" interchangeably). The management fee is an ongoing payment by the LP to the GP, specified as a fraction m (typically 1.5% or 2%) of the committed capital, X_0. The committed capital is the sum of the initial investment, I_0, and the total management fees paid over the life of the fund:

$$X_0 = I_0 + mTX_0. \tag{7}$$

For example, an LP that commits $X_0 = \$125$ to a fund with a management fee of $m = 2\%$ pays an annual management fee of \$2.5. Over ten years, these management fees add up to \$25, leaving the remaining \$100 of the initially committed capital for the initial investment, so $I_0 = \$100$. With leverage of $l = 3$, this initial investment enables the GP to acquire $A_0 = \$400$ worth of PE assets. Hence, leverage enables the GP to charge lower management fees per dollar of PE assets under management. Without leverage, the GP would charge an annual fee of $2.5\% = \$2.5/\$100$ of assets under management. With $l = 3$, this fee per dollar of PE assets declines to 0.625% = \$2.5/\$400, only a quarter of 2.5%. Note that (7) ignores the time value of money, but when we conduct our valuation exercises, obviously, we account for the various effects of both time and risk.

When the fund matures, the final proceeds, A_T, are divided among the creditors, the LP, and the GP according to the "waterfall" schedule. For the creditors, let y denote the (continuously-compounded) yield on the debt. Assuming balloon debt, the payment due to the creditors, at maturity T, is:

$$Z_0 = D_0 e^{yT}, \tag{8}$$

which includes both principal D_0 and interest payments. Any remaining proceeds after repaying the creditors, Z_0, and returning the LP's committed capital, X_0, constitute the fund's profits, given as:

$$A_T - X_0 - Z_0. \tag{9}$$

These profits are divided between the GP and LP, and the GP's share is the carried interest. The LP's total payoff is illustrated in Figure 1. This figure

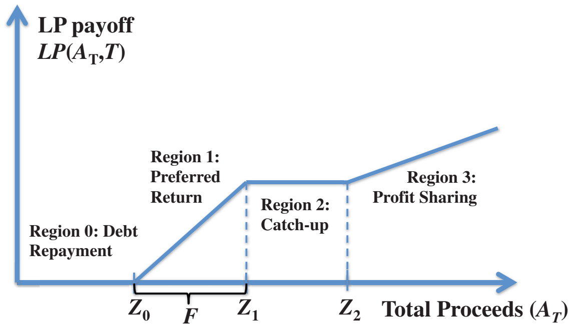

Figure 1
The LP's payoff diagram
This figure illustrates the LP's final payoff, $LP(A_T,T)$, as a function of the total proceeds, A_T, across the four regions of the waterfall structure.

shows the four regions of the waterfall structure, depending on the amount of final proceeds, A_T. These four regions are described next.

1.2.1 Region 0: Debt repayment ($A_T \leq Z_0$). Our model applies to general forms of debt, but for simplicity we consider balloon debt with no intermediate payments. The principal and accrued interest are due at maturity T. Let y denote the yield for the debt, which we derive below to ensure creditors break even. The debt is risky, and at maturity T, the payment to the creditors is:

$$D(A_T,T)=\min\{A_T,Z_0\}. \tag{10}$$

The upper boundary of the debt-repayment region is $Z_0=D_0e^{yT}$. The debt is senior, though, and when the final proceeds, A_T, fall below this boundary, the LP and GP receive nothing.

1.2.2 Region 1: Preferred return ($Z_0 \leq A_T \leq Z_1$) . After the debt is repaid, the LP receives the entire proceeds until the point where the LP's committed capital has been returned, possibly with a preferred ("hurdle") return, denoted h (typically, 8%). This hurdle is defined such that the LP has received the preferred return when the IRR of the LP's cash flows, including both the initial investment and subsequent management fees, equals the hurdle rate. Formally, let F denote the amount that the LP requires to meet the hurdle. This amount is given as:

$$F=I_0e^{hT}+\int_0^T mX_0e^{hs}ds=I_0e^{hT}+\frac{mX_0}{h}(e^{hT}-1). \tag{11}$$

Intuitively, the hurdle amount F is the future value, at maturity T, of the cash flows that the LP has paid to the fund, including management fees and the initial investment, where the future value is calculated using a compounded rate of h. Without a hurdle (i.e., when $h=0\%$), the LP requires just the committed capital, and $F=I_0+mTX_0=X_0$. With a positive hurdle rate, the LP requires the committed capital plus some of the fund's initial profits to meet the hurdle. In either case, the upper boundary for the preferred-return region is:

$$Z_1=F+Z_0. \tag{12}$$

The LP's payoff in this region, at maturity T, is:

$$LP_1(A_T,T)=\max\{A_T-Z_0,0\}-\max\{A_T-Z_1,0\}. \tag{13}$$

This payoff is the difference between the payoffs of two call options with strike prices of Z_0 and Z_1. Therefore, the LP's payoff in (13) resembles "mezzanine debt," where the LP is senior to the GP but junior to the creditors.

1.2.3 Region 2: Catch-up ($Z_1 \le A_T \le Z_2$)**.** With a positive hurdle rate, the LP requires some of the fund's initial profits to meet the hurdle. The catch-up region then awards a large fraction, denoted n (typically, 100%), of the subsequent profits to the GP to "catch up" to the prescribed profit share, denoted k (typically, 20%). The upper boundary of this region, Z_2, is the amount of final proceeds that is required for the GP to fully catch up, and it solves:

$$k(Z_2-(X_0+Z_0))=n(Z_2-Z_1). \tag{14}$$

When $A_T=Z_2$, the left side of (14) is the GP's prescribed profit share, and the right side is the GP's actual carried interest received. Mathematically, (14) has a unique solution if and only if $n>k$, because $Z_1>X_0+Z_0$. When $n<100\%$, the LP receives the residual payoff, resembling a $(1-n)$ share of another mezzanine debt claim,[8] given as:

$$LP_2(A_T,T)=(1-n)[\max\{A_T-Z_1,0\}-\max\{A_T-Z_2,0\}]. \tag{15}$$

With no hurdle (i.e., when $h=0$), the LP only receives the committed capital in the preferred-return region. There is nothing for the GP to catch up on, and the catch-up region disappears.

1.2.4 Region 3: Profit sharing ($A_T>Z_2$)**.** After the GP catches up with the prescribed profit share, k, the waterfall simply divides any remaining proceeds pro rata, with the GP receiving the fraction k. The LP's payoff in this profit-sharing region resembles a junior equity claim with equity stake $(1-k)$, and this payoff is given as:

$$LP_3(A_T,T)=(1-k)\max\{A_T-Z_2,0\}. \tag{16}$$

[8] PE funds usually have catch-up rates of $n=100\%$, leaving nothing for the LP in the catch-up region. For generality, we allow for $n<100\%$, even if it is rare in PE partnerships. Real estate partnerships often use a catch-up rate of $n=80\%$.

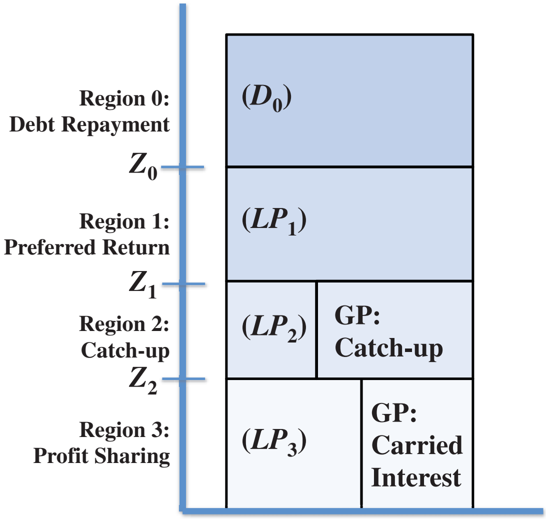

Figure 2
The capital structure of the GP's and LP's total payoffs
This figure illustrates the capital structure and seniority of the four tranches, corresponding to the four regions of the waterfall structure.

1.2.5 Discussion. As illustrated in Figure 2, we can interpret the fund's capital structure as consisting of four tranches, corresponding to the four regions of the waterfall. The LP's partnership interest consists of claims on three of these tranches (the LP does not receive anything in region 0, the debt-repayment region): (1) the LP's claim in the preferred-return region is a mezzanine-type claim that is senior to the GP but junior to the creditors; (2) the LP's claim in the catch-up region corresponds to a $(1-n)$ fraction of another mezzanine-type claim that is junior to the previous one; and (3) the LP's claim in the profit-sharing region corresponds to a $(1-k)$ fraction of a junior equity claim.

At maturity T, the value of the LP's partnership interest is the sum of the values of the LP's individual payoffs in the three regions:

$$LP(A_T,T)=LP_1(A_T,T)+LP_2(A_T,T)+LP_3(A_T,T). \tag{17}$$

With non-spanned volatility and illiquidity, as considered in Section 4, the law-of-one-price no longer holds, and the LP's partnership interest, before maturity T, must be valued as a single combined claim using the LP's certainty-equivalent valuation.

1.3 LP's problem

1.3.1 Preferences. The LP has standard time-separable preferences, represented by:

$$\mathbb{E}\left[\int_0^\infty e^{-\zeta t}U(C_t)dt\right], \tag{18}$$

where $\zeta > 0$ is the LP's subjective discount rate and $U(C)$ is a concave function. For tractability, we choose $U(C) = -e^{-\gamma C}/\gamma$, where $\gamma > 0$ is the coefficient of absolute risk aversion (CARA). Institutional PE investors, such as endowments and pension funds, may have different objectives than individual investors, but we do not model these differences. It is intuitive, though, that even institutional PE investors are averse to fluctuations in their income and expenditures.

1.3.2 Liquid wealth dynamics. Let W_t denote the LP's liquid wealth process, which excludes the value of the LP's partnership interest. The LP allocates Π_t to public equity and the remaining $W_t - \Pi_t$ to the risk-free asset. Over the life of the PE investment, the liquid wealth evolves as:

$$dW_t = (rW_t - mX_0 - C_t)dt + \Pi_t\left((\mu_S - r)dt + \sigma_S dB_t^S\right), \quad t < T. \tag{19}$$

The first term in (19) is the wealth accumulation when the LP is fully invested in the risk-free asset, net of management fees, mX_0, and the LP's consumption/expenditure, C_t. The second term is the excess return from the LP's investment in public equity.

At time T, when the fund is liquidated and the proceeds are distributed, the LP's liquid wealth jumps from W_{T-} to:

$$W_T = W_{T-} + LP(A_T, T), \tag{20}$$

where $LP(A_T, T)$ is the LP's payoff at maturity, given in (17). After the fund is liquidated, the LP only invests in public equity and the risk-free asset, and the liquid wealth process simplifies to:

$$dW_t = (rW_t - C_t)dt + \Pi_t\left((\mu_S - r)dt + \sigma_S dB_t^S\right), \quad t \geq T. \tag{21}$$

2. Solution

After the PE investment matures, the problem reduces to the Merton (1971) problem, and the solution to this problem is summarized in Proposition 1.

Proposition 1. After maturity T, the LP's value function is:

$$J^*(W) = -\frac{1}{\gamma r} e^{-\gamma r(W+b)}, \tag{22}$$

where b is a constant,

$$b = \frac{\eta^2}{2\gamma r^2} + \frac{\zeta - r}{\gamma r^2}. \tag{23}$$

Optimal consumption, C, is:

$$C = r(W + b), \tag{24}$$

and the optimal allocation to public equity, Π, is:

$$\Pi = \frac{\eta}{\gamma r \sigma_S}. \tag{25}$$

2.1 Certainty-equivalent valuation

Let $J(W,A,t)$ be the LP's value function before the PE investment matures. Given $J^*(W)$ from Proposition 1, this value function is:

$$J(W_0,A_0,0)=\max_{C,\Pi}\mathbb{E}\left[\int_0^T e^{-\zeta t}U(C_t)dt+e^{-\zeta T}J^*(W_T)\right]. \tag{26}$$

The LP's optimal consumption and public equity allocation solve the Hamilton-Jacobi-Bellman (HJB) equation:

$$\zeta J(W,A,t)=\max_{\Pi,C}\quad U(C)+J_t+(rW+\Pi(\mu_S-r)-mX_0-C)J_W$$
$$+\frac{1}{2}\Pi^2\sigma_S^2 J_{WW}+\mu_A A J_A+\frac{1}{2}\sigma_A^2 A^2 J_{AA}+\rho\sigma_S\sigma_A \Pi A J_{WA}. \tag{27}$$

In Appendix A, we verify that the value function takes the exponential form:

$$J(W,A,t)=-\frac{1}{\gamma r}\exp[-\gamma r(W+b+V(A,t))]. \tag{28}$$

In this expression, $V(A,t)$ is the LP's certainty-equivalent valuation of the partnership interest, and b is the constant given by (23).

2.2 Consumption and portfolio rules

The LP's optimal consumption is:

$$C(W,A,t)=r(W+V(A,t)+b), \tag{29}$$

which is a version of the permanent-income/precautionary-saving model.[9] Comparing this consumption rule to the rule in equation (24), we see that the LP's total wealth is simply the sum of the liquid wealth, W, and the certainty-equivalent value of the LP's partnership interest, $V(A,t)$.

The LP's optimal allocation to public equity is:

$$\Pi(A,t)=\frac{\eta}{\gamma r\sigma_S}-\beta A V_A(A,t). \tag{30}$$

The first term is the standard mean-variance term from equation (25). The second term is the intertemporal hedging demand, and β is the unlevered beta of the PE asset, given by (4). In option-pricing terminology, we can interpret $V_A(A,t)$ as the "delta" of the value of the LP's partnership interest with respect to the value of the underlying PE asset. Greater values of β and $V_A(A,t)$ create a larger hedging demand for the LP.

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9 Caballero (1991) and Wang (2006) derive explicitly solved optimal consumption rules under incomplete markets with CARA utility. Miao and Wang (2007) derive the optimal American-style growth option exercising problems under incomplete markets. Chen, Miao, and Wang (2010) integrate the incomplete-markets real options framework of Miao and Wang (2007) into Leland (1994) to analyze entrepreneurial default, cash-out, and credit risk implications.

2.3 Certainty-equivalent valuation PDE

The certainty-equivalent valuation of the LP's partnership interest $V(A,t)$, given in (28), solves the partial differential equation (PDE):

$$rV(A,t) = -mX_0 + V_t + (r+\alpha)AV_A + \frac{1}{2}\sigma_A^2 A^2 V_{AA} - \frac{\gamma r}{2}\epsilon^2 A^2 V_A^2, \tag{31}$$

subject to two boundary conditions. First, at maturity T, the value of the LP's claim equals the LP's payoff:

$$V(A_T,T) = LP(A_T,T), \tag{32}$$

where $LP(A_T,T)$ is given in (17). This payoff is net of fees, and the GP's carried interest is captured by this boundary condition. Second, when the value of the underlying PE asset converges to zero, the value of the LP's partnership interest converges to the (negative) PV of the remaining management fees:

$$V(0,t) = -\int_t^T e^{-r(T-s)}(mX_0)ds = -\frac{mX_0}{r}\left(1-e^{-r(T-t)}\right). \tag{33}$$

Regardless of the performance of the PE asset owned by the PE fund, the LP must honor the remaining management fees, and the resulting liability is effectively a risk-free annuity.

2.4 Discussion

The PDE in (31) that values the LP's partnership interest is different from the standard Black-Scholes-Merton PDE. First, the term $-mX_0$ captures the LP's payment of ongoing management fees. Like Metrick and Yasuda (2010), we find that the cost of these management fees is large. Second, the risk-adjusted growth rate is $r+\alpha$, where α is the GP's alpha given in (6). In the standard Black-Scholes-Merton PDE, this risk-adjusted growth rate is r, and the difference arises because our pricing formula values a derivative claim on an underlying asset with a positive alpha. Third, the last term in (31) captures the cost of non-spanned risk and illiquidity. Unlike the standard Black-Scholes-Merton PDE, our PDE is non-linear, because the last term involves V_A^2, and the ϵ in this term is the amount of unspanned risk, given in (5). This non-linear term means that the LP's valuation violates the law-of-one-price. Hence, an LP who values two individual PE investments as V_1 and V_2, independently, will not value the portfolio containing both of the investments at V_1+V_2. Both the non-linear term and the underlying asset with alpha are important departures from the Black-Scholes-Merton pricing formula, which is still a linear PDE and implies the law-of-one-price, despite the non-linear payoffs of the derivatives that it prices.

2.5 Break-even alpha

Following the initial investment, I_0, the LP assumes the liability of the ongoing management fees and receives a claim on the proceeds at maturity, net of the debt repayment and carried interest. The valuation of the LP's partnership interest, $V(A_0,0)$, values this claim. The LP benefits economically from the PE investment when $V(A_0,0)>I_0$, and the LP breaks even, net of fees and accounting for both systematic and unspanned illiquidity risks, when:

$$V(A_0,0)=I_0. \tag{34}$$

The valuation $V(A_0,0)$ is strictly increasing in alpha, and we define the break-even alpha as the alpha that solves (34). This break-even alpha specifies the risk-adjusted excess return that the GP must generate on the underlying portfolio companies, relative to the performance of the public market, to compensate the LP for the GP's compensation and the illiquidity and non-spanned risk of the PE investment. Intuitively, the break-even alpha is the LP's additional cost of capital of the PE investment, in addition to the standard CAPM rate. Hypothetically, if an LP were to evaluate a project proposed by the GP, by discounting the project's future expected free cash flows, the LP should use a discount rate equal to the standard CAPM rate plus the break-even alpha. When the project has a positive NPV using this discount rate, it is in the LP's interest that the GP undertakes the project. When the GP's actual alpha exceeds the break-even value, the PE investment has a positive economic value for the LP.

3. Full Spanning

An important benchmark is the case of full spanning. In this case, the risks of the PE asset and the LP's partnership interest are fully spanned by the public equity, and the risks can be perfectly hedged by the LP by dynamically trading the public equity and the risk-free asset. The PE investment does not involve any remaining non-spanned risk, and hence there is no cost of illiquidity associated with the PE investment.

Under full spanning, we generalize the Black-Scholes (1973) formula to value contingent claims on an underlying asset that earns a positive alpha. Using the new formula, we provide closed-form expressions for the PVs of the LP's partnership interest and the GP's management and incentive fees. Note, in the full-spanning case the valuation of the LP's partnership interest and the GP's compensation is independent of the LP's preferences.[10]

Our assumption of full spanning is different from the usual assumption of complete markets. Under full spanning, the risk of the PE assets is traded in the market, but the PE asset can still earn a positive alpha. In contrast, under complete markets this alpha would be arbitraged away. In our model this arbitrage does not happen because the GP generates the alpha, and the LP can only earn it by investing in the PE fund along with the associated costs. While

10 Technically, the full-spanning case is not a special case of the model in Section 1 with incomplete markets. With full spanning, we need more traded assets to span the PE risk (see Appendix B). As we show in this section, as $\gamma \to 0_+$, the LP's valuation under incomplete markets converges to the valuation under full spanning.

the LP can dynamically hedge the risks associated with the PE asset, the LP cannot invest in the PE asset directly, and the market is formally incomplete. Depending on the relative bargaining power, a skilled GP may capture some or all of the excess return through the compensation contract, as long as the LP remains willing to invest.

3.1 Valuation formulas under full spanning

We first value the PE asset under the GP's management. Recall that A_t grows at the expected rate of μ_A, and it should be discounted at the CAPM rate of $r+\beta(\mu_S-r)$, because the excess return is deterministic and does not covary with the market. Let $EV(A_t,t)$ denote the present value (or economic value) of the PE asset at time t. Using (6), this value is:

$$EV(A_t,t)=\mathbb{E}_t\left[e^{-(r+\beta(\mu_S-r))(T-t)}A_T\right]=e^{\alpha(T-t)}A_t, \quad 0\leq t\leq T. \tag{35}$$

The PV of the PE asset, $EV(A_t,t)$, strictly exceeds A_t when $\alpha>0$. Informally, we can interpret A_t as the PE asset's "mark-to-market" value in an accounting sense. This value is defined as the hypothetical value of the portfolio companies if they were publicly-traded companies instead of being owned by the PE fund, and this mark-to-market value does not include the value of future excess returns earned under GP management. In practice, PE funds mark their companies to market by comparing them to publicly-traded comparables, and the "mark-to-market" interpretation of A_t reflects this practice. In contrast, the economic value, $EV(A_t,t)$, also includes this future value of the GP's management of the PE asset.

3.1.1 Call options on the PE asset. Under full spanning, we can value a contingent claim on the underlying PE asset, with terminal payoff $G(A_T,T)$, as follows:

$$G(A_t,t)=\widetilde{\mathbb{E}}_t\left[e^{-r(T-t)}G(A_T,T)\right], t\leq T. \tag{36}$$

Here, $G(A_t,t)$ is the time-t value of the claim, and $\widetilde{\mathbb{E}}_t[\cdot]$ denotes the expectation under a new measure $\widetilde{P}$, as defined in Appendix B, which allows us to use the risk-free rate r to discount the claim's ultimate payoffs. Specifically, let $\mathcal{C}all(A_t,t;\alpha)$ denote the time-t value of a plain-vanilla European call option with strike price K and terminal payoff $G(A_T,T)=\max\{A_T-K,0\}$ at maturity T. Using (36), we have:

$$\mathcal{C}all(A_t,t;\alpha,K)=\widetilde{\mathbb{E}}_t\left[e^{-r(T-t)}\max\{A_T-K,0\}\right]. \tag{37}$$

In Appendix B, we derive the following explicit solution:

$$\mathcal{C}all(A_t,t;\alpha,K)=A_te^{\alpha(T-t)}N(d_1)-Ke^{-r(T-t)}N(d_2), \tag{38}$$

where $N(\cdot)$ is the cumulative standard normal distribution, and:

$$d_1 = d_2+\sigma_A\sqrt{T-t}, \tag{39}$$

$$d_2 = \frac{\ln\left(\frac{A_t}{K}\right)+\left(r+\alpha-\frac{\sigma_A^2}{2}\right)(T-t)}{\sigma_A\sqrt{T-t}}. \tag{40}$$

These expressions differ from the standard "risk-neutral" Black-Scholes (1973) pricing formula, because the risk-adjusted drift for the underlying asset is $r+\alpha$ instead of r. We can interpret this valuation as the Black-Scholes formula for a call option on an underlying asset with a *negative* dividend yield of $-\alpha$. With a negative dividend yield, the ex-dividend return after the risk adjustment exceeds the risk-free rate by α. As in standard option pricing, the value of the call option (and any other derivative claim) does not depend on the systematic risk, β, of the underlying asset.

3.1.2 Valuation formulas for GP compensation. The value of the GP's compensation, $GP(A_t,t)$, is the sum of management and incentive fees:

$$GP(A_t,t)=MF(A_t,t)+IF(A_t,t). \tag{41}$$

Management fees are senior and resemble a risk-free annuity with an annual payment of mX_0. Thus, MF is given by the standard annuity formula:

$$MF(A_t,t)=\int_t^T e^{-r(s-t)}mX_0 ds=\frac{mX_0}{r}\left(1-e^{-r(T-t)}\right). \tag{42}$$

The GP's incentive fee is a claim on the underlying PE asset, with two parts: the catch-up part, corresponding to region 2 in Figure 2, and the profit-sharing part, corresponding to region 3. In these two regions, the GP's payoffs at maturity T are:

$$GP_2(A_T,T) = n[\max\{A_T-Z_1,0\}-\max\{A_T-Z_2,0\}], \tag{43}$$

$$GP_3(A_T,T) = k\max\{A_T-Z_2,0\}, \tag{44}$$

where Z_1 is the boundary between the preferred-return and catch-up regions, given by (12), and Z_2 is the boundary between the catch-up and profit-sharing regions, given by (14). The values of the GP's claims are then given by our pricing formula as:

$$GP_2(A_t,t) = n[\mathcal{C}all(A_t,t;\alpha,Z_1)-\mathcal{C}all(A_t,t;\alpha,Z_2)], \tag{45}$$

$$GP_3(A_t,t) = k\mathcal{C}all(A_t,t;\alpha,Z_2). \tag{46}$$

Hence, the total value of the GP's incentive fees is:

$$IF(A,t)=GP_2(A_t,t)+GP_3(A_t,t). \tag{47}$$

3.1.3 Valuation formulas for LP's partnership interest. The value of the LP's partnership interest under full spanning is denoted $LP(A_t,t)$, and it has three parts:

$$LP(A_t,t)=LP_1(A_t,t)+LP_2(A_t,t)+LP_3(A_t,t)-MF(A_t,t), \tag{48}$$

where $LP_1(A_t,t)$, $LP_2(A_t,t)$, and $LP_3(A_t,t)$ are the values of the LP's claims in regions 1, 2, and 3, corresponding to the preferred-return, catch-up, and profit-sharing regions, shown in Figure 2. The cost of management fees, $MF(A_t,t)$, is given by (42). Using our formula, the valuations are:

$$LP_1(A_t,t) = \mathcal{C}all(A_t,t;\alpha,Z_0)-\mathcal{C}all(A_t,t;\alpha,Z_1), \tag{49}$$

$$LP_2(A_t,t) = (1-n)[\mathcal{C}all(A_t,t;\alpha,Z_1)-\mathcal{C}all(A_t,t;\alpha,Z_2)], \tag{50}$$

$$LP_3(A_t,t) = (1-k)\mathcal{C}all(A_t,t;\alpha,Z_2). \tag{51}$$

3.1.4 Debt pricing. The debt is also a claim on the PE asset. Recall that D_0 is the initial amount borrowed, and y denotes the yield, so the total amount due at maturity is $Z_0=D_0e^{yT}$. This debt payment is risky, though, and the creditor's actual payment is $D(A_T,T)=\min(A_T,Z_0)$. The value of this claim is:

$$D(A_t,t)=A_te^{\alpha(T-t)}-\mathcal{C}all(A_t,t;\alpha,Z_0). \tag{52}$$

Using the put-call parity (which continues to hold when the underlying asset has alpha), we define $\mathcal{P}ut$ analogously as the value of a put option on an underlying asset with alpha. The debt valuation equation can then be restated in the more familiar form:

$$D(A_t,t)=e^{-r(T-t)}Z_0-\mathcal{P}ut(A_t,t;\alpha,Z_0). \tag{53}$$

The value of the risky debt is a combination of a risk-free asset with time-T payoff $Z_0=D_0e^{yT}$ and a short position in the default put option, which is valued as $\mathcal{P}ut(A_t,t;\alpha,Z_0)$.

Assuming that lenders break even at time 0, we can determine the equilibrium yield y^* by solving for the yield that makes the initial amount borrowed, D_0, equal to the initial valuation of the debt, $D(A_0,0)$. For a given choice of D_0, and substituting D_0e^{yT} for Z_0, the equilibrium yield is defined by the equation:

$$D_0=A_0e^{\alpha T}-\mathcal{C}all(A_0,0;\alpha,D_0e^{y^*T}). \tag{54}$$

Given the equilibrium yield, y^*, the credit spread, cs, is the difference between the yield and the risk-free rate, given as:

$$cs=y^*-r=\frac{1}{T}\ln\left(1+\frac{\mathcal{P}ut(A_0,0;\alpha,D_0e^{y^*T})}{D_0}\right). \tag{55}$$

The debt is exposed to the same non-spanned risk as the underlying PE asset. However, we assume that the debt is held by diversified creditors who do not require any compensation for this risk. Hence, the pricing of the debt is identical under full-spanning and non-spanned risk, and we use the same debt pricing formula in both cases.

3.1.5 Value additivity. Under full spanning, valuations are additive. The sum of the valuations of the LP, GP, and creditors equals the economic value of the PE asset, $EV(A_t,t)$:

$$GP(A_t,t)+LP(A_t,t)+D(A_t,t)=EV(A_t,t)=A_t e^{\alpha(T-t)}. \tag{56}$$

3.2 Results under full spanning

Where possible, we use parameters from Metrick and Yasuda (2010) for our calibration. Metrick and Yasuda find an annual volatility of 60% per individual BO investment, with a pairwise correlation of 20% between any two investments, capturing the diversification in a portfolio of such investments. They report that the average BO fund invests in around 15 BOs (with a median of 12). Using these values we calculate an annual volatility of $\sigma_A=25\%$ for the PE asset. We use an annual risk-free rate of $r=5\%$.

For leverage, Axelson et al. (2011) consider 153 BOs during 1985–2006, and report that, on average, equity accounted for 25% of the purchase price, corresponding to $l=3$. For the compensation contract, we focus on the 2/20 contract, which has a $m=2\%$ management fee, $k=20\%$ of carried interest, and $h=8\%$ in hurdle rate. This contract is widely adopted by PE funds. We also consider typical variations in these contract terms. For the public market, we use an annual volatility of $\sigma_S=20\%$, with an expected return of $\mu_S=11\%$, implying a risk premium of $\mu_S-r=6\%$, and a Sharpe ratio of $\eta=30\%$. Table 1 summarizes the definitions of the variables and the parameter values used in our baseline calculations.

3.2.1 Effects of GP skill. Table 2 reports the effects of the GP's alpha on the valuations and the equilibrium credit spread. The reported numbers are for an initial investment of $I_0=100$, corresponding to a committed capital of $X_0=125$. Management fees are $m=2\%$, carried interest is $k=20\%$, and the hurdle rate is $h=8\%$. Panel A of Table 2 reports unlevered results, and Panel B reports results with $l=3$. In the baseline cases, the value of the LP's partnership interest equals the initial investment of 100, and the LP just breaks even.

With leverage, the value of the LP's partnership interest, LP, is highly sensitive to the GP's alpha. When the GP is unskilled and $\alpha=0$, the present value of the LP's claim is just 64.42, and the LP loses almost 35% of the initial investment of $I_0=100$. When the GP's skill increases to 1.0% annually, the value of the LP's claim improves by 35% to 100, because $\alpha=1.0\%$ is the break-even alpha. As the GP's alpha increases, the value of the LP's partnership interest continues to improve. For example, when $\alpha=2.0\%$, then $LP=138.09$, and when $\alpha=3.0\%$, then $LP=180.46$.

3.2.2 Effects of leverage. Panel B of Table 2 shows valuations with leverage of $l=3$. The break-even alpha declines substantially, from 2.6% annually without leverage to 1.0% with leverage of $l=3$. For a given size of the LP's

Table 1
Summary of key variables and parameters

Variable	Symbol	Parameter	Symbol	Value
LP's consumption or expenditure	C	Risk-free rate	r	5%
LP's value function	J	Expected return of market portfolio	μ_S	11%
LP's value function after exiting illiquid asset	J^*	Expected return of PE asset	μ_A	
LP's certainty equivalent	V	Volatility of market portfolio	σ_S	20%
Market portfolio allocation	Π	Volatility of PE asset	σ_A	25%
Debt	D	Aggregate equity risk premium	$\mu_S - r$	6%
Wealth	W	Market Sharpe ratio	η	30%
Assets	A	Hurdle rate	h	8%
Brownian motion for market return	B^S	Carried interest	k	20%
Brownian motion for PE return	B^A	Management fee	m	2%
Committed capital	X_0	Catch-up rate	n	100%
Invested capital	I_0	Life of PE investment	T	10
Value of LP's partnership interest	LP	Excess return, alpha	α	
Value of carried interest	IF	Correlation between market and PE asset	ρ	0.4
Value of management fees	MF	PE un-levered beta	β	0.5
Total value of GP compensation	GP	Subjective discount rate	ζ	5%
Credit spread	cs	Leverage	l	3
		Unspanned volatility	ϵ	23%
		Coefficient of absolute risk aversion	γ	

This table summarizes the symbols for the key variables in the model and baseline parameter values.

Table 2
The values for the GP and the LP under full spanning for various levels of alpha

α	IF	MF	GP	LP	$GP+LP$	cs	EV
Panel A: Without leverage ($l=0$)							
−1.0%	4.52	19.67	24.19	66.29	90.48	NA	90.48
0.0%	5.73	19.67	25.40	74.60	100.00	NA	100.00
1.0%	7.19	19.67	26.86	83.65	110.51	NA	110.51
2.0%	8.93	19.67	28.60	93.54	122.14	NA	122.14
2.6%[a]	10.14	19.67	29.81	100.00	129.81	NA	129.81
3.0%	10.98	19.67	30.65	104.33	134.99	NA	134.99
Panel B: With leverage ($l=3$)							
−1.0%	9.81	19.67	29.48	32.46	61.93	6.27%	361.94
0.0%	15.91	19.67	35.59	64.42	100.00	4.59%	400.00
1.0%[a]	22.97	19.67	42.64	100.00	142.64	3.46%	442.63
2.0%	30.80	19.67	50.47	138.09	188.56	2.67%	488.56
2.6%	36.17	19.67	55.84	163.45	219.29	2.29%	519.29
3.0%	39.82	19.67	59.49	180.46	239.94	2.07%	539.94

This table reports valuations of the various parts of the waterfall structure under full spanning for a range of alphas. Panels A and B report the cases with and without leverage. The columns refer to incentive fees (IF), management fees (MF), total GP compensation ($GP=IF+MF$), and the LP's partnership interest (LP). The table also presents the equilibrium credit spread (cs) and the economic value of the unlevered PE asset (EV). Parameter values are: $I_0=100$, $m=2\%$, $k=20\%$, $h=8\%$, $n=1$, $T=10$, and $\beta=0.5$.
[a]Indicates the baseline break-even cases.

initial investment, I_0, the main advantage of leverage is that it increases the amount of PE assets, A_0, managed by the GP. This increase enables the GP to earn the alpha on a larger asset base, and it effectively reduces the management fees paid per dollar of assets under management. Hence, a lower alpha, albeit earned on a larger amount of assets, is required for the LP to break even.

Even with a lower break-even alpha, the larger PE asset means that the GP generates more total value. With $\alpha=1.0\%$ annually, the GP increases the value of the PE asset from 400 to 442.64. Since the creditors break even, the total value of the LP and GP's claim increases from 100 to 142.64, and the value of the GP's incentive fees, IF, more than doubles from 10.14 to 22.97, compared to the baseline case without leverage. This increase in the value of the incentive fees arises because the size of the managed PE asset is four times larger due to leverage of $l=3$, and leverage increases the volatility of the GP's carried interest and hence increases its value for the standard optionality reasons. The management fees, MF, remain unchanged despite the increase in the underlying PE assets, because these fees are senior and the LP's committed capital remains constant at $X_0=125$. Hence, the annual management fees also remain constant at 2.5 and their value equals the value of a 10-year annuity discounted at 5%, which is valued at 19.67. Assuming that the GP can hold the alpha constant when leverage increases, the effect of leverage becomes even greater. For an alpha of 2.6% annually, which is the break-even alpha without leverage, the value of the GP's incentives fees more than triples from 10.14 to 36.17 when leverage increases to $l=3$.

Table 3
The values for the GP and the LP under full spanning for a range of credit spreads

cs	D	IF	MF	GP	LP	$GP+LP$	EV	l
Panel A: The case with debt mis-pricing								
0.00%	244.53	31.43	19.67	51.11	147.00	198.10	442.63	3.00
0.50%	252.62	30.20	19.67	49.88	140.14	190.02	442.63	3.00
1.00%	260.72	28.97	19.67	48.64	133.28	181.92	442.63	3.00
1.50%	268.80	27.74	19.67	47.41	126.43	173.84	442.63	3.00
2.00%	276.85	26.51	19.67	46.18	119.61	165.79	442.63	3.00
2.50%	284.84	25.29	19.67	44.96	112.84	157.80	442.63	3.00
3.00%	292.76	24.08	19.67	43.75	106.13	149.88	442.63	3.00
3.46%[a]	300.00	22.97	19.67	42.64	100.00	142.64	442.63	3.00
4.59%	317.22	20.31	19.67	39.99	85.43	125.41	442.63	3.00
Panel B: The case with competitive debt pricing								
0.00%	0.00	7.21	19.67	26.88	83.78	110.66	110.66	0.00
0.50%	54.19	11.63	19.67	31.31	85.13	116.44	170.64	0.54
1.00%	84.99	13.70	19.67	33.37	86.35	119.72	204.72	0.85
1.50%	117.49	15.58	19.67	35.25	87.94	123.19	240.69	1.18
2.00%	153.89	17.39	19.67	37.07	90.01	127.07	280.97	1.54
2.50%	195.89	19.22	19.67	38.89	92.66	131.55	327.44	1.96
3.00%	245.39	21.11	19.67	40.78	96.05	136.83	382.21	2.45
3.46%[a]	300.00	22.97	19.67	42.64	100.00	142.64	442.63	3.00
4.59%	488.19	28.40	19.67	48.08	114.63	162.71	650.90	4.88

This table reports valuations of the various parts of the waterfall structure under full spanning for a range of credit spreads. Panel A presents the case with debt mis-pricing, where both the credit spread and the initial debt amount, $D_0=300$, are exogenously given. Panel B shows the case where the initial debt equals its value, given the credit spread. For both panels, the columns are: the economic value of debt (D), incentive fees (IF), management fees (MF), total GP compensation (GP), the LP's partnership interest (LP), the economic value of the underlying PE asset (EV), and the book leverage $l=D_0/I_0$. Parameter values are: $I_0=100$, $m=2\%$, $k=20\%$, $h=8\%$, $n=1$, $T=10$, $\beta=0.5$, and $\alpha=1.01\%$.
[a]Indicates the baseline break-even cases.

3.2.3 Debt pricing and credit spreads. The debt pricing and credit spreads also depends on the GP's alpha. Panel B of Table 2 reports the equilibrium credit spread, cs, defined as the difference between the equilibrium yield and the risk-free rate. Although the creditors are senior to both the GP and the LP, the credit spread is sensitive to the GP's alpha. A higher alpha leads to a higher expected value of the underlying PE asset and reduces the risk of the fund defaulting on the debt.

To investigate the implications of "cheap" debt, Panel A of Table 3 reports valuations with mispriced debt. The table shows the effects of changes in the yield on the value of the debt D, incentive fees IF, management fees MF, the LP's partnership interest LP, and the combined value $GP+LP$. For all rows in Panel A, the GP borrows $D_0=300$ and makes an initial investment of $I_0=100$ at time 0. The table shows that debt pricing has a substantial effect on the value of the LP's partnership interests, LP, and the incentive fees, IF. For example, if debt is priced with $cs=2\%$, which is below the equilibrium spread of $cs=3.46\%$, the promised total payment to the creditors at maturity T is $300\times e^{0.7}=602.7$ at yield $y=7\%$, which is significantly lower than $300\times e^{0.846}=699.1$ (corresponding to $cs=3.46\%$). Using the pricing equation (52) and $Z_0=602.7$, the time-0 value of the debt with $cs=2\%$ is only $D(300,0)=276.85$. The value of the LP's partnership interest LP increases about 20% from 100 to 119.61, and the value of the GP's incentive fees, IF, increases from 22.97 to 26.51. Intuitively, a lower yield transfers wealth from the creditors to both the LP and GP, which are junior to the creditors in the fund's capital structure, as defined by the waterfall.

Another way to evaluate the role of "cheap" debt is to compare its break-even yield to the required yield for a comparable publicly-traded company, which has zero alpha by definition. Without alpha, the credit spread is 4.59%, as implied by (55) with $D_0 = 300$, $A_0 = 400$, and $\alpha = 0$. Panel A of Table 3 then shows that fixing the credit spread at 4.59% for a GP with an alpha of 1.01% implies a value of the debt of 317.22. The 17.22 increase represents the creditors' value of the GP's alpha. If instead we assume that the creditors just break even, the credit spread declines from 4.59% to the new break-even spread of 3.46%, corresponding to the "cheaper" debt that is available for BO transactions, due to the GP's alpha, relative to the yield charged to publicly-traded comparable companies.

Panel B of Table 3 shows the effect of changes in the credit spread on the amount of leverage that is available to finance the transaction under the assumption that the debt pricing is competitive and rational. For a given spread cs, the amount of debt D_0 provided by rational creditors is given by (54). The top row in Panel B shows that the creditors will not lend anything with a zero credit spread. As the credit spread cs increases, the amount of debt that the creditors will provide increases as well. Given this amount of debt, we calculate the value of the incentive fees, IF, management fees, MF, etc. The value of both incentive fees and the LP's partnership interest increases with the amount of leverage, and the LP loses money if the credit spread is too low and the leverage is too conservative, holding alpha fixed. Indeed, in our model, it is optimal for both the GP and the LP to borrow as much as the creditors will lend, because leverage allows the GP to increase the amount of PE assets, $A_0 = (1+l)I_0$, and earn a positive alpha on the larger A_0. Note though, that in this model there is no cost of leverage (e.g., deadweight, distress, or any other inefficiency).

3.3 The value of GP compensation

Tables 4 and 5 report valuations of the GP's compensation depending on the fee structure. Table 4 presents valuations without leverage. Table 5 contains valuations with leverage of $l = 3$. In both tables, Panel A shows the case with the full waterfall structure, including both a hurdle return and the subsequent catch-up. Panel B shows a simpler compensation structure without the hurdle return ($h = 0\%$) and hence without a catch-up region. With this simpler structure, the profits are simply shared pro rata between the GP and LP, with the GP's share given by the carried interest rate, k. In the baseline case, in Panel A of Table 4, the value of the GP's incentive fees (carried interest), IF, is 10.14. The value of the management fees, MF, is 19.67, and the total value of the GP's

Table 4
The values for the GP and the LP under full spanning without leverage for various management compensation contracts

m	k	GP_2	GP_3	IF	MF	GP	LP
Panel A: Hurdle ($h=8\%$) and catch-up ($n=100\%$)							
1.5%	10%	2.36	3.08	5.45	13.89	19.33	110.48
1.5%	20%	5.04	5.63	10.67	13.89	24.56	105.26
1.5%	30%	8.07	7.54	15.61	13.89	29.50	100.32
2.0%	10%	2.27	2.91	5.18	19.67	24.85	104.96
2.0%[a]	20%	4.83	5.31	10.14	19.67	29.81	100.00
2.0%	30%	7.74	7.09	14.83	19.67	34.50	95.31
2.5%	10%	2.17	2.73	4.89	26.23	31.13	98.69
2.5%	20%	4.61	4.96	9.58	26.23	35.81	94.01
2.5%	30%	7.38	6.62	13.99	26.23	40.22	89.59
Panel B: No hurdle and no catch-up							
1.5%	10%	0	6.79	6.79	13.89	20.68	109.14
1.5%	20%	0	13.58	13.58	13.89	27.46	102.35
1.5%	30%	0	20.37	20.37	13.89	34.25	95.56
2.0%	10%	0	6.51	6.51	19.67	26.18	103.63
2.0%[a]	20%	0	13.02	13.02	19.67	32.69	97.12
2.0%	30%	0	19.53	19.53	19.67	39.20	90.61
2.5%	10%	0	6.21	6.21	26.23	32.44	97.37
2.5%	20%	0	12.42	12.42	26.23	38.65	91.17
2.5%	30%	0	18.62	18.62	26.23	44.85	84.96

This table presents valuations of the different parts of the waterfall structure under full spanning without leverage for various compensation contracts. The columns refer to: management fees (m), incentive fees (k), GP's catch-up (GP_2), GP's profit-sharing region (GP_3), GP's incentive fees (IF), GP's management fees (MF), total GP compensation (GP), and the value of the LP's partnership interest (LP). Parameter values are: $\alpha=2.6\%$, $I_0=100$, $l=0$, $T=10$, and $\beta=0.5$. Panel A reports the baseline case with 100% catch-up and 8% hurdle rate and Panel B reports the "simple" case with no hurdle and no catch up.
[a]Indicates the baseline break-even cases.

compensation is 29.81. Hence, in this case, management fees constitute two-thirds of the GP's total compensation, consistent with the simulation results in Metrick and Yasuda (2010).

When the level of management fees, m, changes, Tables 4 and 5 show that the present value of the management fees, MF, is linear in m. This is natural, because the management fee is simply valued as an annuity by (42), which is proportional to m. The value of the GP's incentive fees (carried interest), IF, is almost, but not exactly, linear in the carried-interest rate, k. We consider three levels of carried interest, of $k=10\%$, $k=20\%$, and $k=30\%$. For example, in Table 5, with leverage $l=3$ and under the 2/20 compensation contract, the value of the GP's incentive fees is $IF=22.97$. This value is almost exactly twice the value of incentive fees with a carried-interest rate of $k=10\%$, of $IF=11.51$. Moreover, when $k=30\%$ the value is almost exactly 1.5 times that of $k=20\%$ and triple the value when $k=10\%$. Intuitively, the "strike" price for the carried interest, a call on the underlying PE asset, increases with m, because the time-T cumulative value of management fees F, given in (11), contributes to the strike price for the carried interest. However, quantitatively,

Table 5
The values for the GP and the LP under full spanning with leverage for various management compensation contracts

m	k	GP_2	GP_3	IF	MF	GP	LP
Panel A: Hurdle ($h=8\%$) and catch-up ($n=100\%$)							
1.5%	10%	2.05	9.60	11.65	13.89	25.54	117.10
1.5%	20%	4.54	18.71	23.25	13.89	37.14	105.50
1.5%	30%	7.63	27.14	34.76	13.89	48.65	93.99
2.0%	10%	2.07	9.44	11.51	19.67	31.19	111.45
2.0%[a]	20%	4.58	18.39	22.97	19.67	42.64	100.00
2.0%	30%	7.69	26.64	34.33	19.67	54.01	88.63
2.5%	10%	2.09	9.27	11.36	26.23	37.59	105.05
2.5%	20%	4.62	18.03	22.65	26.23	48.88	93.76
2.5%	30%	7.76	26.10	33.86	26.23	60.09	82.55
Panel B: No hurdle and no catch-up							
1.5%	10%	0	11.91	11.91	13.89	25.81	116.84
1.5%	20%	0	23.82	23.82	13.89	37.71	104.93
1.5%	30%	0	35.74	35.74	13.89	49.63	93.01
2.0%	10%	0	11.78	11.78	19.67	31.46	111.18
2.0%[a]	20%	0	23.57	23.56	19.67	43.24	99.40
2.0%	30%	0	35.35	35.35	19.67	55.02	87.62
2.5%	10%	0	11.64	11.64	26.23	37.87	104.77
2.5%	20%	0	23.27	23.27	26.23	49.50	93.14
2.5%	30%	0	34.91	34.91	26.23	61.14	81.50

This table reports valuations of the different parts of the waterfall structure under full spanning with leverage for various compensation contracts. The columns refer to: management fees (m), incentive fees (k), GP catch-up (GP_2), GP profit-sharing region (GP_3), GP incentive fees (IF), management fees (MF), total GP compensation (GP), and the value of the LP's partnership interest (LP). Parameter values are: $\alpha=1.01\%$, $I_0=100$, $l=3$, $T=10$, and $\beta=0.5$. The equilibrium credit spread is constant and $cs=2.63\%$, because creditors are senior to the GP and LP. Panel A reports the baseline case with 100% catch-up and 8% hurdle rate and Panel B reports the "simple" case with no hurdle and no catch up.
[a]Indicates the baseline break-even cases.

the effect of management fees m on the present value of the incentive fees, IF, is small. The GP's total compensation consists of management fees and carried interest, and it is valued as $GP=IF+MF$. In the baseline case in Table 5, with leverage, the carried interest and the management fees contribute largely similar amounts to the GP's total compensation and are valued at 22.97 and 19.67, respectively.

3.3.1 Effects of hurdle and catch-up. In Tables 4 and 5, Panels A and B compare the effects of changing the contract to the simpler contract without an 8% hurdle rate and subsequent catch-up region. The hurdle and catch up protect the LP by pushing the GP's claim further down in the capital structure. In the simpler contract, the LP is worse off and the GP is better off, because the value of the GP's claim in the catch-up region, GP_2, vanishes, but the value of the GP's claim in the profit region, GP_3, increases. In Table 4, without leverage, the effects are larger, because the final payoff is more likely to end up in the preferred and catch-up regions. In the baseline case, the value of the GP's incentive fees, IF, increases by about 30%, from 10.14 to 13.02, when moving to the simpler contract. Since the value of the incentive fees is smaller than the value of management fees, the GP's total compensation only increases by about 10%, from 29.81 to 32.69. Conversely, the value of the LP's partnership interest declines from 100 to 97.12 without the hurdle and catch-up. In the case with leverage, in Table 5, it is less likely that the final payoff will end up in the preferred or catch-up regions, and the effects of eliminating these parts of the compensation contract and changing to the simpler structure are even smaller in present-value terms.

4. General Case with Non-Spanned Risk

With non-spanned risks, the risk of the PE asset is not fully spanned by the public market, and the illiquidity of the PE investment is costly for the LP. In this case, the law-of-one-price no longer holds, valuations are no longer additive, and "present value" is not well defined. The LP's certainty-equivalent valuation can still be calculated by numerically solving the PDE from (31). We can then evaluate the LP's costs of illiquidity and various compensation arrangements both in terms of this certainty-equivalent valuation and also by the implied break-even alpha, which is the LP's cost of capital in addition to the cost implied by the standard CAPM.

In contrast to the case with full spanning, the LP's valuation now depends on the beta of the underlying PE asset, the LP's preferencess and risk aversion, as well as the LP's allocation to the PE investment. To calibrate these parameters, we use an unlevered beta of the PE asset of 0.5. This is consistent with evidence from Ljungqvist and Richardson (2003), who match companies involved in PE transactions to publicly-traded companies. They report that the average (levered) beta of the publicly-traded comparable companies is 1.04, suggesting that PE funds invest in companies with average systematic risk exposures. Since publicly-traded companies are typically financed with approximately one-third debt, the unlevered asset beta is around 0.66, and we round it down to an unlevered beta of 0.5. With this beta, the correlation between the PE asset and the public market is $\rho = \beta\sigma_S/\sigma_A = 0.4$.

To determine the LP's risk aversion, γ, and initial investment, I_0, the following invariance result is derived in Appendix C:

Proposition 2. Define $a = A/I_0$, $x_0 = X_0/I_0$, $z_0 = Z_0/I_0$, $z_1 = Z_1/I_0$, and $z_2 = Z_2/I_0$. It is straightforward to verify that $V(A,t) = v(a,t) \times I_0$, where $v(a,t)$ solves the ODE,

$$rv(a,t) = -mx_0 + v_t + (r+\alpha)av_a(a,t) + \frac{1}{2}\sigma_A^2 a^2 v_{aa}(a,t) - \frac{\gamma I_0}{2} r\epsilon^2 a^2 v_a(a,t)^2, \quad (57)$$

subject to the boundary conditions,

$$v(a,T) = \max\{a-z_0,0\}-n\max\{a-z_1,0\}+(n-k)\max\{a-z_2,0\}, \quad (58)$$

$$v(0,t) = -\frac{mx_0}{r}\left(1-e^{-r(T-t)}\right). \quad (59)$$

This invariance proposition shows that the LP's certainty-equivalent valuation, $V(A,t)$, can be normalized with the amount initially invested, I_0. We can then solve for the resulting $v(a,t)$, which gives the LP's certainty-equivalent valuation per dollar initially invested. Proposition 2 shows that $v(a,t)$ depends only on the product γI_0, not on γ and I_0 individually. Hence, the LP's certainty-equivalent valuation $V(A,t)$ is proportional to the invested capital I_0, holding γI_0 constant.

Next, we calibrate the value of the product γI_0. Let γ_R denote the LP's relative risk aversion. In terms of the value function $J(W,A,t)$, the relative risk aversion is defined as:

$$\gamma_R = -\frac{J_{WW}(W,A,t)}{J_W(W,A,t)}W. \quad (60)$$

For a given level of relative risk aversion, we approximate the implied absolute risk aversion, given as $-J_{WW}/J_W$, by adjusting with the initial level of liquid wealth, W_0. Using the FOC with respect to consumption, $U'(C_t)=J_W(W,A,t)$ and $J_{WW}(W,A,t)=-\gamma r J_W(W,A,t)$, we can write γ_R as:

$$\gamma_R = \frac{\gamma r U'(C_t)}{U'(C_t)}W_t = \gamma r W_t. \quad (61)$$

Evaluating γ_R at time 0, we obtain:

$$\gamma I_0 = \frac{\gamma_R}{r}\left(\frac{I_0}{W_0}\right). \quad (62)$$

With this approximation, γI_0 can be determined from the LP's initial relative allocation to PE (in parentheses) and the relative risk aversion, γ_R. Informally, the resulting CARA preferences are a local approximation to the CRRA preferences implied by γ_R. We interpret γI_0 as the LP's effective risk aversion, and an LP with a larger relative risk aversion or greater PE allocation has greater effective risk aversion. When the PE allocation tends to zero or the preferences tend to risk neutral, the effective risk aversion becomes zero. With $r=5\%$, $\gamma_R=1$, and an initial PE allocation as a fraction of liquid wealth W_0 of $I_0/W_0=10\%$, we obtain $\gamma I_0=2$. With a relative risk aversion of $\gamma_R=2.5$, we have $\gamma I_0=5$. Correspondingly, we consider three levels of effective risk aversion: $\gamma I_0 \to 0_+$ for an effectively risk-neutral LP,[11] a "moderate" effective risk aversion of $\gamma I_0=2$, and a "high" effective risk aversion of $\gamma I_0=5$.

11 Formally, our model does not allow the LP to be risk neutral ($\gamma=0$). Since public equity yields a higher expected rate of return than the risk-free rate, a risk-neutral investor would hold an infinite position in the public market portfolio. In the limit, as $\gamma \to 0_+$, the solution for $V(A_t,t)$ remains valid, though, and it converges to the valuation formula for the full-spanning case. We use $\gamma I_0=0_+$ (with subscript "+") to denote the corresponding limit solution. In this limit, the LP is effectively risk neutral and the cost of illiquidity disappears. Formally, for $\gamma=0_+$, the PDE (31) is linear, and the solution is identical to the one from the full-spanning case, given in Section 3.

Table 6
Effects of risk aversion for both the case with no leverage ($l=0$) and the one with leverage of $l=3$ on annual break-even alphas

	$l=0$	$l=3$
$\gamma I_0=0_+$	2.61%	1.01%
$\gamma I_0=2$	3.08%	2.06%[a]
$\gamma I_0=5$	3.74%	3.37%

The table shows break-even alphas for different levels of effective risk aversion γI_0 and leverage l. Other parameter values are $\beta=0.5$, $m=2\%$, $k=0.2$, and $h=8\%$.
[a]Indicates the baseline break-even case.

4.1 Cost of illiquidity

Table 6 shows break-even alphas for various levels of effective risk aversion and leverage. As above, the break-even alpha can be interpreted as the LP's incremental cost of capital of the PE investments, so a higher break-even alpha means a higher cost of capital. The first row of Table 6 shows break-even alphas for an LP with $\gamma I_0=0_+$. Because this LP is effectively risk neutral, there is no additional cost of illiquidity and non-spanned risks, and the break-even alphas of 2.61% annually without leverage and 1.01% with $l=3$ are identical to those in the full-spanning case. These alphas reflect just the costs of management fees and carried interest.

Table 6 shows that higher leverage reduces the break-even alpha. As in the full-spanning case, the benefit of leverage is that it increases the total amount of PE assets, A_0, and enables the GP to earn the alpha on this larger asset base, effectively reducing the management fees paid per dollar of PE assets under management by the GP. A secondary benefit of leverage, with non-spanned risks and holding the total amount of PE assets constant, is that it transfers risk to the creditors, who are better diversified, demand no illiquidity risk, and hence have a lower cost of capital, even after adjusting for the higher beta.

4.1.1 Effects of risk aversion. Table 6 shows that the break-even alpha increases with the LP's effective risk aversion, γI_0. This increase does not arise in the full spanning case where there is no cost of illiquidity and the valuation is independent of the LP's preferences. Intuitively, a more risk-averse LP has a higher cost of illiquidity and non-spanned risks and requires greater compensation, as measured by the break-even alpha. Without leverage, the LP's cost is modest, though, and the break-even alpha just increases from 2.61% to 3.08% and 3.74% annually when γI_0 increases from 0_+ to 2 and 5, respectively. With leverage, the LP's cost of illiquidity is more substantial, because the size of the PE asset is quadrupled, and the break-even alpha more than triples from 1.01% to 2.06% and 3.37% annually.

With leverage, the break-even alpha of 1.01% annually represents just the cost of the GP's management and incentive fees for an effectively risk-neutral LP. The increase in the break-even alpha from 1.01% to 2.06% represents the LP's additional cost of illiquidity and non-spanned risks, with a moderate effective risk aversion ($\gamma I_0=2$). Hence, the LP's cost of the GP's total compensation (the break-even alpha of 1.01%) is comparable to the LP's cost of illiquidity and non-spanned risks, represented by the 1.05% increase in the break-even alpha. For a high level of effective risk aversion ($\gamma I_0=5$), the LP's cost of illiquidity is more than twice the cost of the GP's compensation, using this measure.

4.1.2 Illiquidity discount. While the break-even alpha provides one measure of the LP's costs, we can also evaluate these costs in terms of the LP's certainty-equivalent valuation. In Table 7, column V is the LP's certainty-equivalent valuation, including the costs of non-spanned risks and illiquidity. For comparison, column LP gives the present value of the LP's partnership interest with full-spanning and no cost of illiquidity. The difference between these two valuations is the illiquidity discount, denoted ID:

$$ID=LP(A_0,0)-V(A_0,0). \tag{63}$$

The illiquidity discount is the amount that an LP would be willing to pay for eliminating non-spanned risk and investing under full spanning. In Table 7, column GP values the GP's compensation under full-spanning, and the numbers are identical to those in Table 2, by construction. The equilibrium credit spread is reported in column cs (for the case with leverage), and the total economic value (EV) of the PE asset is in column EV.

In Table 7, Panel A, without leverage, shows a break-even alpha of 3.08%, and the cost of illiquidity is small. Since the LP breaks even, the value of the LP's partnership interest, V, equals the initial investment of $I_0=100$. Without the cost of illiquidity, the LP would value the investment at 105.26, and the LP's illiquidity discount is 5.26, which is 5.26% of the initial invested capital I_0. Without leverage, however, this cost of illiquidity is swamped by the cost of the GP's compensation of 30.84.

With leverage, the cost of illiquidity increases substantially. Panel B of Table 7 shows that the cost of illiquidity now reduces the value of the LP's partnership interest from 140.48 to 100, and the illiquidity discount is 40.48, which is almost as large as the total cost of the GP's compensation of 50.97. Leverage increases the cost of illiquidity, because it increases the size of the PE asset and the LP's illiquidity discount.

Table 7 also shows that the illiquidity discount increases as the alpha increases, because it increases the value of the PE asset and hence it increase the cost of illiquidity, although this increase in ID is modest without leverage. Finally, for a given level of alpha, even with illiquidity and non-spanned risks, the credit spreads remain the same as in the full-spanning case. Creditors are diversified and are senior to both the GP and LP, and the creditors' valuations of their claims remain unchanged when the risk is non-spanned.

4.2 GP compensation and illiquidity

Table 8 shows how changes in the GP's compensation contract affect the illiquidity discount, ID. As above, column m is management fees, k is carried

Table 7
Illiquidity discount due to non-spanned risks for various levels of alpha

α	GP	LP	V	ID	cs	EV
Panel A: Without leverage ($l=0$)						
−1.00%	24.19	66.29	63.68	2.61	NA	90.48
0.00%	25.40	74.60	71.50	3.09	NA	100.00
1.00%	26.86	83.65	79.98	3.67	NA	110.51
2.06%	28.71	94.15	89.74	4.41	NA	122.86
3.08%[a]	30.84	105.26	100.00	5.26	NA	136.10
4.00%	33.05	116.13	109.94	6.19	NA	149.18
Panel B: With leverage ($l=3$)						
−1.00%	29.48	32.46	18.56	13.89	6.27%	361.94
0.00%	35.59	64.42	42.85	21.57	4.59%	400.00
1.00%	42.64	100.00	69.72	30.28	3.46%	442.63
2.06%[a]	50.97	140.48	100.00	40.48	2.63%	491.45
3.08%	60.29	184.11	132.21	51.91	2.03%	544.39
4.00%	69.71	227.03	163.43	63.60	1.61%	596.73

This table presents valuations of the different parts of the waterfall structure with non-spanned risks for various levels of alpha. The columns refer to: total GP compensation (GP), the LP's partnership interest (LP), the LP's certainty-equivalent valuation (V), the illiquidity discount ($ID=LP-V$), the equilibrium credit spread (cs), and the economic value of the PE asset (EV). Parameter values are $\gamma I_0=2$, $I_0=100$, $m=2\%$, $k=20\%$, $T=10$, $\beta=0.5$, and $h=8\%$. Panels A and B report the results for the case with $l=0$ and $l=3$, respectively.
[a]Indicates the baseline break-even cases.

interest, and GP is the value of the GP's compensation. Column V contains the LP's certainty-equivalent valuation with illiquidity costs. For comparison, column LP gives the LP's value under full spanning (i.e., without costs of illiquidity). The difference between V and LP is the illiquidity discount in column ID. The baseline 2/20 contract is identical to the baseline case in Panel B of Table 7. In this case, the combined value of the GP's management fees and carried interest, GP, is 50.97. The value of the LP's partnership interest, without accounting for illiquidity, is 140.48. In contrast, the LP's certainty-equivalent valuation is just 100, and the illiquidity discount is $ID=40.48$. Hence, ignoring the cost of illiquidity, the compounding effect of the GP's alpha means that the GP creates 91.45 of value, in addition to the initial investment of 100. Of the resulting proceeds of 191.45, the GP receives 50.97 and the LP receives the remaining 140.48. Due to the cost of illiquidity, however, the LP's valuation of these proceeds is only 100, which again shows that the illiquidity cost is substantial and comparable to the total costs of management compensation.

As in the full spanning case, when the level of management fees m changes, the value of the GP's management fees MF changes proportionally. The illiquidity discount, however, remains largely constant, because management fees are risk free, and changes in these fees do not affect the risk of the LP's payoff.[12] When the rate of carried interest k increases, the cost of the GP's incentive fees IF increases almost proportionally. An increase in the incentive fee k from 20% to 30% increases the cost of the GP's compensation from 50.97

12 To some extent, the constant illiquidity discount is also a property of CARA utility, because it lacks wealth effects and hence an absolute reduction of wealth does not change the certainty-equivalent valuation.

Table 8
The present values with non-spanned risks for various management compensation contracts

m	k	GP	LP	V	ID
0.0%	0%	0.00	191.45	134.20	57.26
1.5%	10%	29.75	161.70	113.11	48.58
1.5%	20%	45.55	145.90	105.50	40.40
1.5%	30%	61.25	130.20	97.45	32.75
2.0%	10%	35.36	156.09	107.46	48.63
2.0%[a]	20%	50.97	140.48	100.00	40.48
2.0%	30%	66.49	124.96	92.10	32.86
2.5%	10%	41.72	149.73	101.05	48.68
2.5%	20%	57.13	134.32	93.75	40.57
2.5%	30%	72.43	119.02	86.03	32.99

This table present valuations of the different parts of the waterfall structure with non-spanned risks for various levels of management and incentive fees. The columns refer to: total GP compensation (GP), the LP's partnership interest (LP), the LP's certainty-equivalent valuation (V), and the illiquidity discount (ID). Parameter values are: $\gamma I_0 = 2$, $I_0 = 100$, $l = 3$, $\beta = 0.5$, and $h = 8\%$. Moreover, $\alpha = 2.06\%$ and $cs = 2.63\%$, which are the break-even values in the baseline case with a 2/20 compensation contract.
[a]Indicates the baseline break-even case.

to 66.49, and consequently both LP and V decline. This increase in k from 20% to 30%, reduces the illiquidity discount ID from 40.48 to 32.86, because a greater carried interest transfers both rewards and risks from the LP to the GP, which reduces both the LP's certainty equivalent V and the illiquidity discount ID.

4.2.1 GP compensation and break-even alphas. Table 9 reports break-even alphas for three levels of effective risk aversion and for different compensation contracts, holding fixed $\beta = 0.5$, $h = 8\%$, and $l = 3$. The break-even alphas in Table 9 allow us to evaluate the trade-offs between management fees, carried interest, and the illiquidity premium.

Consider the case with no fees, $m = k = 0\%$. In this case, an effectively risk-neutral LP requires no alpha, because the LP is risk neutral and requires no premium for the additional risk arising from the illiquidity and there are no fees. For $\gamma I_0 = 2$, the break-even alpha increases to 1.01%, reflecting the required compensation for illiquidity for an LP with a moderate effective risk aversion. Not surprisingly, increasing either m or k also increases the break-even alpha, for a given level of risk aversion γI_0. Similarly, given m and k, increasing the effective risk aversion γI_0 increases the break-even alpha. For example, in the baseline case with both risk aversion and fees, we have $\gamma I_0 = 2$, $m = 2\%$, and $k = 20\%$, and the break-even alpha increases by 1.05% to 2.06% per year, relative to the case without any GP compensation. Measured in terms of break-even alphas, these results show that the cost of illiquidity, of 1.01%, is comparable to the total cost of the GP's compensation of 1.05%.

Finally, the numbers in Table 9 also allows us to compare typical compensation structures. Increasing the rate of carried interest, k, by 10% leads to an increase in the break-even alpha of 0.18%–0.35%, depending on

Table 9
Effects of risk aversion for different compensation contracts on annual break-even alphas

m	k	$\gamma I_0=0_+$	$\gamma I_0=2$	$\gamma I_0=5$
0.0%	0%	0.00%	1.01%	2.24%
1.5%	10%	0.57%	1.65%	2.96%
1.5%	20%	0.85%	1.87%	3.14%
1.5%	30%	1.19%	2.15%	3.36%
2.0%	10%	0.72%	1.83%	3.19%
2.0%[a]	20%	1.01%	2.06%	3.37%
2.0%	30%	1.34%	2.34%	3.59%
2.5%	10%	0.88%	2.03%	3.44%
2.5%	20%	1.17%	2.26%	3.62%
2.5%	30%	1.52%	2.54%	3.85%

This table presents break-even alphas for different levels of effective risk aversion, γI_0, carried interest k, and management fees, m. Other parameters are $\beta=0.5$, $h=8\%$, and $l=3$.
[a]Indicates the baseline break-even case.

risk aversion. Increasing m by 0.5 percentage-point increases the break-even alpha by 0.15%–0.25%. Hence, the cost of a 50-basis point increase in the management fee, m, is largely similar to the cost of a 10 percentage-point increase in the carried interest, k. To illustrate, a common choice is between a 2/20 and a 1.5/30 compensation contract. Table 9 shows that an LP with low-to-moderate risk aversion slightly prefers the 2/20 contract. In contrast, an LP with higher risk aversion ($\gamma I_0=5$) is indifferent between these two contracts (the break-even alphas are 3.36% and 3.37%), because for this LP the cost of the safer management fees is relatively higher than that of the more risky carried interest. This comparison, however, assumes that the GP's alpha is given and does not depend on the contract. If a higher carried interest can screen for better GPs or incentivize the GPs to produce greater alpha, the trade-off may change.

4.3 Horizon effects, illiquidity, fees, and leverage

Table 10 shows how the break-even alpha varies with the PE investment horizon T. A natural intuition is that an LP with a longer PE investment requires a higher break-even alpha. We find, however, that this simple intuition holds only without leverage and without GP compensation, i.e., in the case with $l=0$ and $m=k=0$, as reported in the first row in Table 10. In this simple case, the LP effectively holds the illiquid PE asset. With full-spanning, the LP's valuation is simply the economic value of the PE asset, i.e., $V_0=A_0e^{\alpha T}$, which implies that V_0 is linear in A_0. In contrast, with unspanned risk, the cost of illiquidity implies that $V_0<LP_0$, and LP is concave in A_t. The break-even value of α increases with the horizon to ensure that the LP is willing to hold the illiquid PE asset with a longer duration.

More generally, when $k>0$, or $m>0$, or $l>0$, the relation between the break-even alpha and the investment horizon becomes more complex, depending on the structure of the GP's compensation and leverage. Changes in the horizon, T, induce several different effects.

Table 10
Effects of different maturities on annual break-even alphas

m	k	$T=0.0001$	$T=0.1$	$T=2$	$T=5$	$T=10$	$T=20$
Panel A: $l=0$							
0%	0%	0.26%	0.26%	0.29%	0.34%	0.46%	0.77%
2%	0%	2.26%	2.26%	2.24%	2.25%	2.34%	2.79%
0%	20%	201.25%	6.12%	2.19%	1.51%	1.20%	1.16%
2%	20%	203.31%	9.80%	4.11%	3.42%	3.08%	3.17%
Panel B: $l=3$							
0%	0%	1.05%	1.05%	0.86%	0.86%	1.01%	1.42%
2%	0%	1.55%	1.55%	1.38%	1.41%	1.63%	2.23%
0%	20%	222.14%	8.10%	2.24%	1.59%	1.44%	1.60%
2%[a]	20%	222.64%	8.60%	2.77%	2.16%	2.06%	2.43%

This table presents break-even alphas for different maturities, T. Other parameters are $\gamma I_0=2$, $\beta=0.5$, and $h=8\%$. Note that the break-even alpha is non-monotonic in the maturity T. For the baseline case, the maturity T at which the break-even alpha is the lowest is about 10, in line with the common industry practice.
[a]Indicates the baseline break-even case.

4.3.1 Short horizons. First consider short horizons. In Table 10, column $T=0.0001$ shows that the break-even alpha does not converge to zero as the horizon shrinks, and the limit behavior of the break-even alpha depends on whether the GP charges carried interest or not.

For the case without carried interest, $k=0$, the PDE in (31) implies that the limit value, when $k=0$ and $T\to 0$, of the break-even alpha is:

$$\lim_{T\to 0}\alpha=\frac{m}{1+l}+\frac{\gamma r I_0}{2}\epsilon^2(1+l). \tag{64}$$

Intuitively, the first term in (64) is the management fee per unit of the unlevered PE asset and the second term captures the illiquidity discount. The intuition for the management fee term is straightforward. By leveraging the LP's invested capital, the alpha is earned on a larger asset base, and the required break-even alpha is reduced from m to $m/(1+l)$. Therefore, without leverage and with $m=2\%$, the contribution of management fees to the break-even alpha is simply 2% per year. With leverage of $l=3$, however, the contribution of $m=2\%$ is only 2%/4=0.50%.

The second term in (64) is the break-even alpha required to compensate the LP for the unspanned risk and illiquidity. This term does not go to zero as T goes to zero, because the term reflects the "flow" cost of illiquidity, which is positive for any finite maturity T, even short ones. In the baseline case, without leverage, the contribution of illiquidity to the break-even alpha is 0.26% annually. With leverage of $l=3$, the contribution of illiquidity quadruples to 1.05% annually, because leverage quadruples the size and risk of the unlevered PE asset.

To summarize, reading down column $T=0.0001$ of Table 10, in the limit when the horizon goes to zero, the limit of the break-even alpha depends on the fees as follows: With $m=0$ and without leverage, the break-even alpha is 0.26% annually, due to the cost of illiquidity. With $m=2\%$ and $l=0$, the break-even alpha increases by 2% to 2.26% annually. With $l=3$ and $m=0$, the break-even alpha is 1.05% annually, which reflects the lower cost of management fees and the higher cost of illiquidity with leverage. Finally, with $l=3$ and $m=2\%$, the break-even alpha increases to 1.55% annually, reflecting a cost of 1.05% of illiquidity and a cost of 0.50% (= 2%/4) of management fees.[13] All these results are without carried interest.

With carried interest, the break-even alpha tends to infinity as the horizon T shrinks to zero. At shorter horizons, the carried interest becomes more important than management fees per unit of time. The cost of management fees is proportional to the horizon, T, but the cost of carried interest is proportional to $\sqrt{T}$, because the incentive fee is effectively an option, and its value depends mainly on the underlying PE asset's volatility, which grows at rate $\sqrt{T}$. For example, with $T=0.0001$, the value of the carried interest is about two orders of magnitude greater than the value of management fees, as $T^{-1/2}=10^2$. The value of the GP's alpha, however, only grows at the rate T. Hence, as $T \to 0$, the $\sqrt{T}$ term dominates and the break-even alpha grows to infinity to compensate the LP for the increasing cost of the optionality of the carried interest. Without management fees and leverage, when $T=0.0001$, the break-even alpha is 201.25% annually! Adding management fees of $m=2\%$, the break-even alpha just increases by 2.06% to 203.31% annually, where the increase mostly reflects $m=2\%$, and the remaining 0.06% percentage-point increase is due to a small nonlinear interaction effect. With leverage of $l=3$, the break-even alpha is 222.14% annually without management fees, and it increases by 0.50% to 222.64% annually with $m=2$. This increase is just the management fees per levered amount of PE assets, of 2%/4=0.5% as above, and in this case the nonlinear interaction term is effectively zero.

4.3.2 Long horizons. With longer horizons, the break-even alpha eventually starts to increase due to the increasing illiquidity of the PE investment. In the simple case, without management fees, carried interest, or leverage, the break-even alpha increases monotonically with T, as the only effect is the increasing cost of illiquidity. Intuitively, the longer the investment horizon, the more distorted is the portfolio allocation, and a higher alpha is needed for the LP to break even. In the case with $l=m=k=0$, the break-even alpha is simply the implied (flow) cost of illiquidity per unit of time.

With management fees, the GP must also return these fees to the LP in the preferred return region, which effectively increases the leverage of the LP's claim. In the case with $m=2\%$, $k=0$, and $l=0$, as the horizon T increases, the optionality (the V_{AA} term in the PDE (31)) becomes more important. As a result, the break-even alpha first declines, albeit only marginally, from 2.26% to 2.24%, as T increases from 0.0001 to 2. As the horizon T increases, the illiquidity effect dominates, and the break-even alpha eventually increases.

[13] We confirm that our numerical solution of the break-even alphas using PDE (31) yields the same result as implied by (64).

Table 10 shows that the U-shaped pattern for the break-even alpha holds generally when either $m > 0$ or $k > 0$. With carried interest, the break-even alpha declines more quickly as the option effect induced by carry quickly weakens as T increases beyond one. With both leverage and fees, the U-shaped pattern becomes more pronounced. Interestingly, in Table 10, the most realistic case is the last one in Panel B, which includes leverage, management fees, and carried interest. In this case, the lowest break-even alpha arises with a horizon of ten years. Hence, holding the GP's alpha fixed, a ten-year horizon may be close to optimal.

5. Empirical Performance Measures

In reality, the alphas generated by actual GPs are difficult to estimate, and more readily available performance measures are used in practice, such as the public market equivalent (PME) and the internal rate of return (IRR). To define these measures, divide the cash flows between the LP and GP into capital calls and distributions: $Call_t$ denotes cash flows paid by the LP to the GP, including the LP's payment of ongoing management fees, and $Dist_t$ are the cash flows returned to the LP from the GP. The IRR is defined as the solution to $1 = \sum \frac{Dist_t}{(1+\text{IRR})^t} / \sum \frac{Call_t}{(1+\text{IRR})^t}$ and $\text{PME} = \sum \frac{Dist_t}{1+R_t} / \sum \frac{Call_t}{1+R_t}$, where R_t is the cumulative realized return on the market portfolio up to time t. Informally, the PME is the present value of the returned (distributed) capital relative to the present value of the invested (called) capital, where the present values are calculated using the realized market returns as the discount rate. Empirical studies typically interpret $\text{PME} > 1$ as PE investments outperforming the market, implicitly assuming a (levered) beta of one, as noted by Kaplan and Schoar (2004).

There are several issues with the PME measure. First, the denominator blends two cash flows: the investment, I_0, and the management fees, mX_0, as noted by Ljungqvist and Richardson (2003). Intuitively, management fees are a risk-free claim and should be discounted at the risk-free rate. Second, the LP only receives the proceeds after debt is fully repaid and shares the profits with the GP, both of which imply that the LP's payoffs has a risk profile that is different from the one for the underlying PE asset, although it is still more risky than the management fees. Hence, the LP's payoff should be discounted at a rate that is different from the rate for the underlying PE asset. Third, the PME does not account for the cost of illiquidity. Finally, the (levered) beta of PE investments may not equal one as assumed for the PME measure.

Due to these issues, it is unclear how to interpret the conventional PME measure. Specifically, it is not true that a PME that exceeds one necessarily implies that the LP is better off, economically. This threshold may be greater (or smaller) than one, and it may change with the leverage and the LP's effective risk aversion. Using our model, we can evaluate the PME levels that are required for LPs to break even.

5.1 Analytical performance measures

In the model, it is straightforward to solve for the analytical counterparts to the empirical performance measures. Given an alpha, the implied PME is:

$$\text{PME} = \frac{\mathbb{E}\left[e^{-\mu_S T} LP(A_T, T)\right]}{I_0 + \mathbb{E}\left[\int_0^T e^{-\mu_S t} m X_0 dt\right]}$$
$$= \frac{e^{(\mu_A - \mu_S)T}\left[EC(A_0; Z_0) - nEC(A_0; Z_1) + (n-k)EC(A_0; Z_2)\right]}{I_0 + \frac{mX_0}{\mu_S}\left(1 - e^{-\mu_S T}\right)}. \qquad (65)$$

Here, $EC(A;K)$ is the *expected payoff*, not the price, of a call option with strike price K under the physical measure P, not the risk-adjusted measure $\widetilde{P}$, as given in (C.3) in Appendix C. Figure 3 shows the relation between the GP's alpha and the expected PME, with a 2/20 compensation contract and leverage of $l = 3$.

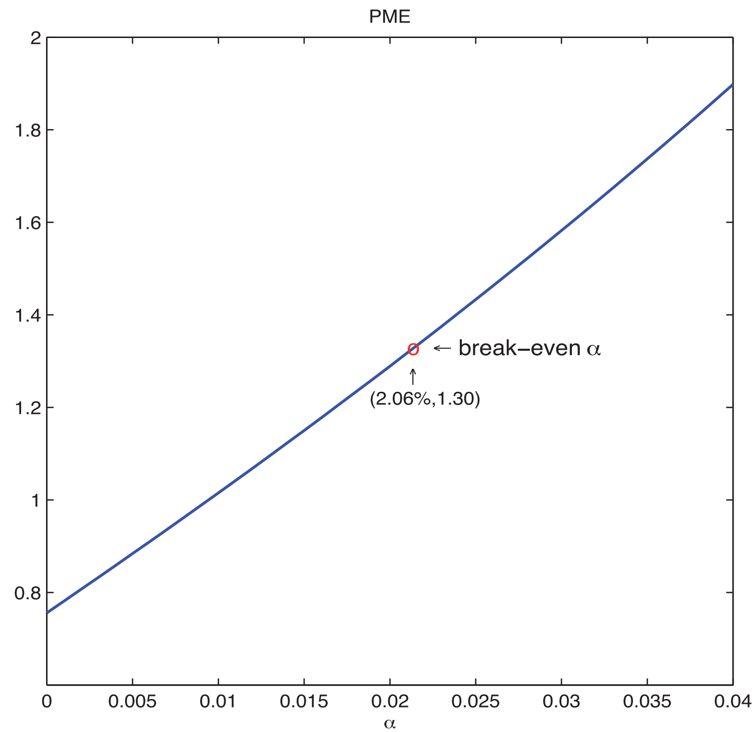

Figure 3
Public market equivalents (PMEs) as a function of the GP's alpha
This figure shows PMEs at different levels of alpha. The circle indicates a PME of 1.3, which is implied by an alpha of 2.06% annually. Other parameters are: $m = 2\%$, $k = 0.2$, $\gamma I_0 = 2$, $\beta = 0.5$, $h = 8\%$, and $l = 3$.

Similarly, let ϕ denote the IRR, which is defined as the solution to:

$$I_0+\int_0^T mX_0 e^{-\phi t}dt = e^{-\phi T}\mathbb{E}[LP(A_T,T)], \tag{66}$$

which simplifies to:

$$I_0+\frac{mX_0}{\phi}\left(1-e^{-\phi T}\right)$$
$$=e^{-(\phi-\mu_A)T}[EC(A_0;Z_0)-nEC(A_0;Z_1)+(n-k)EC(A_0;Z_2)]. \tag{67}$$

5.2 Break-even performance

Given a GP's alpha, we can calculate the expected values of the IRR and PME, using (65) and (66). The alpha is the gross-of-fees performance of the underlying PE asset, but the PME and IRR are based on the LP's cash flows, net of management fees and carried interest, and the expressions for the implied PME and IRR in (65) and (66) explicitly adjust for fees to make the resulting PME and IRR comparable to their empirical counterparts.

Table 11 reports break-even alphas for different levels of effective risk aversion and leverage and the values of the PME and IRR implied by these break-even alphas. This table shows that break-even alphas decrease with leverage, as noted previously. However, the implied break-even values of the IRR and PME increase with leverage. In Panel B, the break-even alpha declines

Table 11
Break-even values of empirical performance measures and the equilibrium credit spread for debt for various levels of effective risk aversion and leverage *l*

Panel A: $\gamma I_0=0_+$					
Leverage (l)	0	1	3	6	9
Alpha (α)	2.61%	1.68%	1.01%	0.63%	0.46%
IRR (ϕ)	7.9%	9.6%	11.2%	12.3%	13.0%
PME	0.75	0.88	1.02	1.13	1.21
Credit spread	N/A	1.05%	3.48%	5.69%	7.14%
Panel B: $\gamma I_0=2$					
Leverage (l)	0	1	3[a]	6	9
Alpha (α)	3.08%	2.46%	2.06%	1.86%	1.77%
IRR (ϕ)	8.4%	10.8%	13.8%	16.8%	19.1%
PME	0.78	0.98	1.30	1.74	2.17
Credit spread	N/A	0.86%	2.63%	3.96%	4.66%
Panel C: $\gamma I_0=5$					
Leverage (l)	0	1	3	6	9
Alpha (α)	3.74%	3.49%	3.37%	3.30%	3.28%
IRR (ϕ)	9.0%	12.3%	16.5%	20.6%	23.5%
PME	0.83	1.13	1.68	2.51	3.33
Credit spread	N/A	0.66%	1.91%	2.70%	3.08%

This table presents break-even valuations of the empirical performance measures and the equilibrium credit spread for debt, as implied by the break-even alphas, for different levels of effective risk aversion, γI_0, and leverage, l. Other parameters are $\beta=0.5$, $m=2\%$, $k=20\%$, and $h=8\%$.
[a]Indicates the baseline break-even case.

from 3.08% to 1.77% as leverage increases from $l=0$ to $l=9$. Yet, the break-even IRR increases from 8.4% to 19.1% annually, and the break-even PME increases from 0.78 to 2.17. In the baseline case, with $l=3$, the LP's break-even IRR is 13.8% annually, and the break-even PME is 1.30. This break-even PME can also be found in Figure 3. The figure shows that a fund's expected PME increases almost linearly in alpha. With an alpha of 2.06% annually, the corresponding PME is 1.30, as indicated by a circle.

It is interesting to compare these theoretical break-even values to their empirical counterparts. Harris, Jenkinson, and Kaplan (2011) summarize estimates of the empirical performance measures across datasets and studies.[14] They report value-weighted average IRRs of 12.3%-16.9%, which is close to the break-even IRR of 13.8% in our baseline case in Table 11. The IRR, however, is an absolute performance measures, which does not adjust for the market performance. For this reason, Harris, Jenkinson, and Kaplan (2011) prefer the PME measure, which is a relative performance measure. They report average value-weighted PMEs of 1.20–1.27, which are close to our theoretical break-even PME of 1.30. While a PME of 1.27 can be interpreted as PE outperforming the market, this outperformance may be just sufficient to compensate the LPs for risk and illiquidity. This is consistent with standard theories of competition and entry in financial markets (e.g., Berk and Green 2004). For a less risk-averse LP, a PME of 1.27 implies positive economic performance.

Table 11 further shows that the credit spread increases with leverage and declines as effective risk aversion increases. A more risk-averse LP requires a greater break-even alpha, which reduces the required equilibrium credit spread. The magnitude of the equilibrium spread is consistent with actual spreads. Table 11 shows that the equilibrium credit spread decreases from 3.48% to 2.63%, due to the higher alpha, as the effective risk aversion γI_0 increases from 0_+ to 2. Ivashina and Kovner (2010) report average and median spreads (to LIBOR) of 3.14% and 3.00% for syndicated loans used to finance PE transactions.

6. Conclusion

To value PE investments, we develop a model of the asset allocation for an institutional investor. The model captures the main institutional features of PE, including: (1) the LP's inability to trade or rebalance its PE investment, and the resulting long-term illiquidity and unspanned risks; (2) GPs creating value and generating alpha by managing the fund's portfolio companies; (3) GP compensation, including management fees and performance-based carried interest; and (4) leverage and the pricing of the resulting risky debt. The

[14] The studies include Ljungqvist and Richardson (2003), Kaplan and Schoar (2005), Jegadeesh, Kraussl, and Pollet (2010), Phalippou and Gottschalg (2009), Korteweg and Sorensen (2010), Metrick and Yasuda (2010), Robinson and Sensoy (2012), and Stucke (2011).

model delivers tractable expressions for the LP's asset allocation and provides an analytical characterization of the certainty-equivalent valuation of the PE investment.

An important benchmark is the full-spanning case where the risk of the PE asset is fully spanned by publicly-traded assets. In this full-spanning case, we can value the individual parts of the waterfall compensation structure, and we derive closed-form expressions for the present values of the GP's compensation, including both management and incentive fees. Our pricing model differs from standard Black-Scholes option pricing, even under full spanning, because it must allow for the GP's value-adding skill, which means that the underlying PE asset earns a positive alpha. In contrast, standard Black-Scholes pricing does not permit risk-adjusted excess returns for any security. Quantitatively, we find that the costs of both management fees and incentive fees are large, in present value terms. This finding corroborates the existing findings for PE compensation from Metrick and Yasuda (2010).

The second important contribution is that our model allows us to evaluate the cost of illiquidity of long-term PE investments. When the risk of the PE asset is not fully spanned by the traded assets, the LP's risk of the PE investments cannot be fully hedged by dynamically trading in the public market. The additional non-spanned risk exposure increases the risk of the LP's overall portfolio. To evaluate the resulting costs of illiquidity and the GP's compensation with non-spanned risks, we calculate the alpha that the GP must generate for the LP to break even, in certainty-equivalent terms. This break-even alpha can be interpreted as the LP's additional cost of capital in addition to the standard CAPM-implied cost of capital, due to the costs of illiquidity and the GP's compensation. Quantitatively, we find that the cost of illiquidity is substantial. Evaluated in terms of break-even alphas, it is comparable to the total cost of the GP's compensation, including both management fees and carried interest. Broadly speaking, the LP's total costs of the PE investment in present-value terms are 50% illiquidity, 25% management fees, and 25% carried interest.

Leverage reduces the break-even alpha. Intuitively, leverage increases the amount of asset managed by the GP, and allows the GP to earn alpha on a greater asset base. Hence, holding management fees fixed, leverage reduces the effective management fee per dollar of PE assets under management by the GP. Additionally, leverage allows better-diversified creditors to bear more of the risk of the underlying PE investment. Both forces cause the break-even alpha to decrease with leverage, which may provide a new justification for the high levels of debt used in PE transactions.

Finally, we use our model to evaluate actual PE performance. The performance of PE funds is typically evaluated in terms of their IRR and PME. Our model gives break-even values of these two performance measures, and we find that the break-even values implied by our model are reasonably close to the actual reported performance for buyout funds, suggesting that LPs in these funds may just break even, on average, which is consistent with Berk and Green

(2004). LPs with lower effective risk aversion and more skilled LPs, who can exploit the performance persistence of PE firms, may still earn economic rents from PE investments.

Appendix

In Appendix A, we sketch out the general incomplete-market solution. We then derive preference-independent results for the full-spanning case in Appendix B. Finally, Appendix C contains the main argument for the invariance result of Proposition 2 and some details for performance measures.

A. The general incomplete-markets solution

We solve the investor's problem by using dynamic programming. First, we report the results for the post-exit stage and then solve the pre-exit stage.

A.1 The post-exit stage and Proposition 1

After exiting from holding the illiquid asset, investors solve a classic Merton-type consumption and portfolio allocation problem by investing in the risk-free asset and public equity. The wealth dynamics is given by:

$$dW_t = (rW_t - C_t)dt + \Pi_t\left((\mu_S - r)dt + \sigma_S dB_t^S\right), \quad t \geq T. \tag{A.1}$$

Let $J^*(W)$ denote investors' value function after time T, i.e.,

$$J^*(W) = \max_{\Pi, C} \mathbb{E}\left[\int_t^\infty e^{-\zeta(s-T)} U(C_s)ds\right], \quad t \geq T. \tag{A.2}$$

Using the standard dynamic programming method, we have the following Hamilton-Jacobi-Bellman (HJB) equation for $J^*(W)$:

$$\zeta J^*(W) = \max_{\Pi, C} U(C) + (rW + \Pi(\mu_S - r) - C)J_W^*(W) + \frac{1}{2}\Pi^2\sigma_S^2 J_{WW}^*(W). \tag{A.3}$$

The FOCs for the portfolio allocation Π and consumption C are:

$$U'(C) = J_W^*(W), \tag{A.4}$$

$$\Pi = -\frac{(\mu_S - r)J_W^*(W)}{\sigma_S^2 J_{WW}^*(W)}. \tag{A.5}$$

We conjecture that $J^*(W)$ is given by (22). And then substituting (22) into (A.4) and (A.5) respectively, we obtain the optimal consumption rule given by (24) and the portfolio allocation rule given by (25) in Proposition 1. Finally, substituting (22), (24), and (25) into the HJB equation (A.3), we obtain the explicit formula for b given in (23).

A.2 Pre-exit stage and details for the key valuation equation (31)

Before exiting from the PE investment, the LP is exposed to illiquidity risk. In this general case, the LP's value function $J(W, A, t)$ solves the following HJB equation,

$$\begin{aligned} \zeta J(W, A, t) &= \max_{\Pi, C} \quad U(C) + J_t + (rW + \Pi(\mu_S - r) - mX_0 - C)J_W \\ &\quad + \frac{1}{2}\Pi^2\sigma_S^2 J_{WW} + \mu_A A J_A + \frac{1}{2}\sigma_A^2 A^2 J_{AA} + \rho\sigma_S\sigma_A \Pi A J_{WA}. \end{aligned} \tag{A.6}$$

We conjecture that $J(W,A,t)$ takes the exponential form as specified in (28). The FOCs for C and Π are:

$$U'(C) = J_W(W,A,t), \tag{A.7}$$

$$\Pi\sigma_S^2 J_{WW}(W,A,t) = -(\mu_S - r)J_W(W,A,t) - \rho\sigma_S\sigma_A A J_{WA}(W,A,t). \tag{A.8}$$

Using the FOC (A.7) for C and the FOC (A.8) for Π, we obtain the consumption and portfolio rules given in (29) and (30), respectively. And then substituting the conjectured value function $J(W,A,t)$ specified in (28) into the HJB equation (A.6), we obtain:

$$-\frac{\zeta}{\gamma r} = \max_{\Pi,C} -\frac{e^{-\gamma(C-r(W+b+V))}}{\gamma} + V_t + rW + \Pi(\mu_S - r) - mX_0 - C$$
$$+\mu_A A V_A + \frac{1}{2}\sigma_A^2 A^2 V_{AA} - \frac{\gamma r}{2}\left(\Pi^2\sigma_S^2 + 2\rho\sigma_S\sigma_A \Pi A V_A + \sigma_A^2 A^2 V_A^2\right). \tag{A.9}$$

Finally, substituting the consumption and portfolio rules given in (29) and (30) into (A.9), we obtain the nonlinear PDE (31) for $V(A,t)$.

B. Full spanning

In this Appendix, we use the dynamic replicating portfolio approach to value a claim with the contingent terminal payoff $G(A_T,T)$. First, without loss of generality, we rewrite the dynamics for A_t as:

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma_A\left(\rho dB_t^S + \sqrt{1-\rho^2}\, dB_t^{S'}\right), \tag{B.1}$$

where $B^{S'}$ is a standard Brownian motion that is orthogonal to the standard Brownian motion B^S representing the risk exposure via the public equity. By construction, the correlation between B_t^S and $B_t^{S'}$ is zero. Full spanning requires that the newly introduced tradable asset whose risk is solely driven by $B_t^{S'}$ commands no risk premium. Let S_t' denote the market value of this tradable asset with no risk premium, whose dynamic evolution is given by:

$$\frac{dS_t'}{S_t'} = r dt + \sigma_{S'} dB_t^{S'}. \tag{B.2}$$

Note that the expected growth rate of $\left\{S_t' : t \geq 0\right\}$ is the risk-free rate under the physical measure as this asset carries no risk premium.

We construct a self-financing portfolio by dynamically trading in the public equity whose value process S_t is given by (1), the newly introduced public asset whose value process S_t' is given in (B.2), and the risk-free asset to replicate the claim whose terminal payoff is given by $G(A_T,T)$. Let Δ_t and Δ_t' denote the number of shares for the publicly-traded assets whose prices are S_t and S_t', respectively. Let Θ_t denote the dollar amount invested in the risk-free asset. The total value of this self-financing portfolio at time t is then given by $Q_t = \Delta_t S_t + \Delta_t' S_t' + \Theta_t$. And the dynamic of the self-financing portfolio is:

$$dQ_t = \Delta_t dS_t + \Delta_t' dS_t' + d\Theta_t, \tag{B.3}$$

$$= rQ_t dt + \Delta_t S_t(\mu_S - r)dt + \Delta_t S_t \sigma_S dB_t^S + \Delta_t' S_t' \sigma_{S'} dB_t^{S'}. \tag{B.4}$$

Let $G(A_t,t)$ denote the constructed shadow value process for the claim with the contingent terminal payoff $G(A_T,T)$. Using Ito's formula, we write the dynamic for the value of contingent claims with terminal payoff $G(A_T,T)$ as follows:

$$dG(A_t,t) = \left(G_t(A_t,t)+G_A(A_t,t)\mu_A A_t+\frac{\sigma_A^2 A_t^2}{2}G_{AA}(A_t,t)\right)dt$$
$$+G_A(A_t,t)\sigma_A A_t\left(\rho dB_t^S+\sqrt{1-\rho^2}dB_t^{S'}\right). \tag{B.5}$$

For the self-financing portfolio to replicate the risk profiles for the contingent claim $G(A_t,t)$, we first match the diffusion parts:

$$\Delta_t S_t\sigma_S=\rho G_A(A_t,t)\sigma_A A_t, \tag{B.6}$$

$$\Delta_t' S_t'\sigma_{S'}=\sqrt{1-\rho^2}G_A(A_t,t)\sigma_A A_t, \tag{B.7}$$

which imply the following delta strategies:

$$\Delta_t = \frac{\rho\sigma_A}{S_t\sigma_S}A_t G_A(A_t,t)=\frac{\beta}{S_t}A_t G_A(A_t,t), \tag{B.8}$$

$$\Delta_t' = \frac{\sqrt{1-\rho^2}\sigma_A}{S_t'\sigma_{S'}}A_t G_A(A_t,t). \tag{B.9}$$

We further equate the drift for $G(A_t,t)$ with that for the self-financing portfolio,

$$rG(A_t,t)+\Delta_t S_t(\mu_S-r)=G_t(A_t,t)+G_A(A_t,t)\mu_A A_t+\frac{\sigma_A^2 A_t^2}{2}G_{AA}(A_t,t). \tag{B.10}$$

Substituting the delta strategies (B.8) and (B.9) into (B.10), we obtain the following PDE for $G(A_t,t)$:

$$rG(A_t,t)=G_t+(r+\alpha)A_t G_A(A_t,t)+\frac{1}{2}\sigma_A^2 A_t^2 G_{AA}(A_t,t), \tag{B.11}$$

where we use the definition of α given in (6). Note that $G(A_t,t)$ is an initial value problem with the terminal boundary condition $G(A_T,T)=\max\{A_T-K,0\}$.

Intuitively, if $G(A_0,0)>Q_0$, then the LP will not hold the asset with terminal payoff $G(A_T,T)$ because the LP can acquire the identical payoff by the dynamic trading strategy discussed above. Alternatively, if $G(A_0,0)<Q_0$, then the LP is strictly better off by buying the asset with the terminal payoff $G(A_T,T)$. Therefore, $G(A_0,0)$ is the valuation at which the LP is indifferent between holding the asset with the terminal payoff $G(A_T,T)$ and choosing the replicating portfolio described above.

By using the Feymann-Kac representation, we can write $G(A_t,t)$ as the conditional expectation specified in (36) under the risk-adjusted probability measure $\widetilde{P}$, where $\widetilde{P}$ is linked to the physical measure P via the following Radon-Nikodym derivative,

$$\frac{d\widetilde{P}}{dP}=\exp\left[-\left(\frac{\widetilde{\eta}\cdot\widetilde{\eta}}{2}T+\widetilde{\eta}\cdot\left(\widetilde{B}_T-\widetilde{B}_0\right)\right)\right], \tag{B.12}$$

where $\widetilde{\eta}$ can be interpreted as the market price of risk given by $\widetilde{\eta}=(\eta,0)^\top$, and $\widetilde{B}=(\widetilde{B}^S,\widetilde{B}^{S'})^\top$ is the standard Brownian motion 2×1 under this new measure $\widetilde{P}$. It is straightforward to show that $\widetilde{B}$ is related to the standard Brownian motion $[B^S,B^{S'}]^\top$ under the original physical measure, via $d\widetilde{B}_t=d\begin{pmatrix}B_t^S\\B_t^{S'}\end{pmatrix}+\begin{pmatrix}\eta\\0\end{pmatrix}dt$. Let $d\widetilde{B}_t^A\equiv\rho d\widetilde{B}_t^S+\sqrt{1-\rho^2}d\widetilde{B}_t^{S'}$. Note that $\widetilde{B}_t^A$ is the standard Brownian motion under this new measure $\widetilde{P}$ and $d\widetilde{B}_t^A=dB_t^A+\rho\eta dt$. And then using the Girsanov theorem, we may write the value of the PE asset A_t under this new measure $\widetilde{P}$ as:

$$dA_t=(r+\alpha)A_t dt+\sigma_A A_t d\widetilde{B}_t^A. \tag{B.13}$$

Importantly, the risk-adjusted drift is $r+\alpha$ rather than r as in the Black-Scholes formula. Finally, note that the systematic risk is fully captured by the change of measure, and we can use the risk-free rate to discount the payoffs after making the appropriate adjustment of the process for A_t from (3) to (B.13).

By integrating under the risk-adjusted measure $\widetilde{P}$, we obtain the following explicit solution:

$$\begin{aligned}\mathcal{C}all(A_t,t;\alpha,K) &= \widetilde{\mathbb{E}}_t\left[e^{-r(T-t)}\max\{A_T-K,0\}\right],\\ &= e^{\alpha(T-t)}\widetilde{\mathbb{E}}_t\left[e^{-(r+\alpha)(T-t)}\max\{A_T-K,0\}\right],\\ &= e^{\alpha(T-t)}\left[A_tN(d_1)-Ke^{-(r+\alpha)(T-t)}N(d_2)\right],\\ &= A_te^{\alpha(T-t)}N(d_1)-Ke^{-r(T-t)}N(d_2), \end{aligned} \tag{B.14}$$

where

$$d_1 = d_2+\sigma_A\sqrt{T-t}, \tag{B.15}$$

$$d_2 = \frac{\ln\left(\frac{A_t}{K}\right)+\left(r+\alpha-\frac{\sigma_A^2}{2}\right)(T-t)}{\sigma_A\sqrt{T-t}}. \tag{B.16}$$

C. Details for Proposition 2 and performance measures

C.1 Details for the invariance result in Proposition 2

Substituting $V(A,t)=v(a,t)\times I_0$ into (31) and using $a=A/I_0$, $x_0=X_0/I_0$, we obtain (57). Substituting (13), (15), and (16) into (17), we have:

$$LP(A,T)=\max\{A-Z_0,0\}-n\max\{A-Z_1,0\}+(n-k)\max\{A-Z_2,0\}. \tag{C.1}$$

Substituting $z_0=Z_0/I_0$, $z_1=Z_1/I_0$, $z_2=Z_2/I_0$ into the maturity T-payoff condition (32), we obtain (58). Finally, substituting $V(A,t)=v(a,t)\times I_0$ and $x_0=X_0/I_0$ into (33), we obtain (59).

C.2 Details for performance measures

Let $EC(A;K)$ denote the *expected discounted payoff*, not value, of a call option with strike price K under the physical measure,

$$EC(A;K) = \mathbb{E}_0\left[e^{-\mu_A T}\max\{A_T-K,0\}\right]. \tag{C.2}$$

Using the standard integration result involving normal random variables, we obtain:

$$EC(A;K) = AN(p_1(A;K))-Ke^{-\mu_A T}N(p_2(A;K)), \tag{C.3}$$

where $p_1(A;K)$ and $p_2(A;K)$ are given by:

$$p_1(A;K) = p_2(A;K)+\sigma_A\sqrt{T}, \tag{C.4}$$

$$p_2(A;K) = \frac{\ln(\frac{A}{K})+\left(\mu_A-\frac{\sigma_A^2}{2}\right)T}{\sigma_A\sqrt{T}}. \tag{C.5}$$

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